Math 135 **Second Exam**

Professor Yuster **November 19, 2007**

> Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, BUT NOTE THIS ON THE FRONT, otherwise the back will not be graded and can be used for scratch work. SHOW ALL WORK- a correct answer alone may not receive credit. NO CALCULATORS. NO CHEATING. You will have until the end of class to complete the exam. GOOD LUCK!!!

Name: ______________________________

RUID # ______________________________

Section: ______________________________

Question	Points	Score
1	12	
2	10	
3	10	
4	18	
5	10	
6	10	
7	10	
8	20	
Total:	100	

RUID # ______________________________

1. Find the derivative of the following functions:

   (a) (6 points) $f(x) = (3x + \sin x)^5$

   Solution: $f'(x) = 5(3x + \sin x)^4(3 + \cos x)$

   (b) (6 points) $g(x) = x^2 e^{-\sqrt{x}}$

   Solution: $g'(x) = x^2 e^{-\sqrt{x}}(\frac{-1}{2\sqrt{x}}) + 2xe^{-\sqrt{x}} = xe^{-\sqrt{x}}(-\frac{\sqrt{x}}{2} + 2)$

2. (10 points) Find the equation of the tangent line to the graph of $xy^2 + x^2y = 6$ at the point $(2, 1)$. Give the equation in slope-intercept form (i.e. $y = mx + b$).

Solution: Use implicit differentiation to find $\frac{dy}{dx}$:
$(x(2y\frac{dy}{dx}) + y^2) + (x^2\frac{dy}{dx} + 2xy) = 0$
$\frac{dy}{dx}(2xy + x^2) = -2xy - y^2$
$\dfrac{dy}{dx} = \dfrac{-2xy - y^2}{2xy + x^2}$. To find the slope of the tangent line at $(2, 1)$, plug $x = 2$, $y = 1$ into $\frac{dy}{dx}$:
$\dfrac{-2(2)(1) - 1^2}{2(2)(1) + 2^2} = -\dfrac{5}{8}$. Now we know the slope of the tangent line, and a point on the line: $(2, 1)$, so we can use the point-slope formula: $y - 1 = -\frac{5}{8}(x - 2)$. Put into slope-intercept form: $y = -\frac{5}{8}x + \frac{9}{4}$.

3. (10 points) Give the equations for all vertical and horizontal asymptotes:

$$f(x) = \frac{3x^2 - 5x + 1}{x^2 - 4}$$

> **Solution:** Horizontal: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2-5x+1}{x^2-4} = \lim_{x\to\infty} \frac{3 - \frac{5}{x} + \frac{1}{x^2}}{1 - \frac{4}{x^2}} = 3$, so $y = 3$ is a horizontal asymptote. Note we get the same limit as $x \to -\infty$.
> Vertical: Find the x's that make the denominator equal to zero:
> $x^2 - 4 = 0$, so $(x-2)(x+2) = 0$. Thus $x = 2$ and $x = -2$ are equations for the vertical asymptotes.

4. Find the following limits:

(a) (6 points) $\lim_{x\to 1} \frac{2\ln x}{x-1}$

> **Solution:** This limit is of type $(\frac{0}{0})$. Thus we apply L'Hôpital's Rule, and find that our limit equals $\lim_{x\to 1} \frac{(\frac{2}{x})}{1} = 2$.

(b) (6 points) $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$

> **Solution:** You CANNOT use L'Hôpital's Rule for this problem, since the limit is not of type $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$. This limit can be solved directly by plugging in $x = \pi$: $\frac{\sin\pi}{1-\cos\pi} = \frac{0}{2} = 0$.

(c) (6 points) $\lim_{x\to\infty} x^{1/x}$

> **Solution:** Call the limit L and take natural logs of both sides:
>
> $$\begin{aligned} \ln L &= \ln \lim_{x\to\infty} x^{1/x} \\ \ln L &= \lim_{x\to\infty} \ln x^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \end{aligned}$$
>
> This limit is of type $(\frac{\infty}{\infty})$, so we can apply L'Hôpital's Rule: $\ln L \overset{L'H}{=} \lim_{x\to\infty} \frac{(\frac{1}{x})}{1} = 0$. Since $\ln L = 0$, therefore $e^0 = L$, and so $L = 1$.

5. (10 points) A 10-foot ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall? Include units in your final answer.

Solution: (Picture coming soon!) We would like to find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 1$ ft/s and $x = 6$ ft. From the Pythagorean Theorem, we know $x^2 + y^2 = 10^2$. Differentiate both sides with respect to t: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Now substitute our specific information, noting that $y = 8$ when $x = 6$:
$2(6)(1) + 2(8)\frac{dy}{dt} = 0$
$\frac{dy}{dt} = -\frac{12}{16} = -\frac{3}{4}$ft/s. Note that $\frac{dy}{dt}$ is negative due to the fact that y is a decreasing quantity.

6. (10 points) You measure the edge of a cube to be 20 cm with a possible error of 0.1 cm. Use differentials to estimate the propagated error in the volume of the cube. Include units in your final answer.

Solution: We write the formula for volume of a cube, then find the differential. Plugging in the measured edge length and maximum error in the measured quantity gives the maximum error in the computed quantity, volume:

$$\begin{aligned} V &= e^3 \\ dV &= 3e^2 de \\ dV &= 3(20)^2(0.1) \\ dV &= 120 \text{ cm}^3 \end{aligned}$$

7. (10 points) A box with a square base and open top must have a volume of 32 cm^3. Find the dimensions of the box that minimize the amount of material used. Include units in your final answer.

Solution: (Picture coming soon!) We would like to minimize *surface area* of the box, so write down an equation for surface area (I'll call it A): $A = b^2 + 4bh$. In order to eliminate a variable, use the fact that the box has a volume of 32 cm^3. This means $b^2h = 32$, and so $h = \frac{32}{b^2}$. Substitute this into our equation for A: $A = b^2 + 4b(\frac{32}{b^2}) = b^2 + \frac{128}{b}$. Now minimize this function by taking its derivative and finding critical numbers: $A' = 2b - \frac{128}{b^2}$. This is undefined at $b = 0$. To find any other critical numbers, set the derivative equal to zero: $2b - \frac{128}{b^2} = 0 \implies 2b^3 - 128 = 0 \implies b^3 - 64 = 0 \implies b^3 = 64 \implies b = 4$. We must verify that $b = 4$ is in fact the minimum. One way to do this is with a number line, where we can see that $A'(b)$ is negative (A decreasing) for $b < 4$, and positive (A increasing) for $b > 4$. So $b = 4$, and we find $h = \frac{32}{4^2} = 2$. The dimensions of our box are 4 cm$\times$ 4 cm$\times$ 2 cm.

8. Consider the function $f(x) = x^4 + 4x^3$.

   (a) (5 points) On which interval(s) is the function strictly increasing?

   Solution: $f'(x) = 4x^3 + 12x^2 = 4x(x + 3)$. This gives critical numbers $x = -3$ and $x = 0$. Put these on a number line, and plug in numbers from each interval to find the sign of f'. We find that f is strictly increasing (i.e. $f'(x)$ is positive) on the intervals $(-3, 0) \cup (0, \infty)$. Note that 0 itself must be excluded since the function is not strictly increasing at $x = 0$.

   (b) (5 points) On which interval(s) is the function concave up?

   Solution: $f''(x) = 12x^2 + 24x = 12x(x + 2)$. This gives critical numbers $x = -2$ and $x = 0$. Put these on a number line, and plug in numbers from each interval to find the sign of f''. We find that f is concave up (i.e. positive second derivative) on the intervals $(-\infty, -2) \cup (0, \infty)$.

   (c) (10 points) Sketch the graph of $y = f(x)$. On your graph, label any relative minima with "rm", label any relative maxima with "RM", and label any inflection points with "IP".

   Solution: Graph coming soon! But there is a relative minimum ("rm") at $(-3, -27)$, and inflection points ("IP") at $(-2, -16)$ and $(0, 0)$. There are no relative maxima.