

Regularity of the distance function to the boundary

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Dedicated to the memory of Luigi Amerio

Abstract

Let Ω be a domain in a smooth complete Finsler manifold, and let G be the largest open subset of Ω such that for every x in G there is a unique closest point from $\partial\Omega$ to x (measured in the Finsler metric). We prove that the distance function from $\partial\Omega$ is in $C_{loc}^{k,\alpha}(G \cup \partial\Omega)$, $k \geq 2$ and $0 < \alpha \leq 1$, if $\partial\Omega$ is in $C^{k,\alpha}$.

1 Introduction

In [1] we studied the singular set of viscosity solutions of some Hamilton-Jacobi equations. This was reduced to the study of the singular set of the distance function to the boundary of a domain Ω — for a Finsler metric. The singular set was defined as the complement of the following open set

$$G := \text{the largest open subset of } \Omega \text{ such that for every } x \text{ in } G \text{ there is a unique closest point from } \partial\Omega \text{ to } x \text{ (measured in the Finsler metric).} \quad (1)$$

*Partially supported by NSF grant DMS-0401118.

In [1] we stated that if $\partial\Omega$ is in $C^{k,\alpha}$, $k \geq 2$ and $0 < \alpha \leq 1$, then the distance function from the boundary belongs to $C^{k-1,\alpha}(G \cup \partial\Omega)$. Recently Joel Spruck asked to see the proof for a Riemannian metric and pointed out that the result would imply that the distance function would be in $C^{k,\alpha}(G \cup \partial\Omega)$. In this paper we provide a proof of that in the Finsler case. This paper can be regarded as an addendum to [1].

We present two proofs of the $C^{k,\alpha}$ result. We use the notation, as in [1],

$$\int_0^T \varphi(\xi(t); \dot{\xi}(t)) dt$$

for the length of a curve $\xi(t)$. $\varphi(\xi; v)$ is homogeneous of degree one in v . For fixed ξ , the level surface $\varphi(\xi; v) = 1$ is smooth, closed, strictly convex, with positive principal curvatures.

The first proof uses very little of [1] and is essentially self-contained. The second proof uses some structure from [1], and may be of some interest to some readers.

We actually prove a more general result here, involving conjugate points from the boundary.

Definition. Conjugate Point. Consider a point y on $\partial\Omega$, and consider the geodesic $\xi(y, s)$ from y going inside Ω “normal” to $\partial\Omega$ (explained below) with s as arclength. The conjugate point to y is the first point \bar{x} on the normal geodesic such that any point x'' on the geodesic beyond \bar{x} has, in any neighborhood of the geodesic, a shorter join from $\partial\Omega$ to x'' than our normal geodesic to x'' .

Normal. A geodesic Γ from a point $y \in \partial\Omega$ is “normal” to $\partial\Omega$ if for x on Γ close to y , the geodesic is the shortest join from $\partial\Omega$ to x .

To obtain the regularity in G we prove a slightly more general result which is local on $\partial\Omega$. Namely, suppose C is a neighborhood on $\partial\Omega$ of a point y and that the normal geodesic Γ from y to a point X in Ω is the unique shortest join from C to X . If the conjugate point to y is beyond X then there is a neighborhood A of X such that the distance from C to any point in A belongs to $C^{k,\alpha}$. See Theorem 1 below.

We shall make use of special coordinates introduced in section 3 of [1] about a given normal geodesic Γ , going from a point $y \in \partial\Omega$ into Ω . In these coordinates y is the origin and the x_n -axis is normal to $\partial\Omega$ there and is the geodesic Γ . Furthermore, in these coordinates, φ has the following properties, see (4.1)-(4.6) in [1]. Here Greek letters α, β range from 1 to $n - 1$, and Latin letters i, j range from 1 to n .

$$\varphi(te_n; e_n) \equiv 1, \tag{2}$$

$$\varphi_{\xi^j}(te_n; e_n) \equiv 0, \tag{3}$$

$$\varphi_{v^\alpha}(te_n; e_n) \equiv 0, \quad \varphi_{v^n}(te_n; e_n) \equiv 1, \quad (4)$$

$$\varphi_{\xi^j v^k}(te_n; e_n) \equiv 0, \quad (5)$$

$$\varphi_{v^j v^n}(te_n; e_n) \equiv 0, \quad (6)$$

$$\varphi_{\xi^j \xi^n}(te_n; e_n) \equiv 0. \quad (7)$$

In these coordinates for $y \in \partial\Omega$ near the origin the geodesic $\xi(y, s)$ from y “normal” to $\partial\Omega$ there satisfies

$$\dot{\xi}(y, 0) = V(y)$$

where $V(y)$ is the unique vector-valued function on $\partial\Omega$ satisfying (here $\nu(y)$ is the Euclidean interior unit normal to $\partial\Omega$ at y)

$$\begin{cases} V(y) \cdot \nu(y) > 0 \\ \varphi(y; V(y)) = 1 \\ \nabla_v \varphi(y; V(y)) \text{ is parallel to } \nu(y). \end{cases} \quad (8)$$

Using these special coordinates, near the origin, $\partial\Omega$ has the form

$$y = (x', f(x')), \quad f(0') = 0, \quad \nabla f(0') = 0'. \quad (9)$$

We assume that $f \in C^{k,\alpha}$, $k \geq 2$, $0 < \alpha \leq 1$.

The result we prove is

Theorem 1 *Assume that the conjugate point of the origin on the geodesic $\Gamma = \{te_n\}$ is beyond e_n , and that there exists a neighborhood C of $0'$ on $\partial\Omega$ such that $\{te_n \mid 0 \leq t \leq 1\}$ is the unique shortest geodesic from C to e_n . Then there exist neighborhoods A of e_n and \mathcal{A} of $0'$ on $\partial\Omega$ such that for any X in A there is a unique $y \in \mathcal{A}$ and geodesic from y to X which is the shortest join from \mathcal{A} to A . Furthermore, if $d(X)$ is its length, then the Jacobian of the map $X \rightarrow (d, y)$ is nonsingular at e_n , and d lies in $C_{loc}^{k,\alpha}$ in A , and y lies in $C_{loc}^{k-1,\alpha}$ in A .*

2 Second Variation

Consider one parameter family of curves $\tau(\epsilon, t)$ from \mathcal{A} to $\bar{t}e_n$, $\bar{t} > 0$, with $\tau(0, t) = te_n$. We look at the second variation of its length $I[\tau(\epsilon, \cdot)]$. For \bar{t} small, it is clearly positive definite. The first \hat{t} for which it fails to be strictly positive definite is the conjugate point. For if $\tilde{t} = \hat{t} + \delta$, $\delta > 0$, then the second variation of curves to \tilde{t} cannot be semipositive definite, and there would then be a shorter connection from \mathcal{A} to $\tilde{t}e_n$ near Γ .

The standard computation of second variation yields

$$\frac{d^2}{d\epsilon^2} I[\tau(\epsilon, \cdot)] \Big|_{\epsilon=0} = J(\tau_\epsilon|_{\epsilon=0}) - f_{x_\alpha x_\beta}(0') \tau_\epsilon^\alpha(0, 0) \tau_\epsilon^\beta(0, 0).$$

Here J is the usual expression of the second variation if the bottom point were kept at the origin. Namely,

$$J(\tau_\epsilon|_{\epsilon=0}) = \int_0^{\bar{t}} \left\{ \varphi_{\xi^\alpha \xi^\beta}(te_n; e_n) \tau_\epsilon^\alpha(0, t) \tau_\epsilon^\beta(0, t) + \varphi_{v^\alpha v^\beta}(te_n; e_n) \dot{\tau}_\epsilon^\alpha(0, t) \dot{\tau}_\epsilon^\beta(0, t) \right\} dt. \quad (10)$$

Note that τ_ϵ^n and $\dot{\tau}_\epsilon^n$ do not occur in J .

3 First proof of Theorem 1

3.1

Recalling (9) we shall denote the normal geodesic from $y = (x', f(x'))$ by $X = \xi(x', s)$; this is a slight change of notation. The geodesic ξ and ξ_s depend smoothly on s and their initial data, while the initial data depend $C^{k-1, \alpha}$ on x' . To prove the theorem, it suffices to show that the Jacobian of the mapping (x', s) to X at $(0', 1)$ is nonsingular. It follows that d and y belong to $C^{k-1, \alpha}$. Since $\nabla_X d = X_s$, it follows that $\nabla_X d$ is in $C^{k-1, \alpha}$ and hence d is in $C^{k, \alpha}$ — as Spruck pointed out to us.

We now prove the Jacobian is nonsingular.

Write $X = (X', X^n)$. Since $X_s(0', 1) = (0', 1)$, the Jacobian of the mapping (x', s) to X at $(0', 1)$ is simply

$$M := \frac{\partial X'}{\partial x'}(0', 1).$$

Assume M is singular, without loss of generality we may suppose that

$$X'_{x_1}(0', 1) = 0'. \quad (11)$$

We construct a perturbation $\tau(\epsilon, t)$ of $\Gamma = \{te_n \mid 0 \leq t \leq 1\}$ such that $\zeta(t) := \tau_\epsilon|_{\epsilon=0}$ satisfies

$$J[\zeta] = f_{x_\alpha x_\beta}(0') \zeta^\alpha(0) \zeta^\beta(0). \quad (12)$$

3.2

Consider the geodesic $\xi(\delta e_1, t)$ of length 1 starting at $(\delta e_1, f(\delta e_1))$, $0 < \delta$ small and “normal” to $\partial\Omega$ there. Set

$$\zeta(t) = \frac{\partial}{\partial \delta} \xi(\delta e_1, t) \Big|_{\delta=0}. \quad (13)$$

By (11),

$$\zeta(1) = 0. \quad (14)$$

We obtain an equation for $\zeta(t)$ by differentiating the geodesic equation

$$\varphi_{\xi^i} = \frac{d}{dt} \varphi_{v^i}(\xi; \dot{\xi})$$

with respect to δ , and setting $\delta = 0$. We find

$$\varphi_{\xi^i \xi^j}(te_n; e_n) \zeta^j = \frac{d}{dt} \left(\varphi_{v^i v^j}(te_n; e_n) \dot{\zeta}^j \right).$$

Here we have used property (5) of our special coordinates. By (7) and (6),

$$\varphi_{\xi^\alpha \xi^\beta} \zeta^\beta = \frac{d}{dt} \left(\varphi_{v^\alpha v^\beta} \dot{\zeta}^\beta \right). \quad (15)$$

We have

$$\zeta^\alpha(0) = \delta_1^\alpha. \quad (16)$$

In addition,

$$\dot{\zeta}(0) = \left. \frac{\partial}{\partial \delta} \dot{\xi}(\delta e_1, 0) \right|_{\delta=0} = \left. \frac{\partial}{\partial \delta} V(\delta e_1) \right|_{\delta=0} = V_{x_1}(0'). \quad (17)$$

By the last formula in (8) we have

$$\nabla_v \varphi((\delta e_1, f(\delta e_1)); V(\delta e_1)) \cdot (e_1 + f_{x_1}(\delta e_1) e_n) = 0.$$

Differentiating in δ and setting $\delta = 0$, we find, using properties of our special coordinates,

$$\varphi_{v^1 v^\beta}(0'; e_n) V_{x_1}^\beta(0') + f_{x_1 x_1}(0') = 0. \quad (18)$$

Now we introduce the perturbation $\tau(\epsilon, t)$ as follows

$$\begin{aligned} \tau^\alpha(\epsilon, t) &= \epsilon \zeta^\alpha(t), \\ \tau^n(\epsilon, t) &= t e_n + (1-t) f(\epsilon e_1). \end{aligned}$$

The definition of τ^n is just to ensure that $\tau(\epsilon, 0)$ lies on $\partial\Omega$.

According to (10),

$$J[\tau_\epsilon|_{\epsilon=0}] = \int_0^1 \left\{ \varphi_{\xi^\alpha \xi^\beta}(te_n; e_n) \zeta^\alpha \zeta^\beta + \varphi_{v^\alpha v^\beta}(te_n; e_n) \dot{\zeta}^\alpha \dot{\zeta}^\beta \right\} dt.$$

Integrating the last expression by parts we find, using (15), (16), (17) and (18),

$$\begin{aligned} J[\tau_\epsilon|_{\epsilon=0}] &= \int_0^1 \left\{ \varphi_{\xi^\alpha \xi^\beta} \zeta^\alpha \zeta^\beta - \zeta^\alpha \frac{d}{dt} \left(\varphi_{v^\alpha v^\beta} \dot{\zeta}^\beta \right) \right\} - \zeta^\alpha(0) \varphi_{v^\alpha v^\beta}(0'; e_n) \dot{\zeta}^\beta(0) \\ &= -\varphi_{v^1 v^\beta}(0'; e_n) V_{x_1}^\beta(0') = f_{x_1 x_1}(0') = f_{x_\alpha x_\beta}(0') \tau_\epsilon^\alpha(0, 0) \tau_\epsilon^\beta(0, 0). \end{aligned}$$

It follows from Section 2 that the second variation is zero.

□

4 Second proof of Theorem 1

Second proof of Theorem 1. For X near e_n and for small $\sigma' = (\sigma_1, \dots, \sigma_{n-1}) \in \mathbb{R}^{n-1}$, let $\tau = \tau(\sigma', X)$ be defined by $\varphi(X; (\sigma', \tau)) = 1$ and $\tau(0', e_n) = 1$. Since $\varphi_{v^n}(e_n; e_n) = 1$, by the implicit function theorem, τ exists as a smooth function of (σ', X) near $(0', e_n)$.

Let, as on page 111 of [1], $\eta = \eta(\sigma', X, t)$ be the unique smooth function of, with $\psi = \varphi^2$,

$$\psi_{\xi^i}(\eta; \dot{\eta}) = \frac{d}{dt} \psi_{v^i}(\eta; \dot{\eta}), \quad t \leq 1,$$

satisfying

$$\eta(\sigma', X, 1) = X, \quad \dot{\eta}(\sigma', X, 1) = (\sigma', \tau(\sigma', X)).$$

As explained in the last two lines of page 108 in [1], $\eta(\sigma', X, t)$ is a geodesic with t the arclength.

In the special coordinates described in Section 1, $\partial\Omega$ has the form (9) near the origin with $f \in C^{k,\alpha}$, $k \geq 2$, $0 < \alpha \leq 1$. Since $\{te_n \mid 0 \leq t \leq 1\}$ is the unique shortest geodesic from C to e_n , we know that for X close to e_n , there exists x' close to $0'$ such that the “normal geodesic” starting from $(x', f(x'))$ will reach X as a shortest join from C to X . It follows that for some σ' close to $0'$ and t close to 0, we have

$$\begin{cases} \eta(\sigma', X, t) - (x', f(x')) = 0, \\ \dot{\eta}^\mu(\sigma', X, t) - V^\mu(x') = 0, \end{cases} \quad (19)$$

where $V(x') := V(x', f(x'))$ satisfies (8). Note that $1 - t$ is the distance from C to X .

To prove Theorem 1, we only need to show that the left hand side of (19), denoted as LHS, has nonsingular Jacobian $\frac{\partial(LHS)}{\partial(t, \sigma', x')}$ at $(t, \sigma', x', X) = (0, 0', 0', e_n)$. Indeed, this would allow the use of the implicit function theorem to show that for X close to e_n and in a neighborhood of $(0, 0', 0')$, there exists a unique $C^{k-1,\alpha}$ solution $(t, \sigma', x') = (t(X), \sigma'(X), x'(X))$ of (19). Thus, Theorem 1 follows as explained at the beginning of Section 3.

Clearly,

$$\frac{\partial(LHS)}{\partial t}(0, 0', 0', e_n) = \begin{pmatrix} \dot{\eta}(0', e_n, 0) \\ (\ddot{\eta}^\mu(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} 0' \\ 1 \\ 0' \end{pmatrix},$$

a $(2n - 1) \times 1$ column vector,

$$\frac{\partial(LHS)}{\partial \sigma'}(0, 0', 0', e_n) = \begin{pmatrix} (\eta_{\sigma_\alpha}(0', e_n, 0)) \\ (\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} (\eta_{\sigma_\alpha}^\mu(0', e_n, 0)) \\ (\eta_{\sigma_\alpha}^n(0', e_n, 0)) \\ (\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0)) \end{pmatrix}, \quad (20)$$

a $(2n - 1) \times (n - 1)$ matrix,

$$\frac{\partial(LHS)}{\partial x'}(0, 0', 0', e_n) = \begin{pmatrix} -I \\ 0 \\ -\nabla V'(0') \end{pmatrix}, \quad (21)$$

a $(2n - 1) \times (n - 1)$ matrix, where I is the $(n - 1) \times (n - 1)$ identity matrix and $\nabla V' := (V_{x_\beta}^\mu)$. Thus

$$\det \left(\frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) = (-1)^{n-1} \det \begin{pmatrix} \frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) & -I \\ \frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) & -\nabla V'(0') \end{pmatrix}, \quad (22)$$

where $\frac{\partial \eta'}{\partial \sigma'} := (\eta_{\sigma_\alpha}^\mu)$ and $\frac{\partial \dot{\eta}'}{\partial \sigma'} := (\dot{\eta}_{\sigma_\alpha}^\mu)$.

By the last line in (8),

$$\nabla_v \varphi((x', f(x')); V(x')) [e_\beta + f_{x_\beta}(x') e_n] = 0,$$

i.e.

$$\varphi_{v\beta}((x', f(x')); V(x')) + \varphi_{vn}((x', f(x')); V(x')) f_{x_\beta}(x') = 0.$$

Differentiating in x_α and setting $x' = 0'$ we find, using properties of our special coordinates (4), (5) and (6),

$$D_{v'}^2 \varphi(0'; e_n) \cdot \nabla V'(0') + D^2 f(0') = 0, \quad (23)$$

where $D_{v'}^2 \varphi := (\varphi_{v^\beta v^\mu})$.

We now evaluate $\frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0)$ and $\frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0)$. It is proved in section 4.4 of [1] that there exists a $C^{2,1}$ function \tilde{f} near $0'$ satisfying

$$\tilde{f}(0') = 0, \quad \nabla \tilde{f}(0') = 0',$$

$$(D^2 \tilde{f}(0') - D^2 f(0')) > 0, \quad (24)$$

$$\eta(\sigma', e_n, 0) = (y', \tilde{f}(y')), \quad (25)$$

$$\dot{\eta}^\mu(\sigma', e_n, 0) = \tilde{V}^\mu(y'), \quad (26)$$

where $\tilde{V}(y') := \tilde{V}((y', \tilde{f}(y')))$ is determined by (8) with f replaced by \tilde{f} , and $y' = y'(\sigma')$ satisfies

$$\det \left(\frac{\partial y'}{\partial \sigma'}(0') \right) \neq 0. \quad (27)$$

Note that (27) is given by (4.9) in [1], while (24) follows from corollary 4.15 in [1] together with the fact that e_n is not a conjugate point.

Differentiating (26) in σ_α and setting $\sigma' = 0'$ we find

$$\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0) = \tilde{V}_{y_\beta}^\mu(0') \frac{\partial y_\beta}{\partial \sigma_\alpha}(0'),$$

i.e.

$$\frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) = \nabla \tilde{V}'(0') \frac{\partial y'}{\partial \sigma'}(0'). \quad (28)$$

Differentiating (25) in σ_α and setting $\sigma' = 0'$ we find

$$\frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) = \frac{\partial y'}{\partial \sigma'}(0'). \quad (29)$$

Since

$$\begin{pmatrix} \frac{\partial y'}{\partial \sigma'}(0') & -I \\ \nabla \tilde{V}'(0') \frac{\partial y'}{\partial \sigma'}(0') & -\nabla V'(0') \end{pmatrix} = \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix} \begin{pmatrix} \frac{\partial y'}{\partial \sigma'}(0') & \\ & I \end{pmatrix},$$

we have, by putting (28) and (29) into (22),

$$\det \left(\frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) = (-1)^{n-1} \det \left(\frac{\partial y'}{\partial \sigma'}(0') \right) \det \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}. \quad (30)$$

The proof of (23), applied to \tilde{f} instead of f , yields

$$D_{v'}^2 \varphi(0'; e_n) \nabla \tilde{V}'(0') + D^2 \tilde{f}(0') = 0. \quad (31)$$

Thus, by (23) and (31),

$$\begin{aligned} \begin{pmatrix} I & -I \\ -D^2 \tilde{f}(0') & D^2 f(0') \end{pmatrix} &= \begin{pmatrix} I & -I \\ D_{v'}^2 \varphi(0'; e_n) \nabla \tilde{V}'(0') & -D_{v'}^2 \varphi(0'; e_n) \nabla V'(0') \end{pmatrix} \\ &= \begin{pmatrix} I & \\ & D_{v'}^2 \varphi(0'; e_n) \end{pmatrix} \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}, \end{aligned}$$

and therefore

$$\det \left(D^2 f(0') - D^2 \tilde{f}(0') \right) = \det D_{v'}^2 \varphi(0'; e_n) \det \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}. \quad (32)$$

Since $D_{v'}^2 \varphi(0'; e_n)$ is positive definite, we deduce from (30), (27) and (32) that

$$\det \left(\frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) \neq 0.$$

□

References

- [1] Y.Y. Li and L. Nirenberg, The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton-Jacobi equations, *Comm. Pure Appl. Math.* 58 (2005), 85-146.