

**Notation:** Let  $G$  be a finite group of order  $N$  whose conjugacy classes are  $C_1, C_2, \dots, C_r$  where  $C_1 = \{1\}$ . We know that  $G$  has exactly  $r$  inequivalent irreducible representations — call them  $\phi_i : G \rightarrow \text{GL}(V_i)$  for  $i = 1, \dots, r$  and let  $\phi_1$  be the trivial representation  $\phi_1(g) = 1 \in \text{GL}(\mathbb{C}) = \mathbb{C}^\times$ . Let  $\dim(V_i) = d_i$  and  $\chi_i$  be the character of  $\phi_i$ . Let  $\mathcal{C} = \{f : G \rightarrow \mathbb{C} \mid f(g) = f(h) \text{ for all } g, h \in C_i\}$  be the vector space of class functions.

**Some Facts About Characters:** Let  $\chi_\phi$  and  $\chi_\rho$  be characters of corresponding representations  $\phi : G \rightarrow \text{GL}(V)$  and  $\rho : G \rightarrow \text{GL}(W)$ .

- $\chi_\phi \in \mathcal{C}$  (Characters are class functions).
- $\phi \cong \rho$  (the representations are equivalent) if and only if  $\chi_\phi = \chi_\rho$ .
- $\chi_\phi(1) = \dim(V)$  and  $\chi_\phi(g^{-1}) = \overline{\chi_\phi(g)}$
- $\chi_\phi(g) = c \in \mathbb{C}$  is an algebraic integer — that is: there exists some monic polynomial  $f \in \mathbb{Z}[x]$  such that  $f(c) = 0$ .

**Regular Representation:** Let  $\mathbb{C}[G]$  be the vector space whose basis is  $\mathbf{G} = \{\mathbf{g} \mid g \in G\}$ .  $G$  acts on  $\mathbb{C}[G]$  via  $g \cdot (\sum_{h \in G} c_h \mathbf{h}) = \sum_{h \in G} c_h \mathbf{gh}$ . Let  $\phi_{\text{reg}} : G \rightarrow \text{GL}(\mathbb{C}[G])$  be the corresponding representation and let  $\chi^{\text{reg}}$  be its character. We know that  $\chi^{\text{reg}}(1) = \dim(\mathbb{C}[G]) = N$  and  $\chi^{\text{reg}}(g) = 0$  for  $g \neq 1$ .

**Big Theorem:** Recall for  $f_1, f_2 \in \mathcal{C}$ ,  $\langle f_1, f_2 \rangle = \frac{1}{N} \sum_{g \in G} \overline{f_1(g)} f_2(g)$ .

- $\langle \chi_i, \chi_j \rangle$  is 1 if  $i = j$  and 0 if  $i \neq j$ . In fact,  $\{\chi_1, \dots, \chi_r\}$  is an orthonormal basis for  $\mathcal{C}$  (the space of class functions on  $G$ ).
- When the regular representation is decomposed into the direct sum of irreducible representations, each irreducible appears with multiplicity equal to its dimension — that is:  $\phi_{\text{reg}} \cong d_1 \phi_1 \oplus d_2 \phi_2 \oplus \dots \oplus d_r \phi_r$ . Which implies that  $\chi^{\text{reg}} = d_1 \chi_1 + d_2 \chi_2 + \dots + d_r \chi_r$ .
- $d_1^2 + d_2^2 + \dots + d_r^2 = N$  and  $d_i$  divides  $N$ .

**Character Table:** Make a table with one row for each irreducible character (i.e. character for an irreducible representation) and one column for each conjugacy class. In the row corresponding to irreducible representation  $\phi_i$  and conjugacy class  $C_j$  record that value  $\chi_i(g)$  where  $g \in C_j$  (since  $\chi_i$  is a class function, every  $g \in C_j$  will give the same answer). This is a **character table** for  $G$ .

- Since the number of inequivalent irreducible characters is equal to the number of conjugacy classes, character tables are square.
- One row will consist entirely of 1's — coming from the trivial representation.
- Since irreducible characters are orthonormal, the rows have “length” one and are “orthogonal” — **IF** — you weight each entry with the size of its corresponding conjugacy class. If  $a_1, \dots, a_r$  and  $b_1, \dots, b_r$  are two rows in the table, then  $|C_1| \bar{a}_1 a_1 + |C_2| \bar{a}_2 a_2 + \dots + |C_r| \bar{a}_r a_r = N$  (remember that our hermitian form has a factor of  $1/N$  in front of the summation) and  $|C_1| \bar{a}_1 b_1 + |C_2| \bar{a}_2 b_2 + \dots + |C_r| \bar{a}_r b_r = 0$ .
- Think of the character table as a (square) matrix. Scale column  $i$  by  $\sqrt{N/|C_i|}$  for  $i = 1, \dots, r$  and call the resulting matrix  $U$ . Then the rows of  $U$  form an orthonormal basis for  $V = \{(c_1, \dots, c_r) \mid c_j \in \mathbb{C}\}$ . Therefore,  $U$  is unitary —  $U^{-1} = U^*$  (inverse = conjugate transpose). This means that the columns of  $U$

are orthonormal as well! Translating back to the character table, we find that given two columns  $a_1, a_2, \dots, a_r$  and  $b_1, b_2, \dots, b_r$ , then  $\bar{a}_1 a_1 + \bar{a}_2 a_2 + \dots + \bar{a}_r a_r = N/|C_i|$  (where  $C_i$  is the conjugacy class corresponding to the columns of  $a$ 's) and  $\bar{a}_1 b_1 + \bar{a}_2 b_2 + \dots + \bar{a}_r b_r = 0$ . Notice in particular that since  $C_1 = \{1\}$  and  $\chi_j(1) = d_j$ , we recover the fact that  $d_1^2 + \dots + d_r^2 = N/1 = N$ .

**Proposition 1:** Let  $\phi : G \rightarrow \text{GL}(V)$  be a representation with character  $\chi$ . Then  $\ker(\phi) = \{g \in G \mid \chi(g) = \chi(1)\}$ .

**Proof:** Let  $g \in \ker(\phi)$  then  $\phi(g) = I_V$  so that  $\chi(g) = \text{tr}(\phi(g)) = \text{tr}(I_V) = \dim(V) = \chi(1)$ . Conversely, assume  $\chi(g) = \chi(1) = \dim(V) = d$ . Let  $\{v_1, \dots, v_d\}$  be a basis for  $V$  consisting of eigenvectors of  $\phi(g)$  (such a basis must exist because we know that  $V$  has a  $G$ -invariant hermitian form which in turn makes  $\phi(g)$  a unitary operator and unitary implies diagonalizable). Let  $\lambda_j \in \mathbb{C}$  be the corresponding eigenvalue of  $v_j$ . We know that since  $g^k = 1$  for some  $k \geq 1$  (since  $G$  is finite),  $\lambda_j$  is a root of unity — that is  $\lambda_j = \cos(\alpha_j) + i \sin(\alpha_j)$  for some  $\alpha_j$  between 0 and  $2\pi$ . But  $d = \chi(g) = \text{tr}(\phi(g)) = \lambda_1 + \dots + \lambda_d$  so that  $(\cos(\alpha_1) + \dots + \cos(\alpha_d)) + i(\sin(\alpha_1) + \dots + \sin(\alpha_d)) = d + i0$ . Therefore,  $\cos(\alpha_1) = \dots = \cos(\alpha_d) = 1$  and  $\sin(\alpha_1) = \dots = \sin(\alpha_d) = 0$ . So that all of the eigenvalues are 1 which means that  $\phi(g)(v_j) = v_j$  for all  $j$ . Hence  $\phi(g) = I_V$  which implies that  $g \in \ker(\phi)$ .

**Proposition 2:** Let  $\phi = \phi_{i_1} \oplus \dots \oplus \phi_{i_\ell}$  (decompose  $\phi$  into a direct sum of irreducibles). Then  $\ker(\phi) = \bigcap_{k=1}^{\ell} \ker(\phi_{i_k})$ .

**Proof:**  $\phi(g)$  is the identity map if and only if it is the identity on every subspace.

**Detecting Normal Subgroups:** Notice that the regular representation is injective. Let  $H$  be a normal subgroup of  $G$ . Consider the the composition of the natural projection map  $\pi : G \rightarrow G/H$  (where  $\pi(g) = gH$ ) and  $\rho_{\text{reg}} : G/H \rightarrow \text{GL}(\mathbb{C}[G/H])$ . Notice that  $\phi = \rho_{\text{reg}} \circ \pi$  is a representation of  $G$  whose kernel is *exactly*  $H$ . Also,  $\det(\mathbb{C}[G/H]) = [G : H]$ . This shows that every normal subgroup  $H$  appears as the kernel of some representation of dimension  $[G : H]$ . We know that every representation can be (uniquely) decomposed into irreducibles and there are finitely many irreducibles and by Proposition 2 the kernel of a representation is equal to the intersection of the kernels of its irreducible components. Therefore, every normal subgroup of  $G$  must appear as the intersection of the kernels of some set of  $\phi_i, i = 1, \dots, r$  (the irreducible reps. of  $G$ ). Now apply Proposition 1 to determine the kernel of each  $\phi_i$  using the character table. Therefore, we can detect all normal subgroups from the character table alone.

**Determining Simplicity:** The only group with a  $1 \times 1$  character table is the trivial group (which is not simple by definition). Let  $G$  be a non-trivial group. If  $G$  is not simple, then  $G$  has a proper non-trivial normal subgroup. This normal subgroup must be the intersection of some kernels of irreducible representations. This means some irreducible representation has a kernel which is a non-trivial proper normal subgroup. Let  $\chi$  be the character of such an irreducible representation. Now we will use Proposition 1. The kernel is proper so  $\chi$  is not the trivial character (otherwise the kernel is everything). So there is a  $g \in G$  such that  $\chi(g) \neq \chi(1) = 1$ . Also, the kernel is not trivial so there is an  $h \in G$  such that  $\chi(h) = \chi(1)$ . Conversely, if there is such a character,  $G$  has a non-trivial proper normal subgroup. To sum up: A non-trivial group  $G$  is simple if and only if for every irreducible character  $\chi$ ,  $\chi(g) = \chi(1)$  implies that either  $g = 1$  or  $\chi$  is the trivial character.