**Math 135** **First Exam**
**Professor Yuster** **October 10, 2007**

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, BUT NOTE THIS ON THE FRONT, otherwise the back will not be graded and can be used for scratch work. NO CALCULATORS. NO CHEATING. You will have until the end of class to complete the exam. GOOD LUCK!!!

Name: ______________________________

RUID # ______________________________

Section: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 19 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 20 | |
| 6 | 15 | |
| 7 | 7 | |
| 8 | 8 | |
| Total: | 100 | |

RUID # ____________________

1. Evaluate the following:

   (a) (2 points) $\sec \frac{\pi}{3}$

   **Solution:** 2

   (b) (2 points) $e^{3 \ln 2}$

   **Solution:** 8

   (c) (2 points) $\log_4 4^7 - \ln e^{0.5}$

   **Solution:** 6.5

2. A particle moving on the $x$-axis has position

$$s(t) = t^3 + 3t^2 - 24t + 5.$$

   (a) (3 points) Find the velocity $v(t)$ of the particle.

   **Solution:** $v(t) = s'(t) = 3t^2 + 6t - 24$

   (b) (3 points) Find the acceleration $a(t)$ of the particle.

   **Solution:** $a(t) = v'(t) = 6t + 6$

   (c) (3 points) After 1 second has elapsed, is the particle speeding up or slowing down? (show your work)

   **Solution:** $v(1) = -15$ (negative), $a(1) = 12$ (positive). Since the velocity and acceleration at $t = 1$ have opposite signs, the particle is **slowing down.**

   (d) (10 points) Find the total distance traveled by the particle in the first three seconds.

   **Solution:** Solve $v(t) = 0$, get $t = -4, t = 2$. Discard $t = -4$. The only time the particle might have changed direction in the first three seconds is at $t = 2$. To find total distance, evaluate $|s(2) - s(0)| + |s(3) - s(2)| = |-23 - 5| + |-13 - (-23)| =$**38 units**

3. Compute the following limits (write DNE if the limit does not exist):

   (a) (5 points) $\lim_{x \to -2} \frac{x^2 + 6x + 8}{x^2 - 4}$

   **Solution:** $= \lim_{x \to -2} \frac{(x+4)(x+2)}{(x-2)(x+2)} = \lim_{x \to -2} \frac{x+4}{x-2} = \frac{2}{-4} = -\frac{\mathbf{1}}{\mathbf{2}}$

   (b) (5 points) $\lim_{x \to 0} \frac{x+2}{e^x(x+9)}$

   **Solution:** $= \frac{0+2}{e^0(0+9)} = \frac{\mathbf{2}}{\mathbf{9}}$

   (c) (5 points) $\lim_{x \to 0} \frac{\sin 5x}{x}$

   **Solution:** $= \lim_{x \to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} = 5 \lim_{x \to 0} \frac{\sin 5x}{5x} = 5 \cdot 1 = \mathbf{5}$

4. (10 points) The formula for continuous compound interest is $A(t) = Pe^{rt}$. A person invests \$5,000 in a bank that compounds interest continuously. If the investment doubles in 10 years, what is the interest rate? (You can leave natural logarithms in your answer)

   **Solution:** $10,000 = 5,000e^{r \cdot 10} \implies 2 = e^{10r} \implies \ln 2 = 10r \implies r = \frac{\ln 2}{10}$

5. (a) (10 points) Find $g'(x)$ using the FORMAL (i.e. limit) DEFINITION of derivative, where

$$g(x) = \frac{5}{x+1}$$

**Solution:**

$$\begin{aligned}
g'(x) &= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{5}{x+\Delta x+1} - \dfrac{5}{x+1}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{5(x+1) - 5(x+\Delta x+1)}{(x+\Delta x+1)(x+1)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5(x+1) - 5(x+\Delta x+1)}{(x+\Delta x+1)(x+1)\Delta x} = \lim_{\Delta x \to 0} \frac{-5\Delta x}{(x+\Delta x+1)(x+1)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-5}{(x+\Delta x+1)(x+1)} = \mathbf{\frac{-5}{(x+1)^2}}
\end{aligned}$$

(b) (10 points) Find the equation of the tangent line to $g(x)$ at the point $(4, 1)$. Give the equation in slope-intercept form (i.e. $y = mx + b$). You may use your work from part (a).

**Solution:** Use template $y = g'(x_0)(x - x_0) + g(x_0)$:
$x_0 = 4$.
$g(x_0) = g(4) = 1$.
$g'(x_0) = g'(4) = \dfrac{-5}{(4+1)^2} = -\frac{1}{5}$.
Now fill in to template: $y = -\frac{1}{5}(x-4) + 1 \implies$
$\mathbf{y = -\frac{1}{5}x + \frac{9}{5}}$

6. Find the first derivative $\frac{dy}{dx}$ for each of the given functions:

(a) (5 points) $y = x\tan x - 1$

**Solution:**

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}\tan x + \frac{d}{dx}x \cdot \tan x + 0 \\ &= x\sec^2 x + 1\tan x \\ \frac{dy}{dx} &= x\sec^2 x + \tan x\end{aligned}$$

(b) (5 points) $y = \frac{2}{\sqrt{x}} + \frac{x^2}{4}$

**Solution:**

$$\begin{aligned}y &= 2x^{-1/2} + \frac{1}{4}x^2 \\ \frac{dy}{dx} &= -x^{-3/2} + \frac{1}{2}x\end{aligned}$$

(c) (5 points) $y = \frac{x^2}{3\ln x + 2}$

**Solution:**

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3\ln x + 2)\frac{d}{dx}x^2 - x^2\frac{d}{dx}(3\ln x + 2)}{(3\ln x + 2)^2} \\ &= \frac{(3\ln x + 2)2x - x^2(\frac{3}{x})}{(3\ln x + 2)^2} = \frac{6x\ln x + 4x - 3x}{(3\ln x + 2)^2} \\ &= \frac{6x\ln x + x}{(3\ln x + 2)^2}\end{aligned}$$

7. (7 points) Show that the function $f(x) = x - \cos x$ has at least one root (i.e. zero) on the interval $(0, \frac{\pi}{2})$.

> **Solution:** $f(0) = 0 - \cos 0 = -1$ is negative, and $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos(\frac{\pi}{2}) = \frac{\pi}{2}$ is positive. The function $f(x)$ is continuous for all $x$, so we can apply the Root Location Theorem (or the Intermediate Value Theorem): since $f(0)$ and $f(\pi)$ have opposite signs, there must be a root (i.e. zero) of $f$ on the interval $(0, \pi)$.

8. Let

$$f(x) = \begin{cases} 2(x+1) & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x^2 + 2 & \text{if } x > 2. \end{cases}$$

(a) (5 points) Find $\lim_{x \to 2} f(x)$, or write DNE if it does not exist.

> **Solution:** We must find the one-sided limits at $x = 2$, and see if they exist and agree:
> $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2(x+1) = 2(2+1) = 6.$
> $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 2 = 2^2 + 2 = 6.$
> Therefore $\lim_{x \to 2} f(x) = 6$. Note that $f(2)$ is not relevant for this limit.

(a) ______

(b) (3 points) Is $f(x)$ continuous? (show your work)

> **Solution:** $f(2) = 4$. Since $f(2) \neq \lim_{x \to 2} f(x)$, the function is discontinuous at $x = 2$. So NO, $f(x)$ is not continuous.

(b) ______