(1) Consider the LPP: maximize \( z = 3x_1 + 4x_2 \) subject to:

\[
\begin{align*}
  x_1 + 2x_2 &\leq 10, \\
  x_1 + x_2 &\leq 8, \\
  3x_1 + 5x_2 &\leq 26, \\
  x_1, x_2 &\geq 0
\end{align*}
\]

(a) Apply the simplex method to this LPP. At each step, write down the matrices \( B \) and \( B^{-1} \).

**Solution:** At each step, if the current basic variables are \( x_i, x_j, \) and \( x_k \) in that order, then the columns of \( B \) are the columns of the *initial* tableau corresponding to \( x_i, x_j, \) and \( x_k \) in that order.

Since the columns corresponding to the slack variables \( x_3, x_4, \) and \( x_5 \) start out as the 3 by 3 identity matrix, and the system has 3 constraints, whatever row operations transform \( B \) into the identity transform these 3 columns into \( B^{-1} \). Thus at each step, \( B^{-1} \) is the 3x3 submatrix which appears in the columns corresponding to the slack variables, \( x_3, x_4, \) and \( x_5 \).

Carrying out the simplex method gives the following:

\[
\begin{bmatrix}
3 & 4 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 3 & 5 & 0 & 0 \\
0 & -3 & -4 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 4 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 3 & 5 & 0 & 0 \\
0 & -3 & -4 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
5 & 0 & 1 \\
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 1 & 0 \\
-\frac{5}{2} & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 4 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 3 & 5 & 0 & 0 \\
0 & -3 & -4 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
5 & 0 & 1 \\
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 1 & 0 \\
-\frac{5}{2} & 0 & 1 \\
\end{bmatrix}
\]
The optimal solution is $x = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, with maximum objective function value $z = 25$.

(b) From the final tableau, find the solution to the dual LPP directly.

**Solution:** We read off the values of the dual variables at an optimal solution to the dual LPP from the objective row of the final simplex tableau of the primal LPP:

$$w_1 = 0, w_2 = \frac{3}{2}, w_3 = \frac{1}{2}.$$  

(c) Use the duality theorem and your solution to the dual LPP to check that your solution to the primal LPP really is optimal.

**Solution:** The dual LPP is to minimize $z_D = 10w_1 + 8w_2 + 26w_3$ subject to

$$w_1 + w_2 + 3w_3 \geq 3$$
$$2w_1 + w_2 + 5w_3 \geq 4$$
$$w_1, w_2, w_3 \geq 0$$

First check that $w_1 = 0, w_2 = \frac{3}{2}, w_3 = \frac{1}{2}$ is a feasible solution of the dual by substituting into all three constraints, which are indeed all satisfied.

Next, compute $z_D(0, \frac{3}{2}, \frac{1}{2}) = 25$, which equals the optimal value of the primal objective function. Since the
primal and dual problems each have a feasible solution which generates the objective function value 25, it follows from the duality theorem that these solutions of the primal and dual are optimal.

(2) Suppose that $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 0$ is an optimal solution of the LPP

Maximize $z = 5x_1 + 2x_2 + x_3 + x_4$ subject to

\[
\begin{align*}
2x_1 + x_2 + x_3 + 2x_4 &\leq 6 \\
3x_1 + x_4 &\leq 15 \\
5x_1 + 4x_2 + x_4 &\leq 24 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

Using the principle of complementary slackness, find which variables of the dual problem (including its surplus variables) must equal 0. Explain your reasoning.

**Solution:** The pairwise correspondence between variables of the primal problem and those of the dual problem is as follows:

\[
\begin{align*}
x_1 &= 3 & w_4 &= 0 \\
x_2 &= 0 & w_5 \\
x_3 &= 0 & w_6 \\
x_4 &= 0 & w_7 \\
x_5 &= 0 & w_1 \\
x_6 &= 6 & w_2 &= 0 \\
x_7 &= 9 & w_3 &= 0
\end{align*}
\]

Notice that decision variables of the primal are paired with slack variables of the dual, and vice versa. The values of the primal slack variables are computed from the given constraints, knowing the values $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 0$. All that really matters, though, in order to answer the question as stated, is to know which variables are equal to 0 and which are not. Those primal variables which are equal to 0 correspond to dual variables which *could* take nonzero values at the optimal solution to the dual, and those primal variables which are not equal to 0 correspond to dual variables which must equal 0.