

# Regularity of the distance function to the boundary

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*Dedicated to the memory of Luigi Amerio*

## Abstract

Let  $\Omega$  be a domain in a smooth complete Finsler manifold, and let  $G$  be the largest open subset of  $\Omega$  such that for every  $x$  in  $G$  there is a unique closest point from  $\partial\Omega$  to  $x$  (measured in the Finsler metric). We prove that the distance function from  $\partial\Omega$  is in  $C_{loc}^{k,\alpha}(G \cup \partial\Omega)$ ,  $k \geq 2$  and  $0 < \alpha \leq 1$ , if  $\partial\Omega$  is in  $C^{k,\alpha}$ .

## 1 Introduction

In [1] we studied the singular set of viscosity solutions of some Hamilton-Jacobi equations. This was reduced to the study of the singular set of the distance function to the boundary of a domain  $\Omega$  — for a Finsler metric. The singular set was defined as the complement of the following open set

$$G := \text{the largest open subset of } \Omega \text{ such that for every } x \text{ in } G \text{ there is a unique closest point from } \partial\Omega \text{ to } x \text{ (measured in the Finsler metric).} \quad (1)$$

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\*Partially supported by NSF grant DMS-0401118.

In [1] we stated that if  $\partial\Omega$  is in  $C^{k,\alpha}$ ,  $k \geq 2$  and  $0 < \alpha \leq 1$ , then the distance function from the boundary belongs to  $C^{k-1,\alpha}(G \cup \partial\Omega)$ . Recently Joel Spruck asked to see the proof for a Riemannian metric and pointed out that the result would imply that the distance function would be in  $C^{k,\alpha}(G \cup \partial\Omega)$ . In this paper we provide a proof of that in the Finsler case. This paper can be regarded as an addendum to [1].

We present two proofs of the  $C^{k,\alpha}$  result. We use the notation, as in [1],

$$\int_0^T \varphi(\xi(t); \dot{\xi}(t)) dt$$

for the length of a curve  $\xi(t)$ .  $\varphi(\xi; v)$  is homogeneous of degree one in  $v$ . For fixed  $\xi$ , the level surface  $\varphi(\xi; v) = 1$  is smooth, closed, strictly convex, with positive principal curvatures.

The first proof uses very little of [1] and is essentially self-contained. The second proof uses some structure from [1], and may be of some interest to some readers.

We actually prove a more general result here, involving conjugate points from the boundary.

**Definition. Conjugate Point.** Consider a point  $y$  on  $\partial\Omega$ , and consider the geodesic  $\xi(y, s)$  from  $y$  going inside  $\Omega$  “normal” to  $\partial\Omega$  (explained below) with  $s$  as arclength. The conjugate point to  $y$  is the first point  $\bar{x}$  on the normal geodesic such that any point  $x''$  on the geodesic beyond  $\bar{x}$  has, in any neighborhood of the geodesic, a shorter join from  $\partial\Omega$  to  $x''$  than our normal geodesic to  $x''$ .

**Normal.** A geodesic  $\Gamma$  from a point  $y \in \partial\Omega$  is “normal” to  $\partial\Omega$  if for  $x$  on  $\Gamma$  close to  $y$ , the geodesic is the shortest join from  $\partial\Omega$  to  $x$ .

To obtain the regularity in  $G$  we prove a slightly more general result which is local on  $\partial\Omega$ . Namely, suppose  $C$  is a neighborhood on  $\partial\Omega$  of a point  $y$  and that the normal geodesic  $\Gamma$  from  $y$  to a point  $X$  in  $\Omega$  is the unique shortest join from  $C$  to  $X$ . If the conjugate point to  $y$  is beyond  $X$  then there is a neighborhood  $A$  of  $X$  such that the distance from  $C$  to any point in  $A$  belongs to  $C^{k,\alpha}$ . See Theorem 1 below.

We shall make use of special coordinates introduced in section 3 of [1] about a given normal geodesic  $\Gamma$ , going from a point  $y \in \partial\Omega$  into  $\Omega$ . In these coordinates  $y$  is the origin and the  $x_n$ -axis is normal to  $\partial\Omega$  there and is the geodesic  $\Gamma$ . Furthermore, in these coordinates,  $\varphi$  has the following properties, see (4.1)-(4.6) in [1]. Here Greek letters  $\alpha, \beta$  range from 1 to  $n - 1$ , and Latin letters  $i, j$  range from 1 to  $n$ .

$$\varphi(te_n; e_n) \equiv 1, \tag{2}$$

$$\varphi_{\xi^j}(te_n; e_n) \equiv 0, \tag{3}$$

$$\varphi_{v^\alpha}(te_n; e_n) \equiv 0, \quad \varphi_{v^n}(te_n; e_n) \equiv 1, \quad (4)$$

$$\varphi_{\xi^j v^k}(te_n; e_n) \equiv 0, \quad (5)$$

$$\varphi_{v^j v^n}(te_n; e_n) \equiv 0, \quad (6)$$

$$\varphi_{\xi^j \xi^n}(te_n; e_n) \equiv 0. \quad (7)$$

In these coordinates for  $y \in \partial\Omega$  near the origin the geodesic  $\xi(y, s)$  from  $y$  “normal” to  $\partial\Omega$  there satisfies

$$\dot{\xi}(y, 0) = V(y)$$

where  $V(y)$  is the unique vector-valued function on  $\partial\Omega$  satisfying (here  $\nu(y)$  is the Euclidean interior unit normal to  $\partial\Omega$  at  $y$ )

$$\begin{cases} V(y) \cdot \nu(y) > 0 \\ \varphi(y; V(y)) = 1 \\ \nabla_v \varphi(y; V(y)) \text{ is parallel to } \nu(y). \end{cases} \quad (8)$$

Using these special coordinates, near the origin,  $\partial\Omega$  has the form

$$y = (x', f(x')), \quad f(0') = 0, \quad \nabla f(0') = 0'. \quad (9)$$

We assume that  $f \in C^{k,\alpha}$ ,  $k \geq 2$ ,  $0 < \alpha \leq 1$ .

The result we prove is

**Theorem 1** *Assume that the conjugate point of the origin on the geodesic  $\Gamma = \{te_n\}$  is beyond  $e_n$ , and that there exists a neighborhood  $C$  of  $0'$  on  $\partial\Omega$  such that  $\{te_n \mid 0 \leq t \leq 1\}$  is the unique shortest geodesic from  $C$  to  $e_n$ . Then there exist neighborhoods  $A$  of  $e_n$  and  $\mathcal{A}$  of  $0'$  on  $\partial\Omega$  such that for any  $X$  in  $A$  there is a unique  $y \in \mathcal{A}$  and geodesic from  $y$  to  $X$  which is the shortest join from  $\mathcal{A}$  to  $A$ . Furthermore, if  $d(X)$  is its length, then the Jacobian of the map  $X \rightarrow (d, y)$  is nonsingular at  $e_n$ , and  $d$  lies in  $C_{loc}^{k,\alpha}$  in  $A$ , and  $y$  lies in  $C_{loc}^{k-1,\alpha}$  in  $A$ .*

## 2 Second Variation

Consider one parameter family of curves  $\tau(\epsilon, t)$  from  $\mathcal{A}$  to  $\bar{t}e_n$ ,  $\bar{t} > 0$ , with  $\tau(0, t) = te_n$ . We look at the second variation of its length  $I[\tau(\epsilon, \cdot)]$ . For  $\bar{t}$  small, it is clearly positive definite. The first  $\hat{t}$  for which it fails to be strictly positive definite is the conjugate point. For if  $\tilde{t} = \hat{t} + \delta$ ,  $\delta > 0$ , then the second variation of curves to  $\tilde{t}$  cannot be semipositive definite, and there would then be a shorter connection from  $\mathcal{A}$  to  $\tilde{t}e_n$  near  $\Gamma$ .

The standard computation of second variation yields

$$\frac{d^2}{d\epsilon^2} I[\tau(\epsilon, \cdot)] \Big|_{\epsilon=0} = J(\tau_\epsilon|_{\epsilon=0}) - f_{x_\alpha x_\beta}(0') \tau_\epsilon^\alpha(0, 0) \tau_\epsilon^\beta(0, 0).$$

Here  $J$  is the usual expression of the second variation if the bottom point were kept at the origin. Namely,

$$J(\tau_\epsilon|_{\epsilon=0}) = \int_0^{\bar{t}} \left\{ \varphi_{\xi^\alpha \xi^\beta}(te_n; e_n) \tau_\epsilon^\alpha(0, t) \tau_\epsilon^\beta(0, t) + \varphi_{v^\alpha v^\beta}(te_n; e_n) \dot{\tau}_\epsilon^\alpha(0, t) \dot{\tau}_\epsilon^\beta(0, t) \right\} dt. \quad (10)$$

Note that  $\tau_\epsilon^n$  and  $\dot{\tau}_\epsilon^n$  do not occur in  $J$ .

### 3 First proof of Theorem 1

#### 3.1

Recalling (9) we shall denote the normal geodesic from  $y = (x', f(x'))$  by  $X = \xi(x', s)$ ; this is a slight change of notation. The geodesic  $\xi$  and  $\xi_s$  depend smoothly on  $s$  and their initial data, while the initial data depend  $C^{k-1, \alpha}$  on  $x'$ . To prove the theorem, it suffices to show that the Jacobian of the mapping  $(x', s)$  to  $X$  at  $(0', 1)$  is nonsingular. It follows that  $d$  and  $y$  belong to  $C^{k-1, \alpha}$ . Since  $\nabla_X d = X_s$ , it follows that  $\nabla_X d$  is in  $C^{k-1, \alpha}$  and hence  $d$  is in  $C^{k, \alpha}$  — as Spruck pointed out to us.

We now prove the Jacobian is nonsingular.

Write  $X = (X', X^n)$ . Since  $X_s(0', 1) = (0', 1)$ , the Jacobian of the mapping  $(x', s)$  to  $X$  at  $(0', 1)$  is simply

$$M := \frac{\partial X'}{\partial x'}(0', 1).$$

Assume  $M$  is singular, without loss of generality we may suppose that

$$X'_{x_1}(0', 1) = 0'. \quad (11)$$

We construct a perturbation  $\tau(\epsilon, t)$  of  $\Gamma = \{te_n \mid 0 \leq t \leq 1\}$  such that  $\zeta(t) := \tau_\epsilon|_{\epsilon=0}$  satisfies

$$J[\zeta] = f_{x_\alpha x_\beta}(0') \zeta^\alpha(0) \zeta^\beta(0). \quad (12)$$

#### 3.2

Consider the geodesic  $\xi(\delta e_1, t)$  of length 1 starting at  $(\delta e_1, f(\delta e_1))$ ,  $0 < \delta$  small and “normal” to  $\partial\Omega$  there. Set

$$\zeta(t) = \frac{\partial}{\partial \delta} \xi(\delta e_1, t) \Big|_{\delta=0}. \quad (13)$$

By (11),

$$\zeta(1) = 0. \quad (14)$$

We obtain an equation for  $\zeta(t)$  by differentiating the geodesic equation

$$\varphi_{\xi^i} = \frac{d}{dt} \varphi_{v^i}(\xi; \dot{\xi})$$

with respect to  $\delta$ , and setting  $\delta = 0$ . We find

$$\varphi_{\xi^i \xi^j}(te_n; e_n) \zeta^j = \frac{d}{dt} \left( \varphi_{v^i v^j}(te_n; e_n) \dot{\zeta}^j \right).$$

Here we have used property (5) of our special coordinates. By (7) and (6),

$$\varphi_{\xi^\alpha \xi^\beta} \zeta^\beta = \frac{d}{dt} \left( \varphi_{v^\alpha v^\beta} \dot{\zeta}^\beta \right). \quad (15)$$

We have

$$\zeta^\alpha(0) = \delta_1^\alpha. \quad (16)$$

In addition,

$$\dot{\zeta}(0) = \left. \frac{\partial}{\partial \delta} \dot{\xi}(\delta e_1, 0) \right|_{\delta=0} = \left. \frac{\partial}{\partial \delta} V(\delta e_1) \right|_{\delta=0} = V_{x_1}(0'). \quad (17)$$

By the last formula in (8) we have

$$\nabla_v \varphi((\delta e_1, f(\delta e_1)); V(\delta e_1)) \cdot (e_1 + f_{x_1}(\delta e_1) e_n) = 0.$$

Differentiating in  $\delta$  and setting  $\delta = 0$ , we find, using properties of our special coordinates,

$$\varphi_{v^1 v^\beta}(0'; e_n) V_{x_1}^\beta(0') + f_{x_1 x_1}(0') = 0. \quad (18)$$

Now we introduce the perturbation  $\tau(\epsilon, t)$  as follows

$$\begin{aligned} \tau^\alpha(\epsilon, t) &= \epsilon \zeta^\alpha(t), \\ \tau^n(\epsilon, t) &= t e_n + (1-t) f(\epsilon e_1). \end{aligned}$$

The definition of  $\tau^n$  is just to ensure that  $\tau(\epsilon, 0)$  lies on  $\partial\Omega$ .

According to (10),

$$J[\tau_\epsilon|_{\epsilon=0}] = \int_0^1 \left\{ \varphi_{\xi^\alpha \xi^\beta}(te_n; e_n) \zeta^\alpha \zeta^\beta + \varphi_{v^\alpha v^\beta}(te_n; e_n) \dot{\zeta}^\alpha \dot{\zeta}^\beta \right\} dt.$$

Integrating the last expression by parts we find, using (15), (16), (17) and (18),

$$\begin{aligned} J[\tau_\epsilon|_{\epsilon=0}] &= \int_0^1 \left\{ \varphi_{\xi^\alpha \xi^\beta} \zeta^\alpha \zeta^\beta - \zeta^\alpha \frac{d}{dt} \left( \varphi_{v^\alpha v^\beta} \dot{\zeta}^\beta \right) \right\} - \zeta^\alpha(0) \varphi_{v^\alpha v^\beta}(0'; e_n) \dot{\zeta}^\beta(0) \\ &= -\varphi_{v^1 v^\beta}(0'; e_n) V_{x_1}^\beta(0') = f_{x_1 x_1}(0') = f_{x_\alpha x_\beta}(0') \tau_\epsilon^\alpha(0, 0) \tau_\epsilon^\beta(0, 0). \end{aligned}$$

It follows from Section 2 that the second variation is zero.

□

## 4 Second proof of Theorem 1

**Second proof of Theorem 1.** For  $X$  near  $e_n$  and for small  $\sigma' = (\sigma_1, \dots, \sigma_{n-1}) \in \mathbb{R}^{n-1}$ , let  $\tau = \tau(\sigma', X)$  be defined by  $\varphi(X; (\sigma', \tau)) = 1$  and  $\tau(0', e_n) = 1$ . Since  $\varphi_{v^n}(e_n; e_n) = 1$ , by the implicit function theorem,  $\tau$  exists as a smooth function of  $(\sigma', X)$  near  $(0', e_n)$ .

Let, as on page 111 of [1],  $\eta = \eta(\sigma', X, t)$  be the unique smooth function of, with  $\psi = \varphi^2$ ,

$$\psi_{\xi^i}(\eta; \dot{\eta}) = \frac{d}{dt} \psi_{v^i}(\eta; \dot{\eta}), \quad t \leq 1,$$

satisfying

$$\eta(\sigma', X, 1) = X, \quad \dot{\eta}(\sigma', X, 1) = (\sigma', \tau(\sigma', X)).$$

As explained in the last two lines of page 108 in [1],  $\eta(\sigma', X, t)$  is a geodesic with  $t$  the arclength.

In the special coordinates described in Section 1,  $\partial\Omega$  has the form (9) near the origin with  $f \in C^{k,\alpha}$ ,  $k \geq 2$ ,  $0 < \alpha \leq 1$ . Since  $\{te_n \mid 0 \leq t \leq 1\}$  is the unique shortest geodesic from  $C$  to  $e_n$ , we know that for  $X$  close to  $e_n$ , there exists  $x'$  close to  $0'$  such that the “normal geodesic” starting from  $(x', f(x'))$  will reach  $X$  as a shortest join from  $C$  to  $X$ . It follows that for some  $\sigma'$  close to  $0'$  and  $t$  close to 0, we have

$$\begin{cases} \eta(\sigma', X, t) - (x', f(x')) = 0, \\ \dot{\eta}^\mu(\sigma', X, t) - V^\mu(x') = 0, \end{cases} \quad (19)$$

where  $V(x') := V(x', f(x'))$  satisfies (8). Note that  $1 - t$  is the distance from  $C$  to  $X$ .

To prove Theorem 1, we only need to show that the left hand side of (19), denoted as LHS, has nonsingular Jacobian  $\frac{\partial(LHS)}{\partial(t, \sigma', x')}$  at  $(t, \sigma', x', X) = (0, 0', 0', e_n)$ . Indeed, this would allow the use of the implicit function theorem to show that for  $X$  close to  $e_n$  and in a neighborhood of  $(0, 0', 0')$ , there exists a unique  $C^{k-1,\alpha}$  solution  $(t, \sigma', x') = (t(X), \sigma'(X), x'(X))$  of (19). Thus, Theorem 1 follows as explained at the beginning of Section 3.

Clearly,

$$\frac{\partial(LHS)}{\partial t}(0, 0', 0', e_n) = \begin{pmatrix} \dot{\eta}(0', e_n, 0) \\ (\dot{\eta}^\mu(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} 0' \\ 1 \\ 0' \end{pmatrix},$$

a  $(2n - 1) \times 1$  column vector,

$$\frac{\partial(LHS)}{\partial \sigma'}(0, 0', 0', e_n) = \begin{pmatrix} (\eta_{\sigma_\alpha}(0', e_n, 0)) \\ (\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} (\eta_{\sigma_\alpha}^\mu(0', e_n, 0)) \\ (\eta_{\sigma_\alpha}^n(0', e_n, 0)) \\ (\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0)) \end{pmatrix}, \quad (20)$$

a  $(2n - 1) \times (n - 1)$  matrix,

$$\frac{\partial(LHS)}{\partial x'}(0, 0', 0', e_n) = \begin{pmatrix} -I \\ 0 \\ -\nabla V'(0') \end{pmatrix}, \quad (21)$$

a  $(2n - 1) \times (n - 1)$  matrix, where  $I$  is the  $(n - 1) \times (n - 1)$  identity matrix and  $\nabla V' := (V'_{x\beta})$ . Thus

$$\det \left( \frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) = (-1)^{n-1} \det \begin{pmatrix} \frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) & -I \\ \frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) & -\nabla V'(0') \end{pmatrix}, \quad (22)$$

where  $\frac{\partial \eta'}{\partial \sigma'} := (\eta'_{\sigma\alpha})$  and  $\frac{\partial \dot{\eta}'}{\partial \sigma'} := (\dot{\eta}'_{\sigma\alpha})$ .

By the last line in (8),

$$\nabla_v \varphi((x', f(x')); V(x')) [e_\beta + f_{x_\beta}(x') e_n] = 0,$$

i.e.

$$\varphi_{v\beta}((x', f(x')); V(x')) + \varphi_{v^n}((x', f(x')); V(x')) f_{x_\beta}(x') = 0.$$

Differentiating in  $x_\alpha$  and setting  $x' = 0'$  we find, using properties of our special coordinates (4), (5) and (6),

$$D_{v'}^2 \varphi(0'; e_n) \cdot \nabla V'(0') + D^2 f(0') = 0, \quad (23)$$

where  $D_{v'}^2 \varphi := (\varphi_{v^\beta v^\mu})$ .

We now evaluate  $\frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0)$  and  $\frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0)$ . It is proved in section 4.4 of [1] that there exists a  $C^{2,1}$  function  $\tilde{f}$  near  $0'$  satisfying

$$\tilde{f}(0') = 0, \quad \nabla \tilde{f}(0') = 0',$$

$$(D^2 \tilde{f}(0') - D^2 f(0')) > 0, \quad (24)$$

$$\eta(\sigma', e_n, 0) = (y', \tilde{f}(y')), \quad (25)$$

$$\dot{\eta}^\mu(\sigma', e_n, 0) = \tilde{V}^\mu(y'), \quad (26)$$

where  $\tilde{V}(y') := \tilde{V}((y', \tilde{f}(y')))$  is determined by (8) with  $f$  replaced by  $\tilde{f}$ , and  $y' = y'(\sigma')$  satisfies

$$\det \left( \frac{\partial y'}{\partial \sigma'}(0') \right) \neq 0. \quad (27)$$

Note that (27) is given by (4.9) in [1], while (24) follows from corollary 4.15 in [1] together with the fact that  $e_n$  is not a conjugate point.

Differentiating (26) in  $\sigma_\alpha$  and setting  $\sigma' = 0'$  we find

$$\dot{\eta}_{\sigma_\alpha}^\mu(0', e_n, 0) = \tilde{V}_{y_\beta}^\mu(0') \frac{\partial y_\beta}{\partial \sigma_\alpha}(0'),$$

i.e.

$$\frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) = \nabla \tilde{V}'(0') \frac{\partial y'}{\partial \sigma'}(0'). \quad (28)$$

Differentiating (25) in  $\sigma_\alpha$  and setting  $\sigma' = 0'$  we find

$$\frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) = \frac{\partial y'}{\partial \sigma'}(0'). \quad (29)$$

Since

$$\begin{pmatrix} \frac{\partial y'}{\partial \sigma'}(0') & -I \\ \nabla \tilde{V}'(0') \frac{\partial y'}{\partial \sigma'}(0') & -\nabla V'(0') \end{pmatrix} = \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix} \begin{pmatrix} \frac{\partial y'}{\partial \sigma'}(0') & \\ & I \end{pmatrix},$$

we have, by putting (28) and (29) into (22),

$$\det \left( \frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) = (-1)^{n-1} \det \left( \frac{\partial y'}{\partial \sigma'}(0') \right) \det \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}. \quad (30)$$

The proof of (23), applied to  $\tilde{f}$  instead of  $f$ , yields

$$D_{v'}^2 \varphi(0'; e_n) \nabla \tilde{V}'(0') + D^2 \tilde{f}(0') = 0. \quad (31)$$

Thus, by (23) and (31),

$$\begin{aligned} \begin{pmatrix} I & -I \\ -D^2 \tilde{f}(0') & D^2 f(0') \end{pmatrix} &= \begin{pmatrix} I & -I \\ D_{v'}^2 \varphi(0'; e_n) \nabla \tilde{V}'(0') & -D_{v'}^2 \varphi(0'; e_n) \nabla V'(0') \end{pmatrix} \\ &= \begin{pmatrix} I & \\ & D_{v'}^2 \varphi(0'; e_n) \end{pmatrix} \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}, \end{aligned}$$

and therefore

$$\det \left( D^2 f(0') - D^2 \tilde{f}(0') \right) = \det D_{v'}^2 \varphi(0'; e_n) \det \begin{pmatrix} I & -I \\ \nabla \tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}. \quad (32)$$

Since  $D_{v'}^2 \varphi(0'; e_n)$  is positive definite, we deduce from (30), (27) and (32) that

$$\det \left( \frac{\partial(LHS)}{\partial(t, \sigma', x')} (0, 0', 0', e_n) \right) \neq 0.$$

□

## References

- [1] Y.Y. Li and L. Nirenberg, The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton-Jacobi equations, *Comm. Pure Appl. Math.* 58 (2005), 85-146.