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Degenerate Conformally Invariant Fully Nonlinear Elliptic Equations

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Abstract

We establish Liouville theorems for $C^{1,1}_{loc}$, entire solutions and locally Lipschitz entire weak solutions to general degenerate conformally invariant fully nonlinear elliptic equations of second order. For applications to local gradient estimates of solutions of general conformally invariant fully nonlinear elliptic equations of second order, see [20].

1. Introduction

There has been much work on conformally invariant fully nonlinear elliptic equations and its applications to geometry and topology. See for instance [24], [5], [4], [9, 10], [12, 16], [11], and the references therein. An important issue in the study of such equations is to classify entire solutions which arise from rescaling blow up solutions. Liouville-type theorems for general conformally invariant fully nonlinear second-order elliptic equations have been obtained in [16]. For previous works on the subject, see [16] for a description. Classification of entire solutions to degenerate equations is also of importance, as demonstrated in [6]. In this paper we give Liouville-type theorems for general degenerate conformally invariant fully nonlinear second-order elliptic equations.

Let $\mathcal{S}^{n\times n}$ denote the set of $n \times n$ real symmetric matrices, $\mathcal{S}^{n\times n}_+$ denote the subset of $\mathcal{S}^{n\times n}$ consisting of positive definite matrices, $O(n)$ denote the set of $n \times n$ real orthogonal matrices, $U \subset \mathcal{S}^{n\times n}$ be an open set, and $F \in C^1(U) \cap C^0(\overline{U})$.

We list below a number of properties of (F, U). Subsets of these properties will be used in various lemmas, propositions and theorems:

$$O^{-1}UO = U, \qquad \forall O \in O(n), \tag{1}$$

$$M \in U \text{ and } N \in \mathcal{S}_+^{n\times n} \text{ implies } M+N \in U, \tag{2}$$

$$M \in U \text{ implies } aM \in U \text{ for all positive constant } a, \tag{3}$$

$$\{aI \mid a > 0\} \cap \partial U = \emptyset, \tag{4}$$

where I denotes the $n \times n$ identity matrix.

Let $F \in C^1(U) \cap C^0(\overline{U})$ satisfy

$$F(O^{-1}MO) = F(M), \quad \forall\, M \in U, \forall\, O \in O(n), \tag{5}$$

$$(F_{ij}(M))) > 0, \qquad \forall\, M \in U, \tag{6}$$

where $F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M)$, and,

$$F > 0 \text{ in } U, \qquad F = 0 \text{ on } \partial U. \tag{7}$$

Examples of such (F, U) include those given by the elementary symmetric functions. For $1 \leqq k \leqq n$, let

$$\sigma_k(\lambda) = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the k-th elementary symmetric function and let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone $\Gamma_n := \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i > 0\}$. Let

$$U_k := \{M \in \mathcal{S}^{n\times n} \mid \lambda(M) \in \Gamma_k\},$$

and

$$F_k(M) := \sigma_k(\lambda(M))^{\frac{1}{k}},$$

where $\lambda(M)$ denotes the eigenvalues of M. Then $(F, U) = (F_k, U_k)$ satisfy all the above listed properties, see for instance [3].

Other, much more general, examples are as follows. Let

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex symmetric cone with vertex at the origin}$$

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 := \left\{\lambda \in \mathbb{R}^n \middle| \sum_i \lambda_i > 0\right\}.$$

Naturally, Γ being symmetric means $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_n}) \in \Gamma$ for any permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$.

Let

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma})$$

satisfy

$$f|_{\partial\Gamma} = 0, \qquad \nabla f \in \Gamma_n \text{ on } \Gamma,$$

and

$$f(s\lambda) = sf(\lambda), \qquad \forall\, s > 0 \text{ and } \lambda \in \Gamma.$$

With such (f, Γ), let

$$U := \{M \in \mathcal{S}^{n\times n} \mid \lambda(M) \in \Gamma\},$$

and

$$F(M) := f(\lambda(M)).$$

Then (F, U) satisfies all the above listed properties. In fact, for all these (F, U), $A^u \in U$ implies $\Delta u \leqq 0$ — see below for the definition of A^u. So for these (F, U), the assumption $\Delta u \leqq 0$ in various theorems in this paper is automatically satisfied. We note that in all these examples, F is actually concave in U, but this property is not needed for results in this paper.

As mentioned above, entire solutions to the general equation

$$F\left(A^u\right) = 1, \; u > 0, \; A^u \in U, \qquad \text{in } \mathbb{R}^n$$

are classified in [16]. Here and throughout the paper we use the notation

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

where ∇u denotes the gradient of u and $\nabla^2 u$ denotes the Hessian of u.

In this paper we classify appropriate weak solutions to

$$F\left(A^u\right) = 0, \; u > 0, \qquad \text{in } \mathbb{R}^n.$$

The techniques developed in [16] play important roles in our studies. As in [16], we make use of the method of moving spheres, a variant of the method of moving planes which fully exploits the conformal invariance of the problem. The method of moving planes has been used in classical works of Gidas, Ni and Nirenberg [7] and Caffarelli, Gidas and Spruck [2], and others—see for instance [16] for a description, to study Liouville-type theorems.

For $x \in \mathbb{R}^n$, $\lambda > 0$, and for some function u, we denote the Kelvin transformation of u with respect to $B_\lambda(x)$ by

$$u_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}}u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Here and throughout the paper we use $B_a(x) \subset \mathbb{R}^n$ to denote the ball of radius a and centered at x, and use B_a to denote $B_a(0)$. Also, unless otherwise stated, the dimension n is bigger than 2.

We first introduce a notion of weak solutions to the degenerate equations.

Definition 1.1. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set and $F \in C^1(U)\cap C^0(\overline{U})$ satisfy* (1), (2), (3), (5), (6), *and* (7). *A positive continuous function u on an open set Ω of $\mathbb{R}^n$ is said to be a weak solution of*

$$F(A^u) = 0 \qquad in\ \Omega \tag{8}$$

if there exist $\{u_i\}$ in $C^2(\Omega)$ and $\{\beta_i\}$ in $C^0(\Omega, \mathcal{S}^{n\times n})$ satisfying: for any compact subset K of Ω, there exists $\bar{i}(K)$ such that

$$u_i > 0, \quad A^{u_i} + \beta_i \in U \qquad in\ K,\ \forall\, i \geqq \bar{i}(K), \tag{9}$$

$$u_i \to u, \quad \beta_i \to 0, \quad in\ C^0(K), \tag{10}$$

$$F\left(A^{u_i} + \beta_i\right) \to 0 \quad in\ C^0(K). \tag{11}$$

In $\mathbb{R}^n$, $n \geqq 2$, we use $C^{0,1}_{loc}(\mathbb{R}^n)$ to denote the set of locally Lipschitz functions in $\mathbb{R}^n$, and use $\mathcal{A}$ to denote the set of functions u with the following properties:

(A1) $u \in C^{0,1}_{loc}(\mathbb{R}^n)$, $u > 0$ in $\mathbb{R}^n$, and $\Delta u \leqq 0$ in $\mathbb{R}^n$ in the distribution sense.
(A2) There exists some $\bar{\delta} > 0$ such that for all $0 < \delta < \bar{\delta}$, all $x \in \mathbb{R}^n$, all $\lambda > 0$, and all bounded open set Ω of $\{y \in \mathbb{R}^n \mid |y - x| > \lambda\}$,

$(1+\delta)u \geqq u_{x,\lambda}$ in Ω and $(1+\delta)u > u_{x,\lambda}$ on $\partial\Omega$ imply $(1+\delta)u > u_{x,\lambda}$ in Ω.

Theorem 1.1. *For any $u \in \mathcal{A}$, there exist $\bar{x} \in \mathbb{R}^n$ and constants $a > 0$ and $b \geqq 0$ such that*

$$u(x) \equiv \left(\frac{a}{1 + b|x - \bar{x}|^2}\right)^{\frac{n-2}{2}}, \qquad \forall\, x \in \mathbb{R}^n. \tag{12}$$

Remark 1.1. *If $b = 0$, then $u \equiv u(0)$.*

Theorem 1.2. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set and $F \in C^1(U)\cap C^0(\overline{U})$ satisfy* (1), (2), (3), (4), (5), (6), *and* (7). *Assume that a positive function $u \in C^{0,1}_{loc}(\mathbb{R}^n)$ is a weak solution of*

$$F(A^u) = 0 \qquad in\ \mathbb{R}^n,$$

and satisfies

$$\Delta u \leqq 0 \ \ in\ \mathbb{R}^n\ in\ the\ distribution\ sense. \tag{13}$$

Then $u \equiv u(0)$ in $\mathbb{R}^n$.

Remark 1.2. *The conclusions of* Theorems 1.1–1.2 *still hold when replacing the assumption $u \in C^{0,1}_{loc}(\mathbb{R}^n)$ by $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ for some $p > n$. This can be seen from the proof, since such u is also differentiable almost everywhere.*

Remark 1.3. Theorem 1.2 *can be viewed as an extension of the classical result which asserts that positive harmonic functions in* $\mathbb{R}^n$ *are constants.*

Remark 1.4. *Our notion of weak solutions includes those arising from rescaling blow up solutions.*

Theorem 1.3. *Let* $U \subset \mathcal{S}^{n\times n}$ *be a convex open set and* $F \in C^1(U)\cap C^0(\overline{U})$ *satisfy* (1), (2), (3), (4), (5), (6), *and* (7). *Assume that a positive function* $u \in C^{1,1}(\mathbb{R}^n)$ *satisfies* (13) *and*

$$F\left(A^u\right) = 0, \text{ or equivalently } A^u \in \partial U, \text{ almost everywhere in } \mathbb{R}^n. \tag{14}$$

Then $u \equiv u(0)$ *in* $\mathbb{R}^n$.

Remark 1.5. *If* $U \subset \{M \in \mathcal{S}^{n\times n} \mid Trace(M) \geqq 0\}$, *then a weak solution or* $C^{1,1}$ *solution* u *of* $F(A^u) = 0$ *automatically satisfies* $\Delta u \leqq 0$. *In particular, this is the case for* $(F, U) = (F_k, U_k)$ *for all* k *in all dimensions* n.

Remark 1.6. *It was proved by* Chang, Gursky *and* Yang *in* [6] *that positive* $C^{1,1}(\mathbb{R}^4)$ *solutions to* $F_2(A^u) = 0$ *are constants.* Aobing Li *proved in* [13] *that positive* $C^{1,1}(\mathbb{R}^3)$ *solutions to* $F_2(A^u) = 0$ *are constants, and, for all* k *and* n, *positive* $C^3(\mathbb{R}^n)$ *solutions to* $F_k(A^u) = 0$ *are constants. Our proof is completely different.*

We give in the following a notion of weak solutions to more general equations.

Definition 1.2. *Let* $U \subset \mathcal{S}^{n\times n}$ *be an open set,* $F \in C^1(U) \cap C^0(\overline{U})$, *and* h *be a continuous function on an open subset* Ω *of* $\mathbb{R}^n$. *A positive function* $u \in C^0(\Omega)$ *is said to be a weak solution of*

$$F\left(A^u\right) \leqq h \quad \text{in } \Omega \tag{15}$$

if there exist $\{u_i\}$ *in* $C^2(\Omega)$ *and* β_i *in* $C^0(\Omega, \mathcal{S}^{n\times n})$ *such that, for any compact subset* K *of* Ω, (9), (10) *hold, and*

$$[F(A^{u_i} + \beta_i) - h]^+ \to 0 \quad \text{in } C^0(K), \tag{16}$$

where we have used the notation $w^+ := \max\{w, 0\}$.

Similarly we define that u *is a weak solution of*

$$F(A^u) \geqq h \quad \text{in } \Omega \tag{17}$$

by changing $[F(A^{u_i} + \beta_i) - h]^+$ *to* $[h - F(A^{u_i} + \beta_i)]^+$ *in* (16).

We say that u *is a weak solution of*

$$F(A^u) = h \quad \text{in } \Omega$$

if it is a weak solution of both (15) *and* (17).

We also establish the following results concerning degenerate equations on $\mathbb{R}^n \setminus \{0\}$.

Theorem 1.4. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set and $F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (1), (2), (3), (4), (5), (6), *and* (7). *Assume that a positive function $u \in C^{0,1}_{loc}(\mathbb{R}^n \setminus \{0\})$ is a weak solution of*

$$F(A^u) = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\},$$

and satisfies

$$\Delta u \leqq 0 \ \text{ in } \mathbb{R}^n \setminus \{0\} \text{ in the distribution sense}. \tag{18}$$

Then

$$u_{x,\lambda}(y) \leqq u(y), \qquad \forall\, 1 < \lambda < |x|, |y - x| \geqq \lambda, y \neq 0. \tag{19}$$

Consequently, u is radially symmetric about the origin and $u'(r) \leqq 0$ for almost all $0 < r < \infty$.

Theorem 1.5. *In addition to the hypotheses on (F, U) in* Theorem 1.4, *we assume that U is convex. Assume that a positive $C^{1,1}(\mathbb{R}^n \setminus \{0\})$ function u satisfies* (18) *and*

$$F\left(A^u\right) = 0, \text{ or equivalently } A^u \in \partial U, \text{ almost everywhere in } \mathbb{R}^n \setminus \{0\}.$$

Then (19) *holds and, consequently, u is radially symmetric about the origin and $u'(r) \leqq 0$ for all $0 < r < \infty$.*

Our proofs of Theorem 1.4 and Theorem 1.5 make use of the following improvement of a result in a companion paper [18]. Let $\varphi(\lambda) \equiv \lambda$ and let, for a function v, and for $x, y \in \mathbb{R}^n$, λ close to 1,

$$\Phi(v, x, \lambda; y) := \varphi(\lambda) v(x + y).$$

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing the origin* 0. *We assume that $u \in C^0(\Omega \setminus \{0\})$, v is $C^{0,1}$ in some open neighborhood of $\overline{\Omega}$,*

$$v > 0 \qquad \text{in } \overline{\Omega}, \tag{20}$$

$$\Delta u \leqq 0 \qquad \text{in } \Omega \setminus \{0\}, \tag{21}$$

$$u \geqq v \qquad \text{in } \Omega \setminus \{0\}. \tag{22}$$

Assume that there exists some $\varepsilon_4 > 0$ such that for any $|x| < \varepsilon_4$ and $|\lambda - 1| < \varepsilon_4$,

$$\inf_{\Omega\setminus\{0\}} [u - \Phi(v, x, \lambda; \cdot)] = 0 \text{ implies } \liminf_{|y|\to 0} [u(y) - \Phi(v, x, \lambda; y)] = 0. \tag{23}$$

Then either

$$\liminf_{|x|\to 0} [u(x) - v(x)] > 0, \tag{24}$$

or $u = v = v(0)$ near the origin.

Remark 1.7. *Replacing "v is $C^{0,1}$ in some open neighborhood of $\overline{\Omega}$" by "v is C^1 in some open neighborhood of $\overline{\Omega}$", the conclusion of* Theorem 1.6 *was established in* [18].

Remark 1.8. *It is easy to see from the proof of* Theorem 1.6 *that the assumption $v \in C^1$ in* Theorems 1.8–1.11 *in* [18] *can be replaced by $v \in C^{0,1}$, and the conclusions still hold.*

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we give some properties of weak solutions and $C^{1,1}$ solutions. In particular, we give comparison principles, see Proposition 3.1 and Proposition 3.2, for weak solutions. A crucial ingredient in our proof of the comparison principles is Lemma 3.7, "the first variation" of the operator A^u. Theorem 1.2 and Theorem 1.3 are proved in Section 3 by first showing that u belongs to $\mathcal{A}$ and then showing that the b in (12) must be zero. In Section 4 we first establish some further comparison principles, Proposition 4.1 and Proposition 4.2, which allow the presence of isolated singularities. Then we prove Theorem 1.4 and Theorem 1.5. Proofs of results in this section rely on Theorem 1.6.

This paper appeared in preprint form in [19]. The result for positive $C^3(\mathbb{R}^n)$ solutions of $F_k(A^u) = 0$ as stated in Remark 1.6 is also independently established by Sheng, Trudinger and Wang in [23]. It is said in [23] that using approximation as in [22] and [25] it can be shown that this holds for continuous positive viscosity solutions.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. By (A1) and the maximum principle,

$$\liminf_{|y|\to\infty} |y|^{n-2} u(y) > 0.$$

As in the proof of lemma 2.1 in [21] or [12], for any x in $\mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that

$$u_{x,\lambda}(y) \leqq u(y), \qquad \forall\, 0 < \lambda < \lambda_0(x), |y - x| \geqq \lambda.$$

Indeed the only change to make in the proof of Lemma 2.1 in [21] is to understand line 16 on page 37 as valid almost everywhere, since u is Lipschitz, or we can simply prove the monotonicity directly using the Lipschitz regularity of u.

For any $\delta \in (0, \bar{\delta})$, we define

$$\bar{\lambda}_\delta(x) := \sup\{\mu > 0 \mid u_{x,\lambda}(y) \leqq (1+\delta)u(y),\ \forall\, 0 < \lambda < \mu, |y - x| \geqq \lambda\}.$$

If $\bar{\lambda}_\delta(x) < \infty$ for some x, then

$$u_{x,\bar{\lambda}_\delta(x)}(y) \leqq (1+\delta)u(y), \qquad \forall |y - x| \geqq \bar{\lambda}_\delta(x). \tag{25}$$

Lemma 2.1. *If* $\bar{\lambda}_\delta(x) < \infty$ *for some* $0 < \delta < \bar{\delta}$ *and* $x \in \mathbb{R}^n$*, then*

$$\liminf_{|y|\to\infty} |y|^{n-2}\left[(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta(x)}(y)\right] = 0.$$

Proof. Suppose the contrary, then for some $0 < \delta < \bar{\delta}$, $x \in \mathbb{R}^n$, $\bar{\lambda}_\delta(x) < \infty$, and for some $R > 1 + \bar{\lambda}_\delta(x)$ and $\varepsilon_1 > 0$, we have

$$|y|^{n-2}\left[(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta}(y)\right] \geqq 2\varepsilon_1, \qquad \forall\, |y-x| \geqq R. \tag{26}$$

As in the proof of (27) in [21], there exists $\varepsilon_2 > 0$ such that

$$(1+\delta)u(y) - u_{x,\lambda}(y) \geqq \frac{\varepsilon_2}{|y|^{n-2}}, \qquad \forall\, |\lambda - \bar{\lambda}_\delta(x)| < \varepsilon_2, |y-x| \geqq R. \tag{27}$$

In fact this can easily be deduced from (26).

Since

$$(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta(x)}(y) = \delta u(y) > 0, \qquad \forall\, |y-x| = \bar{\lambda}_\delta(x), \tag{28}$$

there exists $0 < \varepsilon_3 < \varepsilon_2$ such that

$$(1+\delta)u(y) > u_{x,\lambda}(y), \qquad \forall\, |\lambda - \bar{\lambda}_\delta(x)| < \varepsilon_3,\ |y-x| = \lambda.$$

Let

$$\Omega := \{y \in \mathbb{R}^n \mid \bar{\lambda}_\delta(x) < |y-x| < R\}.$$

We know from (25), (27) and (28) that

$$(1+\delta)u \geqq u_{x,\bar{\lambda}_\delta(x)} \text{ in } \Omega, \quad (1+\delta)u > u_{x,\bar{\lambda}_\delta(x)} \text{ on } \partial\Omega.$$

Thus, by (A2),

$$(1+\delta)u > u_{x,\bar{\lambda}_\delta(x)} \text{ on } \overline{\Omega}. \tag{29}$$

With (27) and (29), the moving plane procedure can go beyond $\bar{\lambda}_\delta(x)$, violating the definition of $\bar{\lambda}_\delta(x)$. Lemma 2.1 is established. □

Lemma 2.2. *For all* $0 < \delta < \bar{\delta}$ *and for all* $x \in \mathbb{R}^n$,

$$\bar{\lambda}_\delta(x)^{n-2}u(x) = (1+\delta)\liminf_{|y|\to\infty} |y|^{n-2}u(y). \tag{30}$$

Proof. If $\bar{\lambda}_\delta(\bar{x}) < \infty$ for some $0 < \delta < \bar{\delta}$ and some $\bar{x} \in \mathbb{R}^n$. By Lemma 2.1,

$$\begin{aligned}(1+\delta) \liminf_{|y|\to\infty} |y|^{n-2} u(y) &= \lim_{|y|\to\infty} |y|^{n-2} u_{\bar{x},\bar{\lambda}_\delta(\bar{x})}(y) \\ &= \bar{\lambda}_\delta(\bar{x})^{n-2} u(\bar{x}) < \infty. \end{aligned} \tag{31}$$

For any $x \in \mathbb{R}^n$,

$$(1+\delta)u(y) \geqq u_{x,\lambda}(y), \qquad \forall\, 0 < \lambda < \bar{\lambda}_\delta(x),\ |y-x| \geqq \lambda.$$

Multiplying the above by $|y|^{n-2}$ and sending $|y|$ to infinity leads to

$$(1+\delta) \liminf_{|y|\to\infty} |y|^{n-2} u(y) \geqq \lambda^{n-2} u(x), \qquad \forall\, 0 < \lambda < \bar{\lambda}_\delta(x). \tag{32}$$

We deduce from (32) and (31) that

$$\bar{\lambda}_\delta(x)^{n-2} u(x) \leqq \bar{\lambda}_\delta(\bar{x}) u(\bar{x}) < \infty, \qquad \forall\, x \in \mathbb{R}^n.$$

Switching the roles of x and $\bar{x}$ leads to (30) in the case that $\bar{\lambda}_\delta$ is not identically equal to infinity. On the other hand, if $\bar{\lambda}_\delta \equiv \infty$ on $\mathbb{R}^n$, we send λ to ∞ in (32) to obtain (30). Lemma 2.2 is established. □

If $\bar{\lambda}_\delta \equiv \infty$ on $\mathbb{R}^n$ for all $0 < \delta < \bar{\delta}$, then

$$(1+\delta)u(y) \geqq u_{x,\lambda}(y), \quad \forall\, 0 < \delta < \bar{\delta}, x \in \mathbb{R}^n, |y-x| \geqq \lambda > 0.$$

Sending δ to 0 in the above yields

$$u(y) \geqq u_{x,\lambda}(y), \quad \forall\, x \in \mathbb{R}^n, |y-x| \geqq \lambda > 0.$$

This implies $u \equiv u(0)$, see lemma 7.1 in [17].

We only need to consider the case that for some $0 < \delta < \bar{\delta}$, $\bar{\lambda}_\delta$ is not identically equal to infinity. According to Lemma 2.2 and Lemma 2.1, $\bar{\lambda}_\delta(x) < \infty$ for all $x \in \mathbb{R}^n$,

$$(1+\delta)u(y) \geqq u_{x,\bar{\lambda}_\delta(x)}(y), \qquad \forall\, x \in \mathbb{R}^n, |y-x| \geqq \bar{\lambda}_\delta(x),$$

and

$$\liminf_{|y|\to\infty} |y|^{n-2} \left[(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta(x)}(y)\right] = 0, \qquad \forall\, x \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$, let

$$\phi_\delta^{(x)}(y) := x + \frac{\bar{\lambda}_\delta(x)^2 (y-x)}{|y-x|^2}, \quad \psi(y) := \frac{y}{|y|^2}, \quad w^{(x)} := \left(u_{\phi_\delta^{(x)}}\right)_\psi = u_{\phi_\delta^{(x)} \circ \psi},$$

where we have used the notation $u_\psi := |J_\psi|^{\frac{n-2}{2n}} (u \circ \psi)$ with J_ψ being the Jacobian of ψ.

Let

$$D := \left\{x \in \mathbb{R}^n \mid u \text{ is differentiable at } x\right\}.$$

Since u is locally Lipschitz in $\mathbb{R}^n$, the Lebesgue measure of $\mathbb{R}^n \setminus D$ is zero. It is easy to see that for $x \in D$, $w^{(x)}(y)$ is differentiable at $y = 0$ and

$$w^{(x)}(0) = \bar{\lambda}_\delta(x)^{n-2}u(x) = (1+\delta) \liminf_{|y|\to\infty} |y|^{n-2}u(y) = \liminf_{|y|\to 0}(1+\delta)u_\psi(y) > 0,$$

$$w^{(x)} \leqq (1+\delta)u_\psi \qquad \text{in } B_{\delta(x)} \setminus \{0\} \text{ for some } \delta(x) > 0.$$

Following the proof of Theorem 1.3 in [16] (see also [14] and [15]), we obtain, for some $V \in \mathbb{R}^n$, and for all $x \in D$,

$$\begin{aligned}\nabla w^{(x)}(0) &= V,\\ \nabla w^{(x)}(0) &= (n-2)\bar{\lambda}_\delta(x)^{n-2}u(x)x + \bar{\lambda}_\delta(x)^n \nabla u(x)\\ &= (n-2)\alpha_\delta x + \alpha_\delta^{\frac{n}{n-2}} u(x)^{\frac{n}{2-n}} \nabla u(x),\end{aligned}$$

where

$$\alpha_\delta := (1+\delta) \liminf_{|y|\to\infty} |y|^{n-2}u(y) > 0.$$

Thus

$$\nabla_x \left(\frac{n-2}{2}\alpha_\delta^{\frac{n}{n-2}} u(x)^{-\frac{2}{n-2}} - \frac{n-2}{2}\alpha_\delta |x|^2 + V \cdot x \right) = 0, \quad \forall\, x \in D.$$

It follows that u is in $C^\infty(\mathbb{R}^n)$ and, for some $\bar{x} \in \mathbb{R}^n$ and $d > 0$,

$$u(y) \equiv \left(\frac{\alpha_\delta^{\frac{2}{n-2}}}{d + |y-\bar{x}|^2} \right)^{\frac{n-2}{2}}, \qquad \forall\, y \in \mathbb{R}^n.$$

Theorem 1.1 is established.

3. Properties of weak solutions and the proof of Theorem 1.2 and Theorem 1.3

We start with some properties of weak solutions.

Lemma 3.1. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2). *Assume that for a positive C^2 function u in some open subset Ω of $\mathbb{R}^n$, there exist $\{u_i\} \subset C^2(\Omega)$ and $\{\beta_i\} \subset C^0(\Omega, \mathcal{S}^{n\times n})$ such that* (9) *and* (10) *hold on any compact subset K of Ω. Then*

$$A^u \in \overline{U} \qquad \text{in } \Omega. \tag{33}$$

Proof. For any $\bar{x} \in \Omega$, we fix some $\delta > 0$ such that $B_{2\delta}(\bar{x}) \subset \Omega$. Consider, for small $\varepsilon > 0$,

$$u^\varepsilon(y) := u(y) - \frac{\varepsilon}{2}|y - \bar{x}|^2, \qquad y \in B_\delta(\bar{x}).$$

We know from (10) that

$$u_i(\bar{x}) = u^\varepsilon(\bar{x}) + \circ(1),$$

$$u_i(y) = u^\varepsilon(y) + \frac{1}{2}\varepsilon\delta^2 + \circ(1), \qquad y \in \partial B_\delta(\bar{x}),$$

$$u_i(y) \geqq u^\varepsilon(y) + \circ(1), \qquad y \in B_\delta(\bar{x}),$$

where $\circ(1) \to 0$ as $i \to \infty$, uniform in y and ε.

It is easy to see that for some $a_i^\varepsilon = 1 + \circ(1)$, $y_i^\varepsilon \in B_\delta(\bar{x})$,

$$u_i \geqq a_i^\varepsilon u^\varepsilon \qquad \text{on } B_\delta(\bar{x}), \tag{34}$$

$$u_i(y_i^\varepsilon) = a_i^\varepsilon u^\varepsilon(y_i^\varepsilon). \tag{35}$$

Passing to a subsequence in (35), $y_i^\varepsilon \to \bar{y}^\varepsilon$, $u(\bar{y}^\varepsilon) = u(\bar{y}^\varepsilon) - \frac{\varepsilon}{2}|\bar{y}^\varepsilon - \bar{x}|^2$. So

$$\lim_{i \to \infty} y_i^\varepsilon = \bar{x}.$$

By (34) and (35),

$$A^{u_i}(y_i^\varepsilon) \leqq A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon). \tag{36}$$

Thus, by (9) and (2), $A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon) \in \overline{U}$. Sending i to ∞, we have, using (10), $A^{u^\varepsilon}(\bar{x}) \in \overline{U}$. Sending ε to 0, we have $A^u(\bar{x}) \in \overline{U}$. Lemma 3.1 is established. □

Lemma 3.2. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and $F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (6). *Assume that for a positive C^2 function u in some open set Ω of $\mathbb{R}^n$, there exist $\{u_i\} \subset C^2(\Omega)$ and $\{\beta_i\} \subset C^0(\Omega, \mathcal{S}^{n\times n})$ such that* (9) *and* (10) *hold for any compact subset K of Ω, and, for some $h \in C^0(\Omega)$,*

$$[h - F(A^{u_i} + \beta_i)]^+ \to 0, \qquad \text{in } C^0_{loc}(\Omega). \tag{37}$$

Then (33) *holds and u is a classical solution of* (17).

Proof. We know from Lemma 3.1 that (33) holds. Following the proof of Lemma 3.1 from the beginning until (36). Then, by (9), (2), (36) and (6),

$$A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon) \in \overline{U},$$

and

$$F\left(A^{u_i}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon)\right) \leqq F\left(A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon)\right). \tag{38}$$

Since

$$\lim_{i\to\infty}\left(A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon)+\beta_i(y_i^\varepsilon)\right)=A^{u^\varepsilon}(\bar{x}),$$

we have $A^{u^\varepsilon}(\bar{x})\in\overline{U}$ and, using the continuity of F on $\overline{U}$,

$$\lim_{i\to\infty}F\left(A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon)+\beta_i(y_i^\varepsilon)\right)=F\left(A^{u^\varepsilon}(\bar{x})\right).$$

Sending i to ∞ in (38) leads to, in view of (37),

$$F\left(A^{u^\varepsilon}(\bar{x})\right)\geqq h(\bar{x}).$$

Sending ε to 0, we obtain

$$F\left(A^{u}(\bar{x})\right)\geqq h(\bar{x}).$$

Lemma 3.2 is established. □

Lemma 3.3. *Let $U\subset\mathcal{S}^{n\times n}$ be an open set satisfying* (2)*, $F\in C^1(U)\cap C^0(\overline{U})$ satisfy* (6)*. Assume that for a positive C^2 function u in some open set Ω of $\mathbb{R}^n$, there exist $\{u_i\}\subset C^2(\Omega)$ and $\{\beta_i\}\subset C^0(\Omega,\mathcal{S}^{n\times n})$ such that* (9)*,* (10) *and*

$$\sup_i\sup_K|\nabla^2 u_i|<\infty \tag{39}$$

hold for any compact subset K of Ω, and, for some $h\in C^0(\Omega)$,

$$\left[F(A^{u_i}+\beta_i)-h\right]^+\to 0,\qquad \textit{in } C^0_{loc}(\Omega). \tag{40}$$

Then (33) *holds and u is a classical solution of* (15).

Proof. We know from Lemma 3.1 that (33) holds. For any $\bar{x}\in\Omega$, we fix some $\delta>0$ such that $B_{2\delta}(\bar{x})\subset\Omega$. Consider, for small $\varepsilon>0$,

$$u^\varepsilon(y):=u(y)+\frac{\varepsilon}{2}|y-\bar{x}|^2,\qquad y\in B_\delta(\bar{x}).$$

Arguing as in the proof of Lemma 3.1, we find $a_i^\varepsilon=1+\circ(1)$, $y_i^\varepsilon\to\bar{x}$,

$$A^{u_i}(y_i^\varepsilon)\geqq A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon). \tag{41}$$

Clearly, there exist $\alpha_\varepsilon>0$ and $\gamma_i>0$ satisfying $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$ and $\gamma_i\to 0$ as $i\to\infty$, such that

$$A^{a_i^\varepsilon u^\varepsilon}(y_i^\varepsilon)+\beta_i(y_i^\varepsilon)\geqq A^u(\bar{x})-\alpha_\varepsilon I-\gamma_i I. \tag{42}$$

We already know that $A^u(\bar{x})\in\overline{U}$. So, by (41), (42), (2) and (6),

$$A^{u_i}(y_i^\varepsilon)+\beta_i(y_i^\varepsilon)+\alpha_\varepsilon+\gamma_i I\in\overline{U}$$

and

$$F\left(A^u(\bar{x})\right)\leqq F\left(A^{u_i}(y_i^\varepsilon)+\beta_i(y_i^\varepsilon)+\alpha_\varepsilon I+\gamma_i I\right).$$

Due to (39), (10) and the positivity and the continuity of u, $A^{u_i}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon) + \alpha_\varepsilon I + \gamma_i I$ remain bounded. Thus, by the continuity of F,

$$F\left(A^u(\bar{x})\right) \leqq F\left(A^{u_i}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon)\right) + \circ(1) + \circ_\varepsilon(1),$$

where $\circ_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, uniform in i, and $\circ(1) \to 0$ as $i \to \infty$, uniform in ε. Sending i to ∞ and then ε to 0, we obtain, using (40),

$$F\left(A^u(\bar{x})\right) \leqq h(\bar{x}). \tag{43}$$

Lemma 3.3 is established. □

Lemma 3.4. *In* Lemma 3.3*, we drop assumption* (39) *but add $A^u \in U$ in Ω. Then u is a classical solution of* (15).

Proof. Follow the proof of Lemma 3.3 until (42). Since we know that $A^u(\bar{x}) \in U$, the right-hand side of (42) is also in U for large i and small ε. Thus

$$F\left(A^u(\bar{x}) - \alpha_\varepsilon I - \gamma_i I\right) \leqq F\left(A^{u_i}(y_i^\varepsilon) + \beta_i(y_i^\varepsilon)\right).$$

Sending i to ∞ and then ε to 0, we obtain (43). Lemma 3.4 is established. □

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that a positive function $u \in C^0(\Omega)$ is a weak solution of* (8)*. Then, for any constant $b > 0$ and for any $x \in \mathbb{R}^n$, the function $v(y) := b^{\frac{n-2}{2}} u(x + by)$ is a weak solution of*

$$F(A^v) = 0 \qquad in\ \widehat{\Omega} := \{y \in \mathbb{R}^n \mid x + by \in \Omega\}.$$

Proof. It is obvious.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that a positive function $u \in C^0(\Omega)$ is a weak solution of* (8)*. Then, for any $x \in \mathbb{R}^n$ and $\lambda > 0$, $u_{x,\lambda}$ is a weak solution of*

$$F\left(A^{u_{x,\lambda}}\right) = 0 \qquad in\ \Omega_{x,\lambda} := \left\{y \in \mathbb{R}^n \;\middle|\; x + \frac{\lambda^2(y-x)}{|y-x|^2} \in \Omega\right\}.$$

Proof. This follows from the conformal invariance of the operator $F(A^u)$, see for example line 9 on page 1431 of [12]. □

The following is a comparison principle for weak solutions.

Proposition 3.1. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3)*, $F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (6) *and* (7)*, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u, v \in C^0(\overline{\Omega})$ satisfy*

$$u, v > 0, \qquad in\ \overline{\Omega}, \tag{44}$$

and

$$u > v \quad \text{on } \partial\Omega. \tag{45}$$

We assume that there exist $\{\beta_i\}$, $\{\tilde{\beta}_i\} \subset C^0(\Omega, \mathcal{S}^{n\times n})$ *and positive functions* $\{u_i\}$, $\{v_i\} \subset C^2(\Omega)$ *such that, for any compact subset* K *of* Ω,

$$A^{u_i} + \beta_i \in U, \; A^{v_i} + \tilde{\beta}_i \in U, \quad \text{in } \Omega, \tag{46}$$

$$u_i \to u, \;\; v_i \to v, \; \beta_i \to 0, \; \tilde{\beta}_i \to 0, \quad \text{in } C^0(K), \tag{47}$$

and

$$F\left(A^{v_i} + \tilde{\beta}_i\right) \to 0 \;\; \text{in } C^0(K). \tag{48}$$

Then

$$u > v \quad \text{on } \overline{\Omega}. \tag{49}$$

To prove Proposition 3.1, we need to produce appropriate approximations to the $\{u_i\}$. This is achieved by studying "the first variation" of the operator A^u.

Writing

$$w = u^{-\frac{2}{n-2}},$$

we have

$$A^u = A_w := w\nabla^2 w - \frac{1}{2}|\nabla w|^2 I.$$

Lemma 3.7. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$, $w \in C^2(\Omega)$ *satisfy, for some constant* $c_1 > 0$,

$$w \geqq c_1, \quad \text{in } \Omega,$$

and let

$$\varphi(y) = e^{\delta|y|^2}. \tag{50}$$

Then there exists some constant $\delta > 0$, *depending only on* $\sup\{|y| \mid y \in \Omega\}$, *and there exists* $\bar{\varepsilon}$, *depending only on* δ, c_1 *and* $\sup\{|y| \mid y \in \Omega\}$, *such that for any* $0 < \varepsilon < \bar{\varepsilon}$,

$$A_{w+\varepsilon\varphi} \geqq \left(1 + \varepsilon\frac{\varphi}{w}\right) A_w + \frac{\varepsilon\delta}{2}\varphi w I \quad \text{in } \Omega. \tag{51}$$

Proof of Lemma 3.7. Let φ be a fixed function, a computation gives

$$A_{w+\varepsilon\varphi} = A_w + \varepsilon\left\{w\nabla^2\varphi + \varphi\nabla^2 w - \nabla w \cdot \nabla\varphi I\right\} + \varepsilon^2 A_\varphi.$$

Replacing $\nabla^2 w$ by $w^{-1}\left(A_w + \frac{1}{2}|\nabla w|^2 I\right)$ in the above, we have

$$A_{w+\varepsilon\varphi} = (1+\varepsilon\frac{\varphi}{w})A_w + \varepsilon\left\{w\nabla^2\varphi + \frac{|\nabla w|^2}{2w}\varphi I - \nabla w\cdot\nabla\varphi I\right\} + \varepsilon^2 A_\varphi. \quad (52)$$

For the φ in (50),

$$\nabla\varphi(y) = 2\delta\varphi(y)y, \quad \nabla^2\varphi(y) = 2\delta\varphi(y)I + 4\delta^2\varphi(y)y\otimes y.$$

It follows that

$$\begin{aligned} & w\nabla^2\varphi + \frac{|\nabla w|^2}{2w}\varphi I - \nabla w\cdot\nabla\varphi I \\ & \geqq \varphi\left\{2\delta w + \frac{|\nabla w|^2}{2w} - 2\delta\nabla w\cdot y\right\} I \\ & \geqq \varphi\left\{2\delta w + \frac{|\nabla w|^2}{2w} - \left(\frac{|\nabla w|}{\sqrt{4w}}\right)\left(2\delta\sqrt{4w}|y|\right)\right\} I \\ & \geqq \varphi\left\{2\delta w + \frac{|\nabla w|^2}{4w} - 4\delta^2 w|y|^2\right\} I. \end{aligned}$$

It is clear that there exists $\delta > 0$, depending only on $\sup\{|y| \mid y \in \Omega\}$, such that

$$w\nabla^2\varphi + \frac{|\nabla w|^2}{2w}\varphi I - \nabla w\cdot\nabla\varphi I \geqq \delta\varphi w I + \frac{|\nabla w|^2}{4w}\varphi I.$$

For this δ, there exists $\bar{\varepsilon} > 0$, depending only on δ, c_1 and $\sup\{|y| \mid y \in \Omega\}$, such that for all $0 < \varepsilon < \bar{\varepsilon}$,

$$A_{w+\varepsilon\varphi} \geqq \left(1+\varepsilon\frac{\varphi}{w}\right)A_w + \varepsilon\delta\varphi w I + \varepsilon^2 A_\varphi \geqq \left(1+\varepsilon\frac{\varphi}{w}\right)A_w + \frac{\varepsilon\delta}{2}\varphi w I,$$

i.e.

$$A^{(u^{-\frac{2}{n-2}}+\varepsilon\varphi)^{-\frac{n-2}{2}}} \geqq \left(1+\varepsilon\frac{\varphi}{w}\right)A_w + \frac{\varepsilon\delta}{2}\varphi w I.$$

Lemma 3.7 is established. □

Proof of Proposition 3.1. Since shrinking Ω slightly will not affect (45), we may assume without loss of generality that (47) and (48) hold with K replaced by $\overline{\Omega}$ — from now on these equations will be understood in this sense.

We prove (49) by a contradiction argument. Suppose the contrary, then there exists some $\bar{x} \in \Omega$ such that

$$u(\bar{x}) \leqq v(\bar{x}).$$

It is clear that there exist $0 < a \leqq 1$ and $\bar{y} \in \Omega$ such that

$$u \geqq av \qquad \text{in } \Omega, \quad (53)$$

$$u(\bar{y}) = av(\bar{y}). \quad (54)$$

Let δ and φ be as in Lemma 3.7,

$$\varepsilon_i := \sup_{y\in\Omega} \sqrt{\|\beta_i(y)\|} \to 0,$$

and

$$\widehat{u_i} := \left(u_i^{-\frac{2}{n-2}} + \varepsilon_i \varphi\right)^{-\frac{n-2}{2}}.$$

By (51),

$$A^{\widehat{u_i}} \geqq \left(1 + \varepsilon_i \frac{\varphi}{w_i}\right) A^{u_i} + \frac{\varepsilon_i \delta}{2} \varphi w_i I, \tag{55}$$

where $w_i = u_i^{-\frac{2}{n-2}}$.

It is easy to see, using (53), (54), (45), and the convergence of $\widehat{u_i}$ to u and v_i to v, that for some $\bar{\delta} > 0$ and some large integer $\bar{I}$, there exist, for $i, j \geqq \bar{I}$, $a_{ij} \in (\frac{a}{2}, \frac{3a}{2})$ and $y_{ij} \in \{y \in \Omega \mid dist(y, \partial\Omega) > \bar{\delta}\}$ such that

$$\widehat{u_i} \geqq a_{ij} v_j \qquad \text{in } \Omega,$$
$$\widehat{u_i}(y_{ij}) = a_{ij} v_j(y_{ij}).$$

It follows that

$$\nabla\left(a_{ij}^{-1}\widehat{u_i}\right)(y_{ij}) = \nabla v_j(y_{ij}), \qquad \left(\nabla^2(a_{ij}^{-1}\widehat{u_i})(y_{ij})\right) \geqq \left((\nabla^2 v_j(y_{ij})\right),$$

and

$$A^{a_{ij}^{-1}\widehat{u_i}}(y_{ij}) \leqq A^{v_j}(y_{ij}).$$

Thus, in view of (55) and the definition of ε_i, for some $\tilde{I} \geqq \bar{I}$, and for all $i, j \geqq \tilde{I}$,

$$\begin{aligned} A^{v_j}(y_{ij}) + \tilde{\beta}_j(y_{ij}) &\geqq (a_{ij})^{\frac{4}{n-2}} A^{\hat{u}_i}(y_{ij}) + \tilde{\beta}_j(y_{ij}) \\ &\geqq (a_{ij})^{\frac{4}{n-2}} \left(1 + \varepsilon_i \frac{\varphi}{w_i}\right) A^{u_i} + (a_{ij})^{\frac{4}{n-2}} \frac{\varepsilon_i \delta}{2} \varphi w_i I + \tilde{\beta}_j(y_{ij}) \\ &\geqq (a_{ij})^{\frac{4}{n-2}} \left(1 + \varepsilon_i \frac{\varphi}{w_i}\right) \left(A^{u_i} + \beta_i\right)(y_{ij}) \\ &\quad + (a_{ij})^{\frac{4}{n-2}} \frac{\varepsilon_i \delta}{4} \varphi w_i I + \tilde{\beta}_j(y_{ij}). \end{aligned}$$

Fixing $i = \tilde{I}$, we have, for large j,

$$\begin{aligned} A^{v_j}(y_{\tilde{I}j}) + \tilde{\beta}_j(y_{\tilde{I}j}) \geqq (a_{\tilde{I}j})^{\frac{4}{n-2}} &\left(1 + \varepsilon_{\tilde{I}} \frac{\varphi}{w_{\tilde{I}}}\right) \left(A^{u_{\tilde{I}}} + \beta_{\tilde{I}}\right)(y_{\tilde{I}j}) \\ &+ (a_{\tilde{I}j})^{\frac{4}{n-2}} \frac{\varepsilon_{\tilde{I}} \delta}{8} \varphi w_{\tilde{I}} I. \end{aligned}$$

By (46), (2), (3), (6) and (7),

$$\begin{aligned}
&F\left(A^{v_j}(y_{\tilde{I}j}) + \tilde{\beta}_j(y_{\tilde{I}j})\right) \\
&\geqq F\left((a_{\tilde{I}j})^{\frac{4}{n-2}}\left(1+\varepsilon_{\tilde{I}}\frac{\varphi}{w_{\tilde{I}}}\right)\left(A^{u_{\tilde{I}}}+\beta_{\tilde{I}}\right)(y_{\tilde{I}j})\right) \\
&\geqq \min_{y\in\Omega, dist(y,\partial\Omega)\geqq\bar{\delta}} \ \min_{(\frac{a}{2})^{\frac{4}{n-2}}\leqq b\leqq(\frac{3a}{2})^{\frac{4}{n-2}}} F\left(b\left(1+\varepsilon_{\tilde{I}}\frac{\varphi}{w_{\tilde{I}}}\right)\left(A^{u_{\tilde{I}}}+\beta_{\tilde{I}}\right)(y)\right) > 0
\end{aligned}$$

Sending j to ∞ leads to, in view of (48), that

$$0 = \lim_{j\to\infty} F\left(A^{v_j}(y_{\tilde{I}j}) + \tilde{\beta}_j(y_{\tilde{I}j})\right) > 0.$$

Impossible. Proposition 3.1 is established. □

Remark 3.1. *If we further assume in* Proposition 3.1 *that*

$$\sup_i \sup_K \left(|\nabla^2 u_i| + |\nabla^2 v_i|\right) < \infty,$$

then modification of the proof of Proposition 3.2 *yields a somewhat different proof. One observation is needed: in addition to* (61)*, we have $D^2 u_{i,\varepsilon} \geqq D^2 v_{i,\varepsilon}$ on $S_{i,\varepsilon}$. So, for small s, using* (2) *and the continuity of $D^2 u_{i,\varepsilon}$ and $D^2 v_{i,\varepsilon}$, we still have* (62).

Proof of Theorem 1.2. We first prove that $u \in \mathcal{A}$. We only need to verify property (A2) since property (A1) has already been assumed. Let Ω be as in the statement of (A2), then, by Lemma 3.5 and Lemma 3.6, both $u_{x,\lambda}$ and $(1+\delta)u$ are weak solutions of (8). Thus, by Proposition 3.1, (A2) is satisfied. So we have proved that $u \in \mathcal{A}$. By Theorem 1.1, (12) holds for some $a > 0$, $b \geqq 0$ and $\bar{x} \in \mathbb{R}^n$. We only need to prove that $b = 0$. Suppose that $b > 0$, then a computation gives, for some positive constant β,

$$A^u \equiv \beta I \qquad \text{in } \mathbb{R}^n. \tag{56}$$

Since u is a weak solution of (8), the hypotheses of Lemma 3.1 with $h \equiv 0$ are satisfied, and therefore, according to Lemma 3.1,

$$A^u \in \overline{U} \qquad \text{in } \mathbb{R}^n. \tag{57}$$

By (56), (57) and (4),

$$A^u \equiv \beta I \in U \quad \text{in } \mathbb{R}^n.$$

Thus, by (7),

$$F(A^u) > 0 \qquad \text{in } \mathbb{R}^n.$$

On the other hand, since $A^u \in U$, we apply Lemma 3.4 to obtain

$$F(A^u) \leqq 0.$$

Impossible. We have proved that $b = 0$ in (12) and therefore $u \equiv u(0)$. Theorem 1.2 is established. □

To prove Theorem 1.3 we need the following comparison principle for $C^{1,1}$ solutions.

Proposition 3.2. *Let $U \subset \mathcal{S}^{n\times n}$ be a convex open set satisfying* (2) *and* (3)*, $F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (6) *and* (7)*, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u, v \in C^{1,1}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfy* (44), (45) *and*

$$F\left(A^u\right) \geqq 0, \quad F\left(A^v\right) = 0, \qquad \textit{almost everywhere in } \Omega. \tag{58}$$

Then (49) *holds.*

Proof. We prove it by a contradiction argument. We assume that $\min_{\Omega}(u - v) \leqq 0$. Let φ be as in (50) for some fixed small $\delta > 0$, and let

$$w := u^{-\frac{2}{n-2}}, \ w_\varepsilon := w + \varepsilon\varphi, \ u_\varepsilon := (w_\varepsilon)^{-\frac{n-2}{2}}.$$

Using Lemma 3.7 and (45), we can find some fixed small positive constants ε and ε_1 such that

$$A^{u_\varepsilon} \geqq \left(1 + \varepsilon\frac{\varphi}{w}\right) A^u + 3\varepsilon_1 I, \tag{59}$$

and

$$u_\varepsilon > v \qquad \text{on } \partial\Omega.$$

Since $A^u \in \overline{U}$ a.e. in Ω, we have, using (2) and the openness of U,

$$A^{u_\varepsilon} + M \in U \text{ a.e. in } \Omega, \qquad \forall\, M \in \mathcal{S}^{n\times n}, \ \|M\| < 2\varepsilon_1. \tag{60}$$

By the contradiction hypothesis, $u \leqq v$ somewhere in Ω, so there exists $a_\varepsilon \in (0, 1]$ such that

$$\begin{aligned} u_\varepsilon &\geqq v_\varepsilon := a_\varepsilon v && \text{in } \Omega, \\ u_\varepsilon &> v_\varepsilon && \text{on } \partial\Omega, \end{aligned}$$

and

$$S_\varepsilon := \{x \in \Omega \mid u_\varepsilon(x) = v_\varepsilon(x)\} \neq \emptyset.$$

Clearly,

$$u_\varepsilon = v_\varepsilon, \ \nabla u_\varepsilon = \nabla v_\varepsilon, \qquad \text{on } S_\varepsilon. \tag{61}$$

Recall that ε has been fixed. Let

$$O_s := \{x \in \Omega \mid dist(x, S_\varepsilon) < s\}.$$

By (61),

$$A^{tu_\varepsilon + (1-t)v_\varepsilon} = tA^{u_\varepsilon} + (1-t)A^{v_\varepsilon} \qquad \text{on } S_\varepsilon.$$

So, in view the convexity and the openness of U, there exists a function $\bar{t}(s)$, $\bar{t}(s) \to 0^+$ as $s \to 0^+$, such that

$$A^{tu_\varepsilon+(1-t)v_\varepsilon} \in U \text{ a.e. in } O_s, \qquad \forall\, \bar{t}(s) < t \leqq 1. \tag{62}$$

Note that we have used, in deriving (62), the fact that A^u is linear in $\nabla^2 u$ and both u and ∇u are continuous.

By (60), there exists some $\varepsilon_2 > 0$, independent of s, such that

$$dist\left(A^{tu_\varepsilon+(1-t)v_\varepsilon}, \partial U\right) > \varepsilon_2, \qquad \forall\, 1-\varepsilon_2 \leqq t \leqq 1.$$

Thus, by (6) and (7), there exists some $\varepsilon_3 > 0$, independent of s, such that

$$F\left(A^{tu_\varepsilon+(1-t)v_\varepsilon}\right) \geqq \varepsilon_3, \left(F_{ij}\left(A^{tu_\varepsilon+(1-t)v_\varepsilon}\right)\right) \geqq \varepsilon_3 I, \quad \forall\, 1-\varepsilon_2 \leqq t \leqq 1. \tag{63}$$

Using the mean value theorem, in view of (63), we have

$$\begin{aligned} &\varepsilon_3 - F\left(A^{\bar{t}(s)u_\varepsilon+(1-\bar{t}(s))v_\varepsilon}\right) \\ &\leqq F\left(A^{u_\varepsilon}\right) - F\left(A^{\bar{t}(s)u_\varepsilon+(1-\bar{t}(s))v_\varepsilon}\right) = \int_{\bar{t}(s)}^1 \left\{\frac{d}{dt} F\left(A^{tu_\varepsilon+(1-t)v_\varepsilon}\right)\right\} dt \\ &=: -\left(\int_{\bar{t}(s)}^1 a_{ij}(x,t)dt\right)\partial_{ij}(u_\varepsilon - v_\varepsilon) + b_i(x)\partial_i(u_\varepsilon - v_\varepsilon) + c(x)(u_\varepsilon - v_\varepsilon), \end{aligned} \tag{64}$$

where $a_{ij}(\cdot, t)$, b_i, c are bounded in L^∞ norm, and, in view of (6) and (63),

$$\left(\int_{\bar{t}(s)}^1 a_{ij}(x,t)dt\right) \geqq \varepsilon_4 I$$

for some $\varepsilon_4 > 0$ independent of s. In view of (64), we can find some small $\bar{s} > 0$ such that

$$0 < \frac{1}{2}\varepsilon_3 \leqq -a_{ij}\partial_{ij}(u_\varepsilon - v_\varepsilon) + b_i(x)\partial_i(u_\varepsilon - v_\varepsilon) + c(x)(u_\varepsilon - v_\varepsilon), \qquad \text{a.e. in } O_{\bar{s}},$$

where a_{ij}, b_i, c are in $L^\infty(O_{\bar{s}})$ and $\left(a_{ij}\right) \geqq \varepsilon_4 I$ a.e. in $O_{\bar{s}}$. We know that

$$u_\varepsilon - v_\varepsilon = 0 \text{ on } S_\varepsilon \subset O_s,\ u_\varepsilon - v_\varepsilon \geqq 0 \text{ in } O_s, u_\varepsilon - v_\varepsilon > 0 \text{ near } \partial O_{\bar{s}}.$$

However, this violates the local maximum principle, see Theorem 9.22 in [8] or Theorem 4.8 in [1]. Proposition 3.2 is established. □

Proof of Theorem 1.3. We first prove that $u \in \mathcal{A}$. We only need to verify property (A2) since property (A1) has already been assumed. Let Ω be as in the statement of (A2), then, by the conformal invariance of A^u, $A^{u_{x,\lambda}}$ and $A^{(1+\delta)u}$ are still in ∂U a.e. in Ω. Thus, by Proposition 3.2, (A2) is satisfied. So we have proved that $u \in \mathcal{A}$. By Theorem 1.1, (12) holds for some $a > 0$, $b \geqq 0$ and $\bar{x} \in \mathbb{R}^n$. We only need to prove that $b = 0$. Suppose that $b > 0$, then A^u is a positive constant multiple of I in $\mathbb{R}^n$, and therefore $A^u \in U$ in $\mathbb{R}^n$ according to (4). This violates $A^u \in \partial U$ a.e. in $\mathbb{R}^n$. Theorem 1.3 is established. □

4. Proof of Theorems 1.4–1.6

We first give the

Proof of Theorem 1.6. This is a modification of the proof of Theorem 1.11 in [18]. Let D denote the set of points at which v is differentiable. Since v is locally Lipschitz, D is of full Lebesgue measure. Start from the proof of Theorem 1.11 in [18]. Under the weaker assumption $v \in C^{0,1}$, Lemmas 3.6–3.8 in [18] still hold with the same proof. We only need to modify in the last paragraph of the proof of Theorem 1.11 in [18] the part "Follow, with obvious modification, the proof of Theorem 1.10 from the line after the proof of Lemma 3.6 until '$\Delta v(x) \geqq 0$ near the origin' towards the end". The changes are made for pages 404–405 there as follows: in (82) and in the second line above (82), add "$x \in D$". In lines 4–5 on page 405, change "$\bar{\lambda}$ is C^1" to "$\bar{\lambda}$ is $C^{0,1}$", and change "we know that ∇v is C^1, so v is C^2" to "we know that ∇v is $C^{0,1}$, so v is $C^{1,1}$. This implies that $\bar{\lambda}$ is C^1, and therefore, by (82) and (83), v is C^2". Theorem 1.6 is established after these changes. □

Now we give a variation of Proposition 3.1 which allows u to have isolated singularities.

Proposition 4.1. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3), *$F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (6) *and* (7), *and let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing m points $S_m := \{P_1, \cdots, P_m\}$, $m \geqq 1$, $u \in C^0(\overline{\Omega} \setminus S_m)$ and $v \in C^{0,1}(\overline{\Omega})$ satisfy*

$$u > 0, \quad \Delta u \leqq 0 \quad in \;\; \Omega \setminus S_m, \quad v > 0 \; in \quad \overline{\Omega}, \tag{65}$$

$$u > v \quad on \; \partial\Omega. \tag{66}$$

We assume that u is a weak solution of

$$F(A^u) \geqq 0 \quad in \; \Omega \setminus S_m,$$

v is a weak solution of

$$F(A^v) = 0 \quad in \; \Omega \setminus S_m.$$

Then

$$\inf_{\Omega\setminus S_m} (u - v) > 0. \tag{67}$$

Proof. We prove it by induction on the number of points m. We start from $m = 0$ with $S_0 = \emptyset$. The result is contained in Proposition 3.1. Now we assume that the result holds for $m - 1$ points, $m - 1 \geqq 0$, and we will prove it for m points.

Let

$$S_m = \{P_1, \ldots, P_m\} \subset \Omega.$$

By the assumption on u and v, there exist $\{\beta_i\}$, $\{\tilde{\beta}_i\} \subset C^0(\Omega \setminus S_m, \mathcal{S}^{n\times n})$ and positive functions $\{u_i\}$, $\{v_i\} \subset C^2(\Omega \setminus S_m)$ satisfying: for any compact subset K of $\Omega \setminus S_m$, there exists $\bar{i}(K)$ such that

$$\begin{aligned} &A^{u_i} + \beta_i \in U,\ A^{v_i} + \tilde{\beta}_i \in U, && \text{in } K,\ \forall\, i \geqq \bar{i}(K), \\ &u_i \to u,\ v_i \to v,\ \beta_i \to 0,\ \tilde{\beta}_i \to 0, && \text{in } C^0(K), \end{aligned}$$

and

$$F\left(A^{v_i} + \tilde{\beta}_i\right) \to 0 \qquad \text{in } C^0(K).$$

We prove (67) by a contradiction argument. Suppose it does not hold, then

$$\inf_{\Omega\setminus S_m} (u - v) \leqq 0.$$

Since $u > 0$ in $\Omega \setminus S_m$, $\Delta u \leqq 0$ in $\Omega \setminus S_m$, we know that $\inf\limits_{\Omega\setminus S_m} u > 0$. Thus, for some $0 < a \leqq 1$,

$$\inf_{\Omega\setminus S_m} (u - av) = 0.$$

Since we can use $a^{-1}u$ instead of u, we may assume without loss of generality that $a = 1$. So we have, in addition,

$$\inf_{\Omega\setminus S_m} (u - v) = 0. \tag{68}$$

Let P_m be the origin, and let

$$\widehat{\Omega} := \Omega \setminus \{P_1, \ldots, P_{m-1}\}.$$

Let φ and Φ be as in Theorem 1.6, with $\widehat{\Omega}$ being the Ω there, are satisfied. Since $\Phi(v, 0, 1; \cdot) = v$ and $u > v$ on $\partial\Omega$, we can fix some small $\varepsilon_4 > 0$ so that $|x| \leqq \varepsilon_4$ and $|\lambda - 1| \leqq \varepsilon_4$ guarantee

$$u > \Phi(v, x, \lambda; \cdot) \qquad \text{on } \partial\Omega. \tag{69}$$

For such x and λ, if we assume both

$$\inf_{\widehat{\Omega}\setminus\{0\}} [u - \Phi(v, x, \lambda; \cdot)] = 0 \tag{70}$$

and

$$\liminf_{|y|\to 0} [u(y) - \Phi(v, x, \lambda; y)] > 0,$$

we would have, for some small $\tilde{\varepsilon}, \hat{\varepsilon} > 0$,

$$u(y) - \Phi(v, x, \lambda; y) > \hat{\varepsilon} > 0, \qquad \forall\, 0 < |y| \leqq \tilde{\varepsilon}. \tag{71}$$

Let

$$\tilde{u}(y) := \varphi(\lambda)^{-1} u(y), \qquad \tilde{v}(y) := v(x + y).$$

We know from (69) and (71) that

$$\tilde{u} > \tilde{v} \qquad \text{on } \partial(\Omega \setminus \overline{B}_{\bar{\varepsilon}}).$$

It is easy to see that the hypotheses of Proposition 4.1, with u replaced by $\tilde{u}$, v replaced by $\tilde{v}$, Ω replaced by $\Omega \setminus \overline{B}_{\bar{\varepsilon}}$, S_m replaced by $S_{m-1} := \{P_1, \ldots, P_{m-1}\}$, are satisfied. By the induction hypothesis,

$$\inf_{(\Omega\setminus\overline{B}_{\bar{\varepsilon}})\setminus S_{m-1}} (\tilde{u} - \tilde{v}) > 0,$$

i.e.

$$\inf_{(\Omega\setminus\overline{B}_{\bar{\varepsilon}})\setminus S_{m-1}} [u - \Phi(v, x, \lambda; \cdot)] > 0.$$

This and (71) violate (70). Impossible. Thus we have proved that (70) implies $\liminf_{|y|\to 0}[u(y) - \Phi(v, x, \lambda; y)] = 0$. Namely, we have verified (23), with Ω replaced by $\widehat{\Omega}$. Therefore, by Theorem 1.6, either (24) holds or $u = v = v(0)$ near the origin.

If (24) holds, then $\inf_{B_\varepsilon\setminus\{0\}} (u - v) > 0$ for some $\varepsilon > 0$. By the induction hypotheses, applied on $\Omega \setminus \overline{B}_\varepsilon$, we obtain $\inf_{(\Omega\setminus\overline{B}_\varepsilon)\setminus\{P_1,\ldots,P_{m-1}\}} (u - v) > 0$. It follows that $\inf_{\Omega\setminus S_m} (u - v) > 0$, violating (68). A contradiction.

If $u = v = v(0)$ near the origin, say in $B_{\bar{\varepsilon}}$ for some small $\bar{\varepsilon} > 0$, we let $\varphi(y) = e^{\delta|y|^2}$ be the function in Lemma 3.7, and let

$$u_\varepsilon := \left(u^{-\frac{2}{n-2}} + \varepsilon\varphi\right)^{-\frac{n-2}{2}}, \qquad a_\varepsilon := \inf_{\Omega\setminus\{P_1,\cdots,P_m\}} \frac{u_\varepsilon}{v}.$$

Since $\dfrac{u_\varepsilon}{v} = \dfrac{u}{v} + O(\varepsilon)$, it is easy to see from (68) that

$$\lim_{\varepsilon\to 0} a_\varepsilon = 1. \tag{72}$$

For $|y| \leqq \bar{\varepsilon}$, $u(y) = v(y) = v(0)$. So

$$\frac{u_\varepsilon(y)}{v(y)} = 1 - \frac{n-2}{2}\varepsilon v(0)^{\frac{2}{n-2}} e^{\delta|y|^2} + O(\varepsilon^2).$$

For $\varepsilon > 0$ small,

$$\frac{u_\varepsilon}{v} > \max_{|y|=\bar{\varepsilon}} \frac{u_\varepsilon(y)}{v(y)} \geqq a_\varepsilon \qquad \text{on } B_{\bar{\varepsilon}/2}. \tag{73}$$

Taking $\varepsilon > 0$ smaller if necessary, we have, in view of (72) and the fact $u > v$ on $\partial\Omega$,

$$\frac{u_\varepsilon}{v} > a_\varepsilon \qquad \text{on } \partial\Omega.$$

Fix this $\varepsilon > 0$ and let $\widehat{\Omega} := \Omega \setminus \overline{B}_{\bar{\varepsilon}/2}$. We know that

$$(a_\varepsilon)^{-1} u_\varepsilon > v \qquad \text{on } \partial\widehat{\Omega}. \tag{74}$$

Let

$$u_i^\varepsilon := \left(u_i^{-\frac{2}{n-2}} + \varepsilon\varphi\right)^{-\frac{n-2}{2}}.$$

Making $\varepsilon > 0$ smaller if necessary, we have, by Lemma 3.7, $A^{(a_\varepsilon)^{-1}u_i^\varepsilon} \in U$ in $\widehat{\Omega}$ for large i. Clearly, $(a_\varepsilon)^{-1}u_i^\varepsilon \to (a_\varepsilon)^{-1}u^\varepsilon$ in $C^0_{loc}(\widehat{\Omega})$. By the induction hypothesis, in view of (74), we obtain

$$\inf_{\widehat{\Omega}\setminus\{P_1,\ldots,P_{m-1}\}} \left((a_\varepsilon)^{-1}u_\varepsilon - v\right) > 0.$$

This and (73) imply

$$\inf_{\Omega\setminus\{P_1,\ldots,P_m\}} \frac{u_\varepsilon}{v} > a_\varepsilon,$$

violating the definition of a_ε. Impossible. Proposition 4.1 is established. □

Similar to Proposition 4.1, we have the following variation of Proposition 3.2.

Proposition 4.2. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3), *$F \in C^1(U) \cap C^0(\overline{U})$ satisfy* (6) *and* (7), *and let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing m points $S_m := \{P_1, \ldots, P_m\}$, $m \geqq 1$, $u \in C^0(\overline{\Omega} \setminus S_m) \cap C^{1,1}_{loc}(\Omega \setminus S_m)$ and $v \in C^{0,1}(\overline{\Omega}) \cap C^{1,1}_{loc}(\Omega \setminus S_m)$ satisfy* (65), (66) *and*

$$F(A^u) \geqq 0, \quad F(A^v) = 0 \qquad \textit{almost everywhere in } \Omega \setminus S_m.$$

Then (67) *holds.*

Proof. It follows from modification of the proof of Proposition 4.1, using Proposition 3.2 instead of Proposition 3.1, and making some other obvious changes. We omit the details. □

Now we give the

Proof of Theorem 1.4. By (18) and the positivity of u,

$$\liminf_{|y|\to 0} u(y) > 0, \qquad \liminf_{|y|\to\infty} |y|^{n-2}u(y) > 0.$$

As usual, for any $x \in \mathbb{R}^n \setminus \{0\}$ and for any $0 < \delta < 1$,

$$\begin{aligned}\bar{\lambda}_\delta(x) = \sup\{0 < \mu < |x| \;|u_{x,\lambda}(y) \\ \leqq (1+\delta)u(y), \forall\, 0 < \lambda < \mu, |y-x| > \lambda, |y| \neq 0\} > 0\end{aligned}$$

is well defined.

By the definition of $\bar{\lambda}_\delta(x)$,

$$u_{x,\bar{\lambda}_\delta(x)}(y) \leqq (1+\delta)u(y), \qquad \forall\, |y-x| \geqq \bar{\lambda}_\delta(x),\, y \neq 0. \tag{75}$$

Lemma 4.1. *If* $\bar{\lambda}_\delta(x) < |x|$ *for some* $0 < \delta < 1$ *and* $x \in \mathbb{R}^n \setminus \{0\}$, *then either*

$$\liminf_{|y|\to 0} \left[(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta(x)}(y)\right] = 0$$

or

$$\liminf_{|y|\to \infty} |y|^{n-2} \left[(1+\delta)u(y) - u_{x,\bar{\lambda}_\delta(x)}(y)\right] = 0$$

Proof. Suppose the contrary, then for some $0 < \delta < \bar{\delta}$, $x \in \mathbb{R}^n \setminus \{0\}$, $\bar{\lambda}_\delta(x) < |x|$, and for some $R > 3|x| + \dfrac{3}{|x| - \bar{\lambda}_\delta(x)}$, $0 < \varepsilon_2 < \frac{1}{9}\min\{|x|, |x| - \bar{\lambda}_\delta(x)\}$, $\varepsilon > 0$,

$$(1+\delta)u(y) - u_{x,\lambda}(y) \geqq \frac{\varepsilon}{|y|^{n-2}}, \qquad \forall\, |\lambda - \bar{\lambda}_\delta(x)| < \varepsilon_2,\ |y - x| \geqq R,$$

$$(1+\delta)u(y) - u_{x,\lambda}(y) \geqq \varepsilon, \qquad \forall\, |\lambda - \bar{\lambda}_\delta(x)| < \varepsilon_2,\ y \in B_{\frac{1}{R}} \setminus \{0\} \cup \partial B_\lambda.$$

Let

$$\Omega := \left\{ y \in \mathbb{R}^n \mid \frac{1}{R} < |y| < R,\ |y - x| > \bar{\lambda}_\delta(x) \right\},$$

By Lemma 3.5 and Lemma 3.6, $(1+\delta)u$ is a weak solution of

$$F\left(A^{(1+\delta)u}\right) = 0 \qquad \text{in } \Omega,$$

and $u_{x,\bar{\lambda}(x)}$ is a weak solution of

$$F\left(A^{u_{x,\bar{\lambda}_\delta(x)}}\right) = 0 \qquad \text{in } \Omega.$$

We also know that

$$(1+\delta)u \geqq u_{x,\bar{\lambda}_\delta(x)} \text{ on } \overline{\Omega} \qquad \text{and} \qquad (1+\delta)u > u_{x,\bar{\lambda}_\delta(x)} \text{ on } \partial\Omega.$$

It follows, using Proposition 3.1, that

$$(1+\delta)u > u_{x,\bar{\lambda}_\delta(x)} \qquad \text{on } \overline{\Omega}.$$

As usual, the moving sphere procedure can go beyond $\bar{\lambda}_\delta(x)$, violating the definition of $\bar{\lambda}_\delta(x)$. Lemma 4.1 is established. □

Lemma 4.2. *For all* $0 < \delta < 1$ *and for all* $x \in \mathbb{R}^n \setminus \{0\}$, $\bar{\lambda}_\delta(x) = |x|$.

Proof. We prove it by a contradiction argument. Suppose the contrary, then $\bar{\lambda}_\delta(x) < |x|$ for some $x \in \mathbb{R}^n \setminus \{0\}$. Let

$$\Omega := B_{\bar{\lambda}_\delta(x)}, \quad P_1 := x - \frac{\bar{\lambda}_\delta(x)^2 x}{|x|^2}, \quad P_2 := x,$$

$$\tilde{u}(y) := (1+\delta) u_{x,\bar{\lambda}_\delta(x)}(y), \qquad \tilde{v}(y) := u(y).$$

Since $0 < \bar{\lambda}_\delta(x) < |x|$, we have $P_1, P_2 \in \Omega$. By (75),

$$\tilde{u} \geqq \tilde{v} \qquad \text{in } \Omega \setminus \{P_1, P_2\}.$$

It is easy to check that the hypotheses of Proposition 4.1, with u replaced by $\tilde{u}$, v replaced by $\tilde{v}$, and with $m = 2$, are satisfied. Thus, by Proposition 4.1,

$$\inf_{\Omega \setminus \{P_1, P_2\}} (\tilde{u} - \tilde{v}) > 0.$$

This implies

$$\liminf_{|y| \to 0} \left[(1+\delta) u(y) - u_{x,\bar{\lambda}_\delta(x)}(y) \right] = \left(\frac{\bar{\lambda}_\delta(x)}{|x|} \right)^{n-2} \liminf_{z \to P_1} \left[\tilde{u}(z) - \tilde{v}(z) \right] > 0,$$

and

$$\liminf_{|y| \to \infty} |y|^{n-2} \left[(1+\delta) u(y) - u_{x,\bar{\lambda}_\delta(x)}(y) \right] = \left(\bar{\lambda}_\delta(x) \right)^{n-2} \liminf_{z \to P_2} \left[\tilde{u}(z) - \tilde{v}(z) \right] > 0,$$

which contradict Lemma 4.1. Impossible. Lemma 4.2 is established.

Now we complete the proof of Theorem 1.4. By Lemma 4.2,

$$\bar{\lambda}_\delta(x) = |x|, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}, \quad \forall\, 0 < \delta < 1,$$

i.e.

$$u_{x,\lambda}(y) \leqq (1+\delta) u(y), \qquad \forall\, 0 < \lambda < |x|, |y - x| \geqq \lambda, y \neq 0.$$

Sending δ to 0 leads to (19). The radial symmetry of u and $u'(r) \leqq 0$ for almost all $0 < r < \infty$ follows from (19), see e.g. [18]. Theorem 1.4 is established.

Proof of Theorem 1.5. The proof is similar to that of Theorem 1.4, using Proposition 3.2 instead of Proposition 3.1 and using Proposition 4.2 instead of Proposition 4.1. We omit the details. □

References

1. Caffarelli, L., Cabre, X.: *Fully Nonlinear Elliptic Equations*. American Mathematical Society Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995
2. Caffarelli, L., Gidas, B., Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42**, 271–297 (1989)
3. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of the eigenvalues of the Hessian. *Acta Math.* **155**, 261–301 (1985)
4. Chang, S.Y.A.: Non-linear Elliptic Equations in Conformal Geometry. *Nachdiplom Lectures Course Notes*, ETH, Zurich, Springer-Birkhäuser, 2004
5. Chang, S.Y.A., Gursky, M., Yang, P.: An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature. *Ann. Math.* 155, 709–787 (2002)
6. Chang, S.Y.A., Gursky, M., Yang, P.: A prior estimate for a class of nonlinear equations on 4-manifolds. *J. Anal. Math.* **87**, 151–186 (2002)
7. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68** , 209–243 (1979)
8. Gilbarg, D., Trudinger, N.: *Elliptic Partial Differential Equations of Second Order*. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin-New York, 1983
9. Guan, P., Wang, G.: Local estimates for a class of fully nonlinear equations from conformal geometry. *Int. Math. Res. Not.* **26**, 1413–1432 (2003)
10. Guan, P., Wang, G.: A fully nonlinear conformal flow on locally conformally flat manifolds. *J. Reine Angew. Math.* **557**, 219–238 (2003)
11. Gursky, M., Viaclovsky, J.: Convexity and singularities of curvature equations in conformal geometry. arXiv:math.DG/0504066 v1 4 Apr 2005
12. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations. *Comm. Pure Appl. Math.* **56**, 1416–1464 (2003)
13. Li, A.: Liouville type theorem for some degenerate conformally invariant fully nonlinear equation, in preparation
14. Li, A., Li, Y.Y.: A general Liouville type theorem for some conformally invariant fully nonlinear equations. arXiv:math.AP/0301239 v1 21 Jan 2003
15. Li, A., Li, Y.Y.: Further results on Liouville type theorems for some conformally invariant fully nonlinear equations. arXiv:math.AP/0301254 v1 22 Jan 2003
16. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe. *Acta Math.* **195**, 117–154 (2005)
17. Li, Y.Y.: Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc.* **6**, 153–180 (2004)
18. Li, Y.Y.: Conformally invariant fully nonlinear elliptic equations and isolated singularities. *J. Funct. Anal.* **233**, 380–425 (2006)
19. Li, Y.Y.: Degenerate conformally invariant fully nonlinear elliptic equations. arXiv:math.AP/0504598 v1 29 Apr 2005, v2 24 May 2005
20. Li, Y.Y.: Local gradient estimates of solutions to some conformally invariant fully nonlinear equations, C. R. Acad. Sci. Paris, Serial I **343**, 249–252 (2006)
21. Li, Y.Y., Zhang, L.: Liouville type theorems and Harnack type inequalities for semilinear elliptic equations. *J. Anal. Math.* **90**, 27–87 (2003)
22. Ma, X.N., Trudinger, N.S., Wang, X.J.: Regularity of potential functions to the optimal transportation problem. *Arch. Ration. Mech. Anal.* **177**, 151–183 (2005)
23. Sheng, W., Trudinger, N.S., Wang, X.J.: The Yamabe problem for higher order curvatures. arXiv:math.DG/0505463 v1 23 May 2005
24. Viaclovsky, J.: Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds. *Comm. Anal. Geom.* **10**, 815–846 (2002)

25. Wang, X.J.: On the design of a reflector antenna. *Inverse problems* **12**, 351–375 (1996)

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