

A Geometric Problem and the Hopf Lemma. II

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(Dedicated to the memory of Shiing-Shen Chern)

Abstract A classical result of A. D. Alexandrov states that a connected compact smooth n -dimensional manifold without boundary, embedded in \mathbb{R}^{n+1} , and such that its mean curvature is constant, is a sphere. Here we study the problem of symmetry of M in a hyperplane $X_{n+1} = \text{constant}$ in case M satisfies: for any two points (X', X_{n+1}) , (X', \widehat{X}_{n+1}) on M , with $X_{n+1} > \widehat{X}_{n+1}$, the mean curvature at the first is not greater than that at the second. Symmetry need not always hold, but in this paper, we establish it under some additional conditions. Some variations of the Hopf Lemma are also presented. Several open problems are described. Part I dealt with corresponding one dimensional problems.

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1 Introduction

1.1

In this sequel to [7], we continue our study on a geometric problem related to a classical result of A. D. Alexandrov. Throughout the paper M is a smooth compact connected embedded hypersurface in \mathbb{R}^{n+1} , its mean curvature is

$$H(X) := \frac{1}{n} [k_1(X) + \cdots + k_n(X)],$$

where $k(X) = (k_1(X), \cdots, k_n(X))$ denote the principle curvatures of M at X with respect to the inner normal. Let G denote the open bounded set bounded by M .

The problem we consider is to prove a symmetry property for M satisfying the following

Main Assumption For any two points $(X', X_{n+1}), (X', \widehat{X}_{n+1}) \in M$ satisfying $X_{n+1} \geq \widehat{X}_{n+1}$ and that $\{(X', \theta X_{n+1} + (1 - \theta)\widehat{X}_{n+1}) \mid 0 \leq \theta \leq 1\}$ lies in \overline{G} ,

$$H(X', X_{n+1}) \leq H(X', \widehat{X}_{n+1}) \tag{1}$$

holds.

It is suggested that the reader first read the introduction of [7].

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If the mean curvature function H is constant on M , then M must be a standard sphere by a classical result in [2]. Under the Main Assumption and assuming that the mean curvature function can be extended to \mathbb{R}^{n+1} as a monotone Lipschitz function, it was proved in [6] that M must be symmetric about some hyperplane $X_{n+1} = \text{constant}$. Examples given in [7] show that the Main Assumption alone is not enough to guarantee the symmetry. It is not difficult to see that the examples can be made so that the mean curvature function can be extended to a monotone function in \mathbb{R}^{n+1} which is Hölder continuous with Hölder exponent α for any $0 < \alpha < 1$. The examples do not satisfy

Condition S *M stays on one side of any hyperplane parallel to the X_{n+1} -axis that is tangent to M .*

Remark 1 It is not difficult to see that Condition S implies that G , the interior of M , is convex in the X_{n+1} direction. The converse is not true.

We make the following

Conjecture *Any smooth compact connected embedded hypersurface M in \mathbb{R}^{n+1} satisfying the Main Assumption and Condition S must be symmetric about some hyperplane $X_{n+1} = \text{constant}$.*

The Conjecture in dimension $n = 1$ was proved in [7]. A crucial ingredient in the proof was later established in [3] by a simpler method. In the present paper we present results concerning the Conjecture in dimensions $n \geq 2$. Our main result in the present paper for higher dimensions requires a further condition:

Condition T *Any line parallel to the X_{n+1} -axis that is tangent to M has contact of finite order.*

If $\nu(X) = (\nu_1(X), \dots, \nu_{n+1}(X))$ denotes the inner unit normal of M at X , we will consider the set

$$T := \{X \in M \mid \nu_{n+1}(X) = 0\},$$

i.e., the set of points on M where the tangent planes are parallel to the X_{n+1} -axis.

For a point \bar{X} in T , we often work in a new coordinate system which is orthogonal to the original one. The new coordinate system is centered at \bar{X} and denoted by $(y_1, \dots, y_{n-1}, t, y_{n+1})$, with y_{n+1} -axis pointing in the direction of the inner normal of M at \bar{X} , t -axis pointing to the opposite direction of the X_{n+1} -axis, and the (y_1, \dots, y_{n-1}, t) -coordinate plane is the tangent plane of M at \bar{X} . In this new coordinate system, let $v = v(t, y)$, $y = (y_1, \dots, y_{n-1})$ denote the smooth function whose graph is M near the origin. Clearly $v(0, 0) = 0$ and $\nabla v(0, 0) = 0$. With this notation, Condition T means: For any $\bar{X} \in T$, there exists some integer $k \geq 2$ such that

$$\partial_t^k v(0, 0) \neq 0. \quad (2)$$

Our main theorem, Theorem 1 below, also assumes that M is locally convex in the X_{n+1} -direction near T in the following sense:

Condition LC *For every point \bar{X} in T , if we view M locally as the graph of a function defined on the tangent plane, the function is convex in the X_{n+1} direction near the point. Namely, the above defined function v satisfies $v_{tt} \geq 0$ near the origin for every \bar{X} in T .*

Remark 2 Neither of the Condition S and LC implies the other.

Here is our main result.

Theorem 1 *Let M satisfy the Main Assumption and Conditions T and LC. Then M must be symmetric with respect to some hyperplane $X_{n+1} = \text{constant}$.*

Corollary 1 *Let M be a smooth compact convex hypersurface in \mathbb{R}^{n+1} satisfying the Main Assumption and Condition T. Then M must be symmetric with respect to some hyperplane $X_{n+1} = \text{constant}$.*

In particular, we have

Corollary 2 *Let M be a real analytic compact convex hypersurface in \mathbb{R}^{n+1} satisfying the Main Assumption. Then M must be symmetric with respect to some hyperplane $X_{n+1} = \text{constant}$.*

The conclusion of Theorem 1 still holds when the mean curvature function is replaced by more general curvature functions. Let M satisfy Conditions T and LC, and let $g(k_1, k_2, \dots, k_n)$ be a C^3 function, symmetric in (k_1, \dots, k_n) , defined in an open neighborhood Γ of $\{(k_1(X), \dots, \dots(X)) \mid X \in M\}$, and satisfying in Γ

$$\frac{\partial g}{\partial k_i} > 0, \quad 1 \leq i \leq n \quad \text{and} \quad \frac{\partial^2 g}{\partial k_i \partial k_j} \eta^i \eta^j \leq 0, \quad \forall \eta \in \mathbb{R}^n.$$

Theorem 2 *Let M and g be as above. We assume that for any two points $(X', X_{n+1}), (X', \widehat{X}_{n+1}) \in M$ satisfying $X_{n+1} \geq \widehat{X}_{n+1}$ with $\{(X', \theta X_{n+1} + (1 - \theta)\widehat{X}_{n+1}) \mid 0 \leq \theta \leq 1\}$ lying in \overline{G} ,*

$$g(k(X', X_{n+1})) \leq g(k(X', \widehat{X}_{n+1})) \tag{3}$$

holds. Then M must be symmetric with respect to some hyperplane $X_{n+1} = \text{constant}$.

In [7] we mentioned extension by A. Ros of Alexandrov's result to other symmetric functions of the principal curvatures. There was earlier work [5] by P. Hartman.

Elementary symmetric functions satisfy the above properties of g in appropriate regions: For $1 \leq m \leq n$, let

$$\sigma_m(k_1, \dots, k_n) = \sum_{1 \leq i_1 < \dots < i_m \leq n} k_{i_1} \cdots k_{i_m}$$

be the m -th elementary symmetric functions, and let

$$g_m := (\sigma_m)^{\frac{1}{m}}.$$

It is well known that g_m satisfies the above properties in

$$\Gamma_m := \{(k_1, \dots, k_n) \in \mathbb{R}^n \mid \sigma_j(k_1, \dots, k_n) > 0 \text{ for } 1 \leq j \leq m\}.$$

1.2

Theorem 1 is proved in Section 2; our method of proof begins as in that of A. D. Alexandrov, using the method of moving planes. As indicated in [7] one is led to the need for extensions of

the classical Hopf Lemma. Here we also present some variations of the Hopf Lemma and the strong maximum principle. In [7] these were studied in one dimension.

The Hopf Lemma is a local result. We have not been able to prove the analogous local result for our problem. Our proof of Theorem 1, which uses the maximum principle, is via a global argument.

Here are some plausible variations of the Hopf Lemma adapted for our problem. Consider

$$\Omega = \{(t, y) \mid y \in \mathbb{R}^{n-1}, |y| < 1, 0 < t < 1\}, \tag{4}$$

$$u, v \in C^\infty(\overline{\Omega}), \tag{5}$$

$$u \geq v \geq 0 \quad \text{in } \Omega \tag{6}$$

$$u(0, y) = v(0, y), \quad \forall |y| < 1; \quad u(0, 0) = v(0, 0) = 0, \tag{7}$$

$$u_t(0, 0) = 0, \tag{8}$$

$$u_t > 0 \quad \text{in } \Omega, \tag{9}$$

and

$$\begin{cases} \text{whenever } u(t, y) = v(s, y), \quad 0 < s < 1, \quad |y| < 1, \text{ then there} \\ H(\nabla u, \nabla^2 u)(t, y) \leq H(\nabla v, \nabla^2 v)(s, y), \end{cases} \tag{10}$$

where

$$H(\nabla u, \nabla^2 u) := \frac{1}{n} \nabla \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

gives the mean curvature of the graph of u .

The followings are some plausible variations of the Hopf Lemma.

Open Problem 1 Assume the above. Is it true that either

$$u \equiv v \quad \text{near } (0, 0) \tag{11}$$

or

$$v_t \equiv 0 \quad \text{near } (0, 0)? \tag{12}$$

A weaker version is

Open Problem 2 In addition to the assumption in Open Problem 1, we further assume that

$$w(t, y) := \begin{cases} v(t, y), & t \geq 0, \quad |y| < 1, \\ u(-t, y), & t < 0, \quad |y| < 1 \end{cases} \text{ is } C^\infty \text{ in } \{(t, y) \mid |t| < 1, \quad |y| < 1\}. \tag{13}$$

Is it true that either (11) or (12) holds?

If the answer to Open Problem 2 is affirmative, then the Conjecture can be proved by modification of the arguments in [7] and the present paper. The answer to Open Problem 1 in dimension $n = 1$ is affirmative, as proved in [7]. On the other hand, the answer to Open Problem 2 in higher dimensions is not known even under an additional hypothesis that $\frac{\partial^k v}{\partial t^k}(0, 0) > 0$ for some integer $k \geq 2$.

Though our knowledge about the problems above concerning variations of the Hopf Lemma is very limited, here is a simple variation of the strong maximum principle.

Theorem 3 For $n \geq 2$, let Ω be in (4), and let $u, v \in C^2(\Omega)$ satisfy (10),

$$u \geq v \quad \text{in } \Omega, \tag{14}$$

and

$$\max\{u_t, v_t\} > 0 \quad \text{in } \Omega. \tag{15}$$

Then either

$$u > v \quad \text{in } \Omega, \tag{16}$$

or

$$u \equiv v \quad \text{in } \Omega. \tag{17}$$

A more general result, Theorem 4, is proved in Section 4.

Remark 3 The analogue of Theorem 3 in dimension $n = 1$ was proved in [7]. The same conclusion of Theorem 3 holds when the mean curvature operator $H(\nabla u, \nabla^2 u)$ is replaced by any elliptic operator $F(u, \nabla u, \nabla^2 u)$, see Theorem 4 in Section 4.

Another weaker form of Open Problem 1 is

Open Problem 3 Let u and v satisfy (5), (8), (9), (10),

$$u \geq v > 0 \quad \text{in } \Omega, \tag{18}$$

and

$$u(0, y) = v(0, y) = 0, \quad \forall |y| < 1. \tag{19}$$

Is it true that (11) holds?

In Open Problem 3, one may also replace the mean curvature operator by other elliptic operators including the Laplacian.

Theorems 1 and 2 are proved in Sections 2 and 3. Section 5 contains some partial results on the open problems 1-3 and variations of the Hopf Lemma. We think that they are of independent interest.

2 Proof of Theorem 1

This is the main section of the paper.

2.1

We start with the method of moving planes.

Proof of Theorem 1 Without loss of generality, we assume that

$$\max\{X_{n+1} \mid (X_1, \dots, X_{n+1}) \in M \text{ for some } X_1, \dots, X_n\} = 0.$$

For $\lambda < 0$, let $S_\lambda := \{X \in M \mid X_{n+1} > \lambda\}$ denote the portion of M above the hyperplane $\{X_{n+1} = \lambda\}$, and S'_λ denote the mirror image of S_λ with respect to $\{X_{n+1} = \lambda\}$. It is obvious that for $\lambda < 0$ but close to 0,

$$S'_\lambda \text{ lies in } G, \text{ the interior of } M, \quad S'_\lambda \cap M = \emptyset, \quad (20)$$

and for all $X \in \partial S'_\lambda$,

$$\nu_{n+1}(X) < 0. \quad (21)$$

Let $(\lambda_0, 0)$ denote the largest open interval such that (20) and (21) hold for all $\lambda \in (\lambda_0, 0)$. To prove the theorem we need only to show that

$$M = S_{\lambda_0} \cup \overline{S'_{\lambda_0}}, \quad (22)$$

where $\overline{S'_{\lambda_0}} = S'_{\lambda_0} \cup \partial S'_{\lambda_0}$.

It is easy to see from the definition of λ_0 that

$$\nu_{n+1}(X) < 0, \quad \forall X \in S_{\lambda_0}, \quad (23)$$

and that at least one of the following two cases occurs:

$$\text{there exists some } \tilde{X} \in S'_{\lambda_0} \cap M \text{ with } \nu_{n+1}(\tilde{X}) > 0, \quad (24)$$

$$\text{there exists some } \tilde{X} \in \partial S'_{\lambda_0} \cap M \text{ with } \nu_{n+1}(\tilde{X}) = 0. \quad (25)$$

If (24) occurs, S'_{λ_0} and M near \tilde{X} can be represented as graphs of smooth functions u and v :

$$u = u(X_1, \dots, X_n), \quad v = v(X_1, \dots, X_n) \quad \text{for } (X_1, \dots, X_n) \text{ close to } (\tilde{X}_1, \dots, \tilde{X}_n).$$

Clearly

$$u(\tilde{X}_1, \dots, \tilde{X}_n) = v(\tilde{X}_1, \dots, \tilde{X}_n), \quad \text{and } u \geq v \quad \text{near } (\tilde{X}_1, \dots, \tilde{X}_n). \quad (26)$$

By the Main Assumption,

$$\nabla \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \leq \nabla \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \quad \text{near } (\tilde{X}_1, \dots, \tilde{X}_n).$$

It follows, using the mean value theorem, that

$$L(u - v) := a_{ij} \partial_{ij}(u - v) + b_i \partial_i(u - v) \leq 0 \quad \text{near } (\tilde{X}_1, \dots, \tilde{X}_n),$$

where (a_{ij}) is some smooth positive definite $n \times n$ matrix function and $\{b_i\}$ are some smooth functions, both near $(\tilde{X}_1, \dots, \tilde{X}_n)$. By the strong maximum principle, in view of (26), $u \equiv v$ near $(\tilde{X}_1, \dots, \tilde{X}_n)$. Since the argument applies to every point \tilde{X} satisfying (24), we obtain (22) in this case.

Now we treat the much more delicate case (25), and we will assume below that (24) does not occur. Consider

$$M_{\lambda_0} := \{X = (X_1, \dots, X_{n+1}) \in M \mid X_{n+1} < \lambda_0\}, \quad (27)$$

the part of M below the hyperplane T_{λ_0} . For $X \in M_{\lambda_0}$, let

$$\bar{Y}(X) := (X_1, \dots, X_n, \lambda_0), \tag{28}$$

$$O := \{X \in M_{\lambda_0} \mid \{\text{the segment between } X \text{ and } \bar{Y}(X)\} \cap S'_{\lambda_0} \neq \emptyset\}. \tag{29}$$

For $X \in O$, we define

$$Y(X) := \{\text{the segment between } X \text{ and } \bar{Y}(X)\} \cap S_{\lambda'_0}. \tag{30}$$

It is clear that $Y(X)$ is uniquely defined for $X \in O$, and it is a smooth function on O .

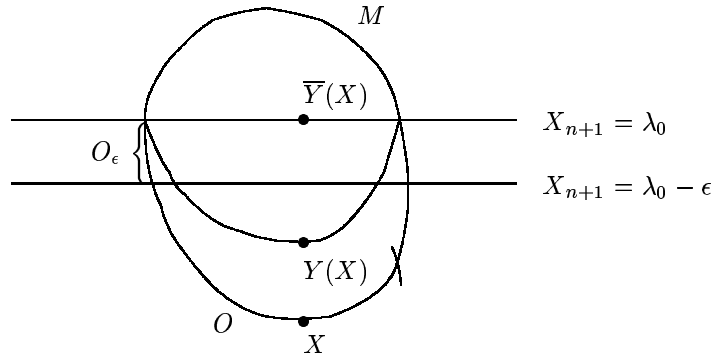


Figure 1

Let

$$\tau(X) := \text{dist}(X, Y(X)), \quad \bar{\tau}(X) := \text{dist}(X, \bar{Y}(X)) = \lambda_0 - X_{n+1}, \quad X \in O, \tag{31}$$

where dist denotes the Euclidean distance between the two points. Both $\tau(X)$ and $\bar{\tau}(X)$ are smooth functions on O , and they can be extended continuously to the closure of O .

Since we have assumed that (24) does not occur,

$$\tau(X) > 0, \quad \forall X \in O. \tag{32}$$

Clearly,

$$\tau(X) = \bar{\tau}(X), \quad \forall X \in \partial O. \tag{33}$$

2.2

The main step in our proof of Theorem 1 is to establish

Proposition 1 *Assume (32). Then there exist some constants $\epsilon, c > 0$ such that*

$$\tau(X) \geq c\bar{\tau}(X), \quad \forall X \in O_\epsilon := \{X \in O \mid \bar{\tau}(X) = \lambda_0 - X_{n+1} < \epsilon\}. \tag{34}$$

Remark 4 Proposition 1 holds without assuming Condition T.

Proof of Proposition 1 If $O_\epsilon = \emptyset$ for some $\epsilon > 0$, (34) is considered to hold trivially. So we assume that $O_\epsilon \neq \emptyset$ for all $\epsilon > 0$. In fact, Condition T guarantees that O_ϵ is not empty,

as shown towards the end of the proof of Theorem 1. For small $\epsilon > 0$, by (32) and (33), there exists some small number $c = c(\epsilon) \in (0, \frac{1}{8})$, depending on ϵ , such that

$$\tau(X) \geq 4c\bar{\tau}(X) \geq 2c(\bar{\tau}(X) + \bar{\tau}(X)^{\frac{3}{2}}), \quad \forall X \in \partial O_\epsilon. \quad (35)$$

For sufficiently small ϵ we will prove (34) with $c = c(\epsilon)$ arguing by contradiction.

Here is a sketch of how the argument goes.

From our Main Assumption (1) it follows that τ satisfies a second order linear differential inequality on O_ϵ , though we do not write it down at a general point. If (35) fails, there is a point \tilde{X} where

$$\sigma := \tau - 2c(\bar{\tau} + \bar{\tau}^{\frac{3}{2}})$$

has a negative minimum in O_ϵ . But

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in O_\epsilon} \text{dist}(x, T \cap \{X_{n+1} = \lambda_0\}) = 0. \quad (36)$$

Thus \tilde{X} has a closest point \bar{X} in $(T \cap \{X_{n+1} = \lambda_0\})$. We then use our special coordinates, taking \bar{X} as origin, and compute, near \bar{X} , the differential inequality. $L\tau < 0$. In addition, we find that

$$L(\bar{\tau} + \bar{\tau}^{\frac{3}{2}}) > 0.$$

But then, \tilde{X} cannot be a minimum point for σ .

We now proceed with the argument. First we write down the linear inequality $L\tau < 0$ near any point \bar{X} in $(T \cap \{X_{n+1} = \lambda_0\})$, working in the new coordinate system $(y_1, \dots, y_{n-1}, t, y_{n+1})$ as described earlier.

Let, with $y = (y_1, \dots, y_{n-1})$ and $\delta > 0$ some small universal number,

$$\begin{aligned} \Omega &= \{(t, y) \mid 0 < t < \delta, y \in \mathbb{R}^{n-1}, |y| < \delta\}, \\ \Omega^+ &= \{(s, y) \in \Omega \mid \text{there exists some } 0 < t < s \text{ such that } u(t, y) = v(s, y)\}. \end{aligned} \quad (37)$$

Throughout the paper, a number is said to be universal if it depends only on M . We note that $(s, y) \in \Omega^+$ if and only if $(s, y, v(s, y))$ lies in O .

By (32) and (23),

$$u(t, y) := v(-t, y) \quad \text{in } \Omega \quad (38)$$

satisfies

$$u(t, y) > v(t, y), \quad (t, y) \in \Omega,$$

and

$$u_t(t, y) > 0, \quad (t, y) \in \Omega. \quad (39)$$

With (39), an application of the implicit function theorem yields that for any $(s, y) \in \Omega^+$, there exists a unique $t = t(s, y) \in (0, s)$ satisfying

$$u(t(s, y)) = v(s, y), \quad (40)$$

and the function $t(s, y)$ is smooth in Ω^+ .

By the Main Assumption,

$$H(\nabla u, \nabla^2 u)(t(s, y), y) \leq H(\nabla v, \nabla^2 v)(s, y), \quad \forall (s, y) \in \Omega^+. \quad (41)$$

Set

$$\tau(s, y) = s - t(s, y), \quad (s, y) \in \Omega^+. \quad (42)$$

Differentiating (40), we have, with $1 \leq \alpha, \beta \leq n-1$,

$$v_s(s, y) = u_t(t, y) - u_t(t, y)\tau_s(s, y), \quad (43)$$

$$v_{y_\alpha}(s, y) = u_{y_\alpha}(t, y) - u_t(t, y)\tau_{y_\alpha}(s, y), \quad (44)$$

$$v_{ss}(s, y) = u_{tt}(t, y) - \tau_s(s, y)[2 - \tau_s(s, y)]u_{tt}(t, y) - u_t(t, y)\tau_{ss}(s, y), \quad (45)$$

$$\begin{aligned} v_{sy_\alpha}(s, y) &= v_{y_\alpha s}(s, y) = u_{ty_\alpha}(t, y) - u_{tt}(t, y)\tau_{y_\alpha}(s, y)[1 - \tau(s, y)] \\ &\quad - u_{ty_\alpha}(t, y)\tau_s(s, y) - u_t(t, y)\tau_{sy_\alpha}(s, y), \end{aligned} \quad (46)$$

$$\begin{aligned} v_{y_\alpha y_\beta}(s, y) &= u_{y_\alpha y_\beta}(t, y) - u_{ty_\alpha}(t, y)\tau_{y_\beta}(s, y) - u_{ty_\beta}(t, y)\tau_{y_\alpha}(s, y) \\ &\quad + u_{tt}(t, y)\tau_{y_\alpha}(s, y)\tau_{y_\beta}(s, y) - u_t(t, y)\tau_{y_\alpha y_\beta}(s, y). \end{aligned} \quad (47)$$

By the mean value theorem, we have, with $t = t(s, y)$ and $(s, y) \in \Omega^+$ and using (43) and (44),

$$\begin{aligned} &H(\nabla v(s, y), \nabla^2 v(s, y)) - H(\nabla u(t, y), \nabla^2 v(s, y)) \\ &= \left(\int_0^1 H_p(\theta \nabla v(s, y) + (1-\theta)\nabla u(t, y), \nabla^2 v(s, y)) d\theta \right) \cdot (\nabla v(s, y) - \nabla u(t, y)) \\ &= [O(1)\tau_s(s, y) + O(1) \cdot \nabla_y \tau(s, y)]u_t(t, y), \end{aligned} \quad (48)$$

where $O(1)$ satisfies $|O(1)| \leq C$ for some universal constant C .

Next we have, using (45), (46) and (47),

$$\begin{aligned} &H(\nabla u(t, y), \nabla^2 v(s, y)) - H(\nabla u(t, y), \nabla^2 u(t, y)) \\ &= -u_t(t, y) \left[H_{00}\tau_{ss}(s, y) + 2 \sum_{\alpha=1}^{n-1} H_{0\alpha}\tau_{sy_\alpha} + \sum_{\alpha, \beta=1}^{n-1} H_{\alpha\beta}\tau_{y_\alpha y_\beta} \right] \\ &\quad - H_{00}(2 - \tau_s)u_{tt}\tau_s - 2 \sum_{\alpha=1}^{n-1} H_{0\alpha}u_{ty_\alpha}\tau_s + \eta \cdot \nabla_y \tau, \end{aligned} \quad (49)$$

where H_{ij} denotes $\frac{\partial H(\nabla u(t, y), N)}{\partial N_{ij}}$ which are independent of the matrix N , and η denotes some vector-valued function in $L_{\text{loc}}^\infty(\Omega^+)$ which may vary from line to line. Note that here and in the following, ∇u denotes $\nabla u(t, y)$, etc., ∇v denotes $\nabla v(s, y)$, H_{00} denotes $\frac{\partial H}{\partial u_{tt}}$, etc.

We deduce from (41), (48) and (49) that

$$\begin{aligned} 0 &\leq -u_t \left[H_{00}\tau_{ss} + 2 \sum_{\alpha=1}^{n-1} H_{0\alpha}\tau_{sy_\alpha} + \sum_{\alpha, \beta=1}^{n-1} H_{\alpha\beta}\tau_{y_\alpha y_\beta} \right] \\ &\quad - H_{00}(2 - \tau_s)u_{tt}\tau_s + O(1)u_t\tau_s - 2 \sum_{\alpha=1}^{n-1} H_{0\alpha}u_{ty_\alpha}\tau_s + \eta \cdot \nabla_y \tau. \end{aligned} \quad (50)$$

Since

$$H_{0\alpha} = -(1 + |\nabla u|^2)^{-\frac{3}{2}} u_t u_{y_\alpha}, \quad 1 \leq \alpha \leq n-1,$$

we have

$$\begin{aligned} 0 \leq & -u_t \left[H_{00} \tau_{ss} + 2 \sum_{\alpha=1}^{n-1} H_{0\alpha} \tau_{s y_\alpha} + \sum_{\alpha, \beta=1}^{n-1} H_{\alpha\beta} \tau_{y_\alpha y_\beta} \right] \\ & - H_{00} (2 - \tau_s) u_{tt} \tau_s + O(1) u_t \tau_s + \eta \cdot \nabla_y \tau. \end{aligned} \quad (51)$$

Define

$$L := H_{00} \partial_{ss} + 2 \sum_{\alpha=1}^{n-1} H_{0\alpha} \partial_{s y_\alpha} + \sum_{\alpha, \beta=1}^{n-1} H_{\alpha\beta} \partial_{y_\alpha y_\beta} + H_{00} (2 - \tau_s) \frac{u_{tt}}{u_t} \partial_s - O(1) \partial_s - \frac{\eta_\alpha}{u_t} \partial_{y_\alpha}. \quad (52)$$

We know from (51) that

$$L\tau \leq 0 \quad \text{in } \Omega^+. \quad (53)$$

Let

$$\hat{\tau}(s, y) := s + s^{1+\bar{\epsilon}}, \quad (54)$$

where $\bar{\epsilon} = \frac{1}{2}$.

A calculation gives

$$L\hat{\tau} = H_{00} (2 - \tau_s) \frac{u_{tt}}{u_t} \frac{d}{ds} [s + s^{1+\bar{\epsilon}}] + H_{00} (1 + \bar{\epsilon}) \bar{\epsilon} s^{\bar{\epsilon}-1} - O(1) [1 + (1 + \bar{\epsilon}) s^{\bar{\epsilon}}].$$

Lemma 1 *There exists some universal constant $\delta' > 0$ such that*

$$\tau_s(s, y) < 1, \quad \forall (s, y) \in \Omega^+, \quad |(s, y)| < \delta'.$$

Proof In view of (43) and the positivity of u_t in Ω^+ , we only need to show that $v_s(s, y) > 0$. We prove this by contradiction. Suppose that $v_s(s, y) = 0$ for some small (s, y) , $s > 0$. Recall that $u(t, y) = v(s, y)$, $0 < t = t(s, y) < s$. Since M satisfies Condition LC,

$$v(0, y) \geq v(s, y) + v_s(s, y)(-s) = v(s, y). \quad (55)$$

It follows that

$$u(0, y) = v(0, y) \geq v(s, y) = u(t, y),$$

which violates $u_t > 0$ in Ω^+ .

We will assume from now on, making δ smaller if necessary, that $\delta \leq \delta'$. Since M satisfies Condition LC, we have, making δ smaller if necessary,

$$u_{tt} \geq 0 \quad \text{in } \Omega^+. \quad (56)$$

It follows, using (39), (56) and Lemma 1, that

$$L\hat{\tau} \geq (1 + \bar{\epsilon}) \bar{\epsilon} H_{N_{00}} s^{\bar{\epsilon}-1} + O(1).$$

Thus, making δ smaller if necessary,

$$L\hat{\tau} > 0 \quad \text{in } \Omega^+. \tag{57}$$

Now the value of δ is fixed; it works for every \bar{X} in $T \cap \{X_{n+1} = \lambda_0\}$. We see from (36) that for $\epsilon > 0$ small,

$$\sup_{x \in O_\epsilon} \text{dist}(x, T \cap \{X_{n+1} = \lambda_0\}) < \frac{\delta}{2}. \tag{58}$$

As we described above, we fix such an ϵ now and take $c = c(\epsilon)$ the one in (35). We will prove (34) with this value of c arguing by contradiction. Suppose that (34) does not hold, then there exists $\tilde{X} \in \bar{O}_\epsilon$ such that

$$(\tau - c(\bar{\tau} + \bar{\tau}^{\frac{3}{2}}))(\tilde{X}) = \min_{\bar{O}_\epsilon}(\tau - c(\bar{\tau} + \bar{\tau}^{\frac{3}{2}})) < 0. \tag{59}$$

Because of (35), $\tilde{X} \in O_\epsilon$. Namely, \tilde{X} is an interior local minimum point of $\tau - c(\bar{\tau} + \bar{\tau}^{\frac{3}{2}})$ in O_ϵ . Let \bar{X} be a closest point in $T \cap \{X_{n+1} = \lambda_0\}$ to \tilde{X} . We know from (58) that $\text{dist}(\bar{X}, \tilde{X}) \leq \frac{\delta}{2}$. With this \bar{X} and the function v and u defined earlier, \tilde{X} corresponds to some (\tilde{s}, \tilde{y}) in Ω^+ , with $0 < \tilde{s} < \epsilon$. Clearly (\tilde{s}, \tilde{y}) is an interior local maximum point of $\tau - c\hat{\tau}$ in Ω^+ . Thus

$$L(\tau - c\hat{\tau}) \geq 0 \quad \text{at } (\tilde{s}, \tilde{y}).$$

On the other hand, by (53) and (57),

$$L(\tau - c\hat{\tau}) < 0 \quad \text{at } (\tilde{s}, \tilde{y}).$$

A contradiction. Proposition 1 is established.

Now we use Condition T to show that (25) cannot hold if (24) does not occur.

Since we are treating case (25), we let \bar{X} be a point satisfying (25), $v = v(t, y)$ be the function defined earlier for the point, and, in view of Condition T, let $k \geq 2$ be the smallest k satisfying (2). Set $u(t, y) := v(-t, y)$ for $t \geq 0$. By the definition of λ_0 and by the assumption that case (24) does not occur, $u(t, y) > v(t, y)$ for (t, y) small and $t > 0$. Since M satisfies Condition LC, $v_{tt}(t, 0) \geq 0$ for small t . So k is even, $\partial_t^k v(0, 0) > 0$ and therefore $v(t, 0) > 0$ for small $t > 0$ which clearly implies that $(t, 0) \in \Omega^+$ for small $t > 0$.

Now

$$v(t, 0) = \frac{1}{k!} \partial_t^k v(0, 0) t^k + O(t^{k+1}), \quad u(t, y) = \frac{1}{k!} \partial_t^k v(0, 0) t^k + O(t^{k+1}).$$

From $u(t(s, 0), 0) = v(s, 0)$ and the above, we see easily that

$$\lim_{s \rightarrow 0^+} \frac{t(s, 0)}{s} = 1. \tag{60}$$

Since

$$\bar{\tau}(s, 0, v(s, 0)) = s, \quad \tau(s, 0, v(s, 0)) = \tau(s, 0) = s - t(s, 0),$$

and $(s, 0, v(s, 0)) \in O$, we know from Proposition 1 that for some constant $c > 0$ and for all $s > 0$ small,

$$s - t(s, 0) = \tau(s, 0, v(s, 0)) \geq c\bar{\tau}(s, 0, v(s, 0)) = cs.$$

This is contradicted by (60). Theorem 1 is established.

3 Proof of Theorem 2

Proof of Theorem 2 We follow the proof of Theorem 1 until (26). Let $A(\nabla u, \nabla^2 u) := (A_{il}(\nabla u, \nabla^2 u))$ denote the second fundamental form of S'_{λ_0} with respect to its first fundamental form. Then, see Lemma 1.1 of [4],

$$A_{il}(\nabla u, \nabla^2 u) = \frac{1}{w} \left\{ u_{il} - \frac{u_i u_j u_{jl}}{w(1+w)} - \frac{u_l u_k u_{ki}}{w(1+w)} + \frac{u_i u_l u_j u_k u_{jk}}{w^2(1+w)^2} \right\},$$

where

$$w = \sqrt{1 + |\nabla u|^2}.$$

Let $\mathcal{S}^{n \times n}$ denote the set of real symmetric $n \times n$ matrices, and let $O(n)$ denote the set of $n \times n$ real orthogonal matrices. For $A \in \mathcal{S}^{n \times n}$ we use $k(A)$ to denote $(k_1(A), \dots, k_n(A))$ where $k_1(A), \dots, k_n(A)$ are the n eigenvalues of A . We define a function G on

$$U := \{A \in \mathcal{S}^{n \times n} \mid k(A) \in \Gamma\}$$

by

$$G(A) := g(k(A)).$$

By the properties of g , $G \in C^3(U)$,

$$\begin{aligned} O^{-1}UO &= U, & \forall O \in O(n), \\ G_{A_{ij}}(A)\eta^i\eta^j &> 0, & \forall A \in U, \eta \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \quad (61)$$

$$G(O^{-1}AO) = G(A), \quad \forall A \in U \text{ and } O \in O(n), \quad (62)$$

$$G_{A_{ij}A_{kl}}(A)\xi^{ij}\xi^{kl} \leq 0, \quad \forall A \in U, \forall \xi \in \mathcal{S}^{n \times n}. \quad (63)$$

By (3),

$$A(\nabla u, \nabla^2 u), A(\nabla v, \nabla^2 v) \in U \quad \text{near } (\tilde{X}_1, \dots, \tilde{X}_n),$$

and

$$G(A(\nabla u, \nabla^2 u)) \leq G(A(\nabla v, \nabla^2 v)).$$

Using the mean value theorem as usual we have, by (61),

$$L(u - v) := a_{ij}\partial_{ij}(u - v) + b_i\partial_i(u - v) \leq 0 \quad \text{near } (\tilde{X}_1, \dots, \tilde{X}_n),$$

where (a_{ij}) is some smooth positive definite $n \times n$ matrix function and $\{b_i\}$ are some smooth functions, both near $(\tilde{X}_1, \dots, \tilde{X}_n)$. We obtain (22) in this case as in the proof of Theorem 1.

Now we treat the much more delicate case (25), and we will assume below that (24) does not occur. We follow from (27) until (33), and we give the

Proof of Proposition 1 under the Hypotheses of Theorem 2 Follow from the beginning of the proof of Proposition 1 until (40). Instead of (41), we have

$$F(\nabla u, \nabla^2 u)(t(s, y), y) \leq F(\nabla v, \nabla^2 v)(s, y), \quad (64)$$

where we have used notation

$$F(\nabla u, \nabla^2 u) = G(A(\nabla u, \nabla^2 u)).$$

With $\tau(s, y)$ defined in (42), we still have (43)–(47). Similarly to (48), we have

$$\begin{aligned} & F(\nabla v(s, y), \nabla^2 v(s, y)) - F(\nabla u(t, y), \nabla^2 v(s, y)) \\ &= [O(1)\tau_s(s, y) + O(1) \cdot \nabla_y \tau(s, y)]u_t(t, y). \end{aligned} \tag{65}$$

Since we only work in regions where u_t and ϵ are very small, there

$$\{A(\nabla u(t, y), \theta \nabla^2 v(s, y) + (1 - \theta) \nabla^2 u(t, y)) \mid 0 \leq \theta \leq 1\} \subset U.$$

Since $G(A)$ is concave in A and $A(p, N)$ is linear in N , $F(p, N)$ is concave in N . So we have

$$\begin{aligned} & F(\nabla u(t, y), \nabla^2 v(s, y)) - F(\nabla u(t, y), \nabla^2 u(t, y)) \\ & \leq F_{jk}(\nabla u(t, y), \nabla^2 u(t, y)) [v_{jk}(s, y) - u_{jk}(t, y)], \end{aligned}$$

where

$$F_{jk}(p, N) := \frac{\partial F(p, N)}{\partial N_{jk}}.$$

It is easy to see from (61) that for some universal constant $C_1 > 1$,

$$\frac{1}{C_1} |\xi|^2 \leq F_{jk}(\nabla u(t, y), \nabla^2 u(t, y)) \xi_j \xi_k \leq C_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Next, still with $t = t(s, y)$ and $(s, y) \in \Omega^+$ and using (45), (46) and (47), we have

$$\begin{aligned} & F(\nabla u(t, y), \nabla^2 v(s, y)) - F(\nabla u(t, y), \nabla^2 u(t, y)) \\ & \leq -u_t(t, y) \left[F_{00} \tau_{ss}(s, y) + 2 \sum_{\alpha=1}^{n-1} F_{0\alpha} \tau_{sy_\alpha} + \sum_{\alpha, \beta=1}^{n-1} F_{\alpha\beta} \tau_{y_\alpha y_\beta} \right] \\ & \quad - F_{00} (2 - \tau_s) u_{tt} \tau_s - 2 \sum_{\alpha=1}^{n-1} F_{0\alpha} u_{0\alpha} \tau_s + \eta \cdot \nabla_y \tau, \end{aligned} \tag{66}$$

where $u_{0\alpha} = u_{ty_\alpha}$, F_{jk} denotes $F_{jk}(\nabla u(t, y), \nabla^2 u(t, y))$, and η denotes some vector-valued function in $L_{loc}^\infty(\Omega^+)$ which may vary from line to line.

The term $-F_{00}(2 - \tau_s)u_{tt}\tau_s$ can be handled as in the proof of Theorem 1, by using Lemma 1 and Condition LC. We mainly need to show that

$$\sum_{\alpha=1}^{n-1} F_{0\alpha} u_{0\alpha} = O(1)u_t. \tag{67}$$

For $1 \leq \alpha \leq n - 1$,

$$F_{0\alpha} u_{0\alpha} = \sum_{0 \leq i, l \leq n-1} G_{il} \cdot \frac{\partial A(\nabla u, N)_{il}}{\partial N_{0\alpha}} u_{0\alpha}.$$

Observe that

$$\frac{\partial A(\nabla u, N)_{il}}{\partial N_{0\beta}} = \frac{1}{w} (\delta_{i0} \delta_{l\beta} + O(1)u_t), \quad 1 \leq \beta \leq n - 1. \tag{68}$$

It follows that

$$\begin{aligned} \sum_{\alpha=1}^{n-1} F_{0\alpha} \cdot u_{0\alpha} &= \sum_{\alpha \geq 1} \sum_{l \geq 0} G_{0l} \frac{\partial A(\nabla u, N)_{0l}}{\partial N_{0\alpha}} u_{0\alpha} + O(1)u_t \\ &= \sum_{\alpha, \beta \geq 1} G_{0\beta} \frac{\partial A(\nabla u, N)_{0\beta}}{\partial N_{0\alpha}} u_{0\alpha} + O(1)u_t. \end{aligned} \tag{69}$$

Since $\{A(\nabla u, N)\}_{\beta \geq 1}$ is linear in $\{N_{0\alpha}\}_{\alpha \geq 1}$, we have, using Lemma 6,

$$\begin{aligned} \sum_{\alpha \geq 1} F_{0\alpha} \cdot u_{0\alpha} &= \sum_{\beta \geq 1} G_{0\beta}(A)A_{0\beta} + O(1)u_t = O(1) \sum_{\beta \geq 1} |A_{0\beta}|^2 + O(1)u_t \\ &= O(1) \sum_{\beta \geq 1} |u_{0\beta}|^2 + O(1)u_t. \end{aligned}$$

Since $u_t \geq 0$ and $\nabla_y^2 u_t = O(1)$ in 2Ω , we have, using some well known inequality, see [8], for some universal constant C ,

$$\sum_{j \geq 1} |u_{ty_j}(t, y)| \leq C\sqrt{u_t(t, y)}, \quad \forall (t, y) \in \Omega. \tag{70}$$

With this, we obtain (67).

We deduce from (64)–(67) that

$$L\tau \leq 0 \quad \text{in } \Omega^+, \tag{71}$$

where

$$L := F_{00}\partial_{ss} + 2 \sum_{\alpha=1}^{n-1} F_{0\alpha}\partial_{sy_\alpha} + \sum_{\alpha, \beta=1}^{n-1} F_{\alpha\beta}\partial_{y_\alpha y_\beta} + F_{00}(2 - \tau_s)\frac{t_{tt}}{u_t}\partial_s - O(1)\partial_s - \frac{\eta_\alpha}{u_t}\partial_{y_\alpha}.$$

Using Condition S and Lemma 1, as in the proof of Theorem 1, we have

$$F_{00}(2 - \tau_s)\frac{u_{tt}}{u_t} \geq 0 \quad \text{in } \Omega. \tag{72}$$

Let $\hat{\tau}(s, y)$ be in (54) with $\bar{\epsilon} = \frac{1}{2}$, we derive from (72) that for $\delta > 0$ small,

$$L\hat{\tau} > 0 \quad \text{in } \Omega^+. \tag{73}$$

With (71) and (73), the rest of the proof of Theorem 2 follows as in the proof of Theorem 1.

4 A Variation of the Strong Maximum Principle

In this section we establish a result more general than Theorem 3. We consider $F \in C^1(\mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfying

$$\frac{\partial F}{\partial N_{ij}}(s, p, N)\xi_i \xi_j > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \forall (s, p, N).$$

Theorem 4 For $n \geq 2$, let F be as above, and let Ω be in (4). We assume that $u, v \in C^2(\Omega)$ satisfy (14), (15) and

$$\begin{cases} \text{if } u(t, y) = v(s, y), \quad 0 < s < 1, \quad |y| < 1, \text{ then there} \\ F(u, \nabla u, \nabla^2 u)(t, y) \leq F(v, \nabla v, \nabla^2 v)(s, y). \end{cases} \tag{74}$$

Then either (16) or (17) holds.

Remark 5 The analogue of Theorem 4 in dimension $n = 1$ was proved in [7].

Proof Suppose that (16) does not hold, then $u(\bar{s}, \bar{y}) = v(\bar{s}, \bar{y})$ for some $(\bar{s}, \bar{y}) \in \Omega$. Clearly, $u_t(\bar{s}, \bar{y}) = v_t(\bar{s}, \bar{y}) > 0$ and, by the implicit function theorem, for (s, y) close to (\bar{s}, \bar{y}) there exists a unique C^2 function $t = t(s, y)$ such that $u(t(s, y), y) = v(s, y)$. Thus $F(u, \nabla u, \nabla^2 u)(t(s, y), y) \leq F(v, \nabla v, \nabla^2 v)(s, y)$. As in the proof of Theorem 1, (43)–(47) hold near (\bar{s}, \bar{y}) with $\tau(s, y) = s - t(s, y)$. As usual these lead to $L\tau \leq 0$ near (\bar{s}, \bar{y}) where $L = a_{ij}\partial_{ij} + b_i\partial$ with (a_{ij}) positive definite. Since $\tau(\bar{s}, \bar{y}) = 0$ and $\tau \geq 0$ near (\bar{s}, \bar{y}) , we have, by the strong maximum principle, $\tau \equiv 0$ near (\bar{s}, \bar{y}) . Namely $u \equiv v$ near (\bar{s}, \bar{y}) . Theorem 4 is established.

5 Partial Results on Open Problems 1–3

In this section we give some partial results on or related to Open Problems 1–3 and variations of the Hopf Lemma.

5.1

Theorem 5 Let Ω be as in (4), and let u and v satisfy (5), (7), (8), (18), (9), (10) and

$$u_{tt} \geq 0 \quad \text{in } \Omega. \tag{75}$$

We assume that for some open set $0 \in \omega \subset \bar{\omega} \subset \{y \in \mathbb{R}^n \mid |y| < 1\}$,

$$\frac{\partial v}{\partial t}(0, y) < 0, \quad \forall y \in \partial\omega. \tag{76}$$

Then

$$\frac{\partial^k u}{\partial t^k}(0, 0) = 0, \quad \forall k \geq 2.$$

Remark 6 It is clear from the proof that the conclusion of Theorem 5 still holds when the mean curvature operator is replaced by the more general curvature operators in Theorem 4.

Proof We prove it by contradiction argument. Suppose the contrary, then for some integer $k \geq 2$,

$$\frac{\partial^k u}{\partial t^k}(0, 0) \neq 0, \quad \frac{\partial^i u}{\partial t^i}(0, 0) = 0, \quad 1 \leq i \leq k - 1. \tag{77}$$

By (76), (9) and Theorem 3, $u > v$ in Ω . Let Ω^+ be as in (37). Clearly for some $\epsilon_1 > 0$, $(t, 0) \in \Omega^+$ for all $0 < t < \epsilon_1$. On the other hand, in view of (76) and (9), there exists some $\epsilon_2 > 0$ such that $\{(t, y) \mid 0 < t < \epsilon_2\} \cap \Omega^+ = \emptyset, \forall y \in \partial\omega$. For $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$, let

$$\Omega_\epsilon^+ := \{(s, y) \in \Omega^+ \mid 0 < s < \epsilon\}.$$

Then Ω_ϵ^+ is a nonempty open set satisfying

$$\Omega_\epsilon^+ \subset \{(s, y) \mid 0 < s < \epsilon, y \in \omega\}. \tag{78}$$

Let $t(s, y)$ and $\tau(s, y)$ be defined as in (40) and (42). Then $\tau = s$ on $\partial\Omega_\epsilon^+ \cap \{0 < s < \epsilon\}$, and $\tau > 0$ on $\partial\Omega_\epsilon^+ \cap \{s = \epsilon\}$. Thus, for some constant $c = c(\epsilon) \in (0, \frac{1}{4})$,

$$\tau - c(s + s^{\frac{3}{2}}) \geq 0 \quad \text{on } \partial\Omega_\epsilon^+. \tag{79}$$

Let L be defined in (52). By (75), we still have (55), and therefore we still have $\tau_s < 1$ in Ω^+ . Making ϵ smaller if necessary, we have, as established in the proof of Theorem 1, $L\tau \leq 0 < L(s + s^{\frac{3}{2}})$ in Ω_ϵ^+ . Thus

$$L(\tau - c(s + s^{\frac{3}{2}})) < 0 \quad \text{in } \Omega_\epsilon^+.$$

It follows that $\tau \geq c(s + s^{\frac{3}{2}}) \geq cs$ in Ω_ϵ^+ . With (77), we reach a contradiction by using the argument towards the end of the proof of Theorem 1.

5.2

Let Ω be as in (4), and let

$$f \in C^\infty([-1, 1]^{n-1} \times (0, \infty)), \tag{80}$$

$$u \in C^\infty(\bar{\Omega}), \quad u > 0 \quad \text{in } \Omega, \tag{81}$$

$$u(0, y) = 0, \quad \forall |y| < 1, \tag{82}$$

and

$$\Delta u(t, y) = f(y, u(t, y)) \quad \text{in } \Omega. \tag{83}$$

Assume, for some integer $k \geq 1$,

$$u(t, y) = t^k a_k(y) + O(t^{k+1}), \tag{84}$$

where

$$a_k(y) > 0, \quad \forall |y| \leq 1. \tag{85}$$

Theorem 6 *Let Ω and f be as above, and let u be a solution of (83) satisfying (81), (82), (84) and (85).*

- (i) *If $k = 1$, then all $\left\{ \frac{\partial^l}{\partial t^l} u(0, y) \right\}_{l \geq 2}$ are determined by f and $a_1(y)$.*
- (ii) *If $k \geq 2$, then both k and $\left\{ \frac{\partial^l}{\partial t^l} u(0, y) \right\}_{l \geq k}$ are determined by f .*
- (iii) *If both u and v are solutions of (83) satisfying (81), (82), (84) and (85), so that by (i) and (ii),*

$$\frac{\partial^l}{\partial t^l} u(0, y) = \frac{\partial^l}{\partial t^l} v(0, y), \quad \forall |y| < 1, \quad l \geq k, \tag{86}$$

and $u \geq v$ in Ω , then $v \equiv u$ in Ω .

Remark 7 The f in Theorem 6 is not assumed to be smooth up to $u = 0$, otherwise the conclusion follows from classical results.

Lemma 2 Assume (80)–(85) with $k \geq 2$. Then, for some constant $C > 0$,

$$\sup_{|y| \leq 1, 0 < s < 1} |f(y, s) - a_k(y)^{\frac{2}{k}} k(k-1) s^{\frac{k-2}{k}}| s^{\frac{1-k}{k}} < \infty, \quad (87)$$

$$\lim_{t \rightarrow 0^+} \frac{f(y, u(t, y))}{\left[\frac{u(t, y)}{t^2}\right]} = k(k-1), \quad \text{uniform in } |y| \leq 1. \quad (88)$$

Consequently, both k and $a_k(y)$ are determined by f .

Proof Write

$$u(t, y) = t^k a_k(y) + O(t^{k+1}). \quad (89)$$

Then

$$\Delta u(t, y) = k(k-1)t^{k-2} a_k(y) + O(t^{k-1}).$$

Set

$$s = u(t, y) = t^k a_k(y) + O(t^{k+1}).$$

We have

$$t = \left[\frac{s}{a_k(y)}\right]^{\frac{1}{k}} [1 + O(s^{\frac{1}{k}})], \quad t^{k-2} = \left[\frac{s}{a_k(y)}\right]^{\frac{k-2}{k}} + O(s^{\frac{1}{k}}),$$

$$\Delta u(t, y) = k(k-1)a_k(y)^{\frac{2}{k}} s^{\frac{k-2}{k}} + O(s^{\frac{k-1}{k}}).$$

Estimate (87) follows from this and (83). It is easy to see from (87) that k is determined by f . In turn, again from (87), $a_k(y)$ is determined by f .

By (87), we have, for some constant $C > 0$,

$$|f(y, u(t, y)) - k(k-1)a_k(y)^{\frac{2}{k}} u(t, y)^{\frac{k-2}{k}}| \leq C u(t, y)^{\frac{k-1}{k}}, \quad \forall |y| \leq 1. \quad (90)$$

By (89),

$$|u(t, y) - t^k a_k(y)| \leq C t^{k+1}, \quad \forall |y| \leq 1, \quad 0 < t < 1. \quad (91)$$

By (90) and (91),

$$\lim_{t \rightarrow 0^+} \frac{f(y, u(t, y))}{u(t, y)^{\frac{k-2}{k}}} = k(k-1)a_k(y)^{\frac{2}{k}}, \quad \forall |y| \leq 1, \quad (92)$$

$$\lim_{t \rightarrow 0^+} \frac{u(t, y)^{\frac{2}{k}}}{t^2} = a_k(y)^{\frac{2}{k}}, \quad \forall |y| \leq 1. \quad (93)$$

Estimate (88) follows from (92) and (93). Lemma 2 is established.

Proof of Part (i) and (ii) of Theorem 6 Because of Lemma 2, we only need to prove that $\left\{\frac{\partial^l}{\partial t^l} u(0, y)\right\}_{l \geq k+1}$ are determined by f and $a_k(y)$. We will prove it by induction.

Write

$$u(t, y) = t^k a_k(y) + t^{k+1} a_{k+1}(y) + \cdots + t^{m-1} a_{m-1}(y) + t^m a_m(y) + O(t^{m+1}), \quad (94)$$

and we assume that $m \geq k + 1$, and $a_k(y), \dots, a_{m-1}(y)$ are determined by f . We will prove that $a_m(y)$ is also determined by f and $a_k(y)$.

Let

$$s := u(t, y) = t^k a_k(y) + t^{k+1} a_{k+1}(y) + \dots + t^m a_m(y) + O(t^{m+1}).$$

Then

$$\lambda := \left[\frac{s}{a_k(y)} \right]^{\frac{1}{k}} = t \left\{ 1 + t \frac{a_{k+1}}{a_k} + \dots + t^{m-k} \frac{a_m}{a_k} + O(t^{m-k+1}) \right\}^{\frac{1}{k}}.$$

It follows that

$$\lambda = t \left\{ 1 + \dots + t^{m-k-1} b_{m-k-1} + t^{m-k} \frac{a_m}{k a_k} + O(t^{m-k+1}) \right\},$$

where $\{b_i(y)\}_{i \leq m-k-1}$ are determined by f and $a_k(y)$.

Clearly, $\lim_{t \rightarrow 0} \frac{\lambda}{t} = 1$. We know that $\frac{d\lambda}{dt} \Big|_{t=0}, \frac{d^2\lambda}{dt^2} \Big|_{t=0}, \dots, \frac{d^{m-k}\lambda}{dt^{m-k}} \Big|_{t=0}$ are determined by f and $a_k(y)$. We now write t in terms of λ . First

$$\frac{dt}{d\lambda} \frac{d\lambda}{dt} = 1.$$

Applying $\frac{d}{d\lambda}$ to the above $m - k + 1$ times, we have

$$\begin{aligned} \frac{d^2 t}{d\lambda^2} \frac{d\lambda}{dt} + \left(\frac{dt}{d\lambda} \right)^2 \frac{d^2 \lambda}{dt^2} &= 0, \quad \frac{d^3 t}{d\lambda^3} \frac{d\lambda}{dt} + \dots + \left(\frac{dt}{d\lambda} \right)^3 \frac{d^3 \lambda}{dt^3} = 0, \quad \dots, \\ \frac{d^{m-k+1} t}{d\lambda^{m-k+1}} \frac{d\lambda}{dt} + \dots + \left(\frac{dt}{d\lambda} \right)^{m-k+1} \frac{d^{m-k+1} \lambda}{dt^{m-k+1}} &= 0. \end{aligned}$$

Set $\lambda = 0$ in the above. All the “ \dots ” contribute to quantities determined by f and $a_k(y)$. Therefore $\frac{dt}{d\lambda} \Big|_{\lambda=0}, \frac{d^2 t}{d\lambda^2} \Big|_{\lambda=0}, \dots, \frac{d^{m-k} t}{d\lambda^{m-k}} \Big|_{\lambda=0}$ are determined by f , and $\frac{d^{m-k+1} t}{d\lambda^{m-k+1}} \Big|_{\lambda=0} + \frac{d^{m-k+1} \lambda}{dt^{m-k+1}} \Big|_{t=0}$ is determined by f and $a_k(y)$. We also note that $\frac{d^{m-k+1} \lambda}{dt^{m-k+1}} \Big|_{t=0} = (m - k + 1)! \frac{a_m(y)}{k a_k(y)}$. It follows that

$$t = \lambda + \lambda^2 c_2(y) + \dots + \lambda^{m-k} c_{m-k}(y) - \lambda^{m-k+1} \frac{a_m(y)}{k a_k(y)} + \lambda^{m-k+1} c_{m-k+1}(y) + O(\lambda^{m-k+2}),$$

where $c_2(y), \dots, c_{m-k+1}(y)$ are determined by f and $a_k(y)$.

Applying Δ to (94) yields

$$\Delta u(t, y) = \sum_{j=k}^{m-1} j(j-1)t^{j-2} \alpha_j(y) + m(m-1)t^{m-2} a_m(y) + O(t^{m-1}),$$

where $\{\alpha_j(y)\}_{k \leq j \leq m-1}$ are determined by f . Since $f(y, u) = \Delta u$, we have

$$f(y, s) = \sum_{j=k}^{m-1} j(j-1)t^{j-2} \alpha_j(y) + m(m-1)t^{m-2} a_m(y) + O(t^{m-1}).$$

For $k \leq j \leq m - 1$,

$$\begin{aligned} t^{j-2} &= \lambda^{j-2} \left\{ 1 + \lambda c_2 + \cdots + \lambda^{m-k-1} c_{m-k} - \lambda^{m-k} \frac{a_m(y)}{ka_k(y)} + O(\lambda^{m-k+1}) \right\}^{j-2} \\ &= \lambda^{j-2} \left\{ 1 + \lambda d_2 + \cdots + \lambda^{m-k-1} d_{m-k} - \lambda^{m-k} \frac{j-2}{ka_k(y)} a_m(y) + O(\lambda^{m-k+1}) \right\} \\ &= \lambda^{j-2} + \lambda^{j-1} d_2 + \cdots + \lambda^{m+j-k-3} d_{m-k} - \frac{j-2}{ka_k(y)} \lambda^{m+j-k-2} a_m(y) + O(\lambda^{m+j-k-1}), \end{aligned}$$

where d_2, \dots, d_{m-k} are determined by f and $a_k(y)$.

The coefficient of $a_m(y)$ in the above expansion of t^{k-2} is of order $\lambda^{m-2} \sim t^{m-2}$, while the coefficients of $a_m(y)$ in the expansions of t^{j-2} for $k < j \leq m - 1$ are of higher order. Thus

$$\begin{aligned} f(y, s) &= \lambda^{k-2} e_{k-2}(y) + \cdots + \lambda^{m-3} e_{m-3}(y) - (k-1)(k-2) \lambda^{m-2} a_m(y) \\ &\quad + m(m-1) \lambda^{m-2} a_m(y) + \lambda^{m-2} e_{m-2}(y) + O(\lambda^{m-1}), \end{aligned}$$

where $\{e_j(y)\}_{k-2 \leq j \leq m-2}$ are determined by f and $a_k(y)$. Since $m \geq k + 1$, we have $m(m-1) > (k-1)(k-2)$. Therefore $a_m(y)$ is also determined by f and $a_k(y)$. Part (i) and (ii) of Theorem 6 are established.

To prove Part (iii) of Theorem 6, we can make use of the following

Theorem 7 *Let $w \in C^\infty(\bar{\Omega})$ satisfy*

$$\begin{aligned} w &\geq 0 \quad \text{in } \Omega, \\ \partial^\alpha w(0, y) &= 0, \quad \forall |y| \leq 1, \forall \alpha, \end{aligned}$$

and, for some positive constant C_0 ,

$$\Delta w \leq C_0 \frac{w}{t^2} \quad \text{in } \Omega.$$

Then

$$w \equiv 0 \quad \text{in } \Omega.$$

Theorem 7 is an immediate corollary of the following kind of Hopf Lemma.

Theorem 8 *Consider a domain Ω in \mathbb{R}^n with C^2 boundary, and a positive function w in Ω , $w \in C^\infty(\bar{\Omega})$, satisfying: for some positive constant C_0 ,*

$$\Delta w(x) \leq C_0 \frac{w(x)}{\text{dist}(x, \partial\Omega)^2}. \tag{95}$$

Suppose $w = 0$ at some boundary point P . Then, along the inner normal to $\partial\Omega$ at P , close to P ,

$$w(x) \geq a|x - P|^k, \tag{96}$$

where a is a positive constant and $k > n$ satisfies

$$k(k-n) = C_0. \tag{97}$$

Proof Let B_R be an open ball in Ω whose boundary touches $\partial\Omega$ only at P . We may suppose that its center is the origin and that $P = (-R, 0, \dots, 0)$. Set $|x| = r$. By (95), w satisfies

$$\Delta w(x) \leq C_0 \frac{w(x)}{(R-r)^2} \quad \text{in } B_R. \tag{98}$$

We construct a comparison function $h = (R-r)^k$ with k satisfying (97). In the region

$$K := \left\{ x \in B_R \mid x_1 < -\frac{R}{2} \right\}$$

we have

$$\frac{R-r}{r} < 1.$$

Then, in K ,

$$\Delta h = (R-r)^{k-2} \left[k(k-1) - (n-1)k \frac{R-r}{r} \right] \geq (R-r)^{k-2} k(k-n).$$

Thus

$$\Delta h \geq C_0 \frac{h}{(R-r)^2}.$$

Since $w > 0$ in Ω , on the straight part of ∂K , $w \geq ch$ for some constant $c > 0$. This same inequality holds on the curved part of ∂K since, there, $h = 0$. By the maximum principle it follows that

$$w \geq ch \quad \text{in } K,$$

and so (96) follows.

By choosing K much narrower one sees that (96) holds provided $k(k-1) > C_0$; of course a depends on k . An immediate consequence of this is the following kind of Hopf Lemma, in which we may take $k < 2$.

Corollary 3 *In a domain Ω in \mathbb{R}^n with C^2 boundary, let $w \geq 0$, $w \in C^2(\bar{\Omega})$, satisfy (95) near $\partial\Omega$, with $C_0 < 2$. Suppose that at some boundary point P , w and its normal derivative vanish. Then $w \equiv 0$.*

Remark 8 The proof of Theorem 8 applies also to a function $w > 0$ satisfying an elliptic inequality

$$Lw(x) \leq C_0 \frac{w(x)}{\text{dist}(x, \partial\Omega)^2}.$$

Here $Lw = a_{ij}w_{ij} + b_iw_i + cw$ is uniformly elliptic with bounded coefficients. The value of k is, of course, different.

Returning to Theorem 6, we derive some further properties of f .

Lemma 3 *Assume (80)–(85) with $k = 1$. Then*

$$\sup_{0 < s_1 < s_2 < 1, |y| < 1} \frac{|f(y, s_1) - f(y, s_2)|}{|s_1 - s_2|} < \infty. \tag{99}$$

Proof Write $u(t, y) = ta_1(y) + t^2a_2(y) + t^3a_3(y) + O(t^4)$. Then

$$u_t(t, y) = a_1(y) + O(t), \quad \Delta u_t(t, y) = 6a_3(y) + \Delta_y a_1(y) + O(t).$$

Applying ∂_t to (83) yields

$$f_u(y, u(t, y)) = \frac{\Delta u_t(t, y)}{u_t(t, y)} = \frac{6a_3(y) + \Delta_y a_1(y) + O(t)}{a_1(y) + O(t)}.$$

Let $s = u(t, y) = ta_1(y) + O(t^2)$. Then

$$t = \frac{s}{a_1(y)} + O(s^2).$$

It follows that

$$f_u(y, s) = \frac{6a_3(y) + \Delta_y a_1(y) + O(s)}{a_1(y) + O(s)}.$$

This implies (99). Lemma 3 is established.

Lemma 4 *If $k \geq 2$, there exists some positive constant C such that*

$$f_u(y, s) \leq Cs^{-\frac{2}{k}}, \quad \forall |y| \leq 1, \quad 0 < s < 1.$$

Proof For $k \geq 3$,

$$u_t(t, y) = ka_k(y)t^{k-1} + O(t^k), \quad \Delta u_t(t, y) = k(k-1)(k-2)a_k(y)t^{k-3} + O(t^{k-2}).$$

Applying ∂_t to (83) gives

$$\Delta u_t(t, y) = f_u(y, u)u_t(t, y). \tag{100}$$

Write $s = u = a_k(y)t^k + O(t^{k+1})$. We have

$$t = \left[\frac{s}{a_k(y)} \right]^{\frac{1}{k}} [1 + O(s^{\frac{1}{k}})].$$

Thus

$$s^{\frac{2}{k}} f_u(y, s) = s^{\frac{2}{k}} \frac{\Delta u_t(t, y)}{u_t(t, y)} = \frac{k(k-1)(k-2)t^{k-3} + O(t^{k-2})}{kt^{k-1} + O(t^k)} \rightarrow (k-1)(k-2) \quad \text{as } s \rightarrow 0^+.$$

For $k = 2$, write $u(t, y) = a_2(y)t^2 + a_3(y)t^3 + O(t^4)$. Applying ∂_t to the above gives

$$u_t(t, y) = 2a_2(y)t + O(t^2), \quad \Delta u_t(t, y) = 6a_3(y) + O(t).$$

We still have (100). Write $s = u(t, y) = a_2(y)t^2 + O(t^3)$. We have

$$s^{\frac{1}{2}} f_u(y, s) = s^{\frac{1}{2}} \frac{\Delta u_t(t, y)}{u_t(t, y)} = s^{\frac{1}{2}} \frac{6a_3(y) + O(t)}{2ta_2(y) + O(t^2)} \rightarrow \frac{3a_3(y)}{\sqrt{a_2(y)}} \quad \text{as } s \rightarrow 0^+.$$

Lemma 4 is established.

Proof of Part (iii) of Theorem 6 For $k = 1$, it follows from Lemma 3 that

$$\Delta(u - v) = O(1)(u - v) \quad \text{in } \Omega.$$

Since

$$u - v \geq 0 \quad \text{in } \Omega, \quad \frac{\partial}{\partial t}(u - v)(0, y) = 0, \quad \forall |y| < 1,$$

we have, by the Hopf Lemma and the strong maximum principle, that $u \equiv v$ in Ω .

Now we assume that $k \geq 2$. Clearly, for any $\epsilon \in (0, 1)$, there exists some positive constant C such that

$$\frac{1}{C}t^k \leq u, v \leq Ct^k \quad \text{in } (1 - \epsilon)\Omega. \tag{101}$$

Using the equation satisfied by u and v , Lemma 4 and (101), we have

$$\begin{aligned} \Delta(u - v) &= f(y, u) - f(y, v) = \int_0^1 f_u(y, \theta u + (1 - \theta)v) d\theta(u - v) \\ &= O(1)(u - v) \int_0^1 [\theta u + (1 - \theta)v]^{-\frac{2}{k}} = O(1)\frac{u - v}{t^2} \quad \text{in } (1 - \epsilon)\Omega. \end{aligned}$$

If $u \geq v$ in Ω , then, by Theorem 7, $u \equiv v$ in $(1 - \epsilon)\Omega$. Part (iii) of Theorem 6, where we have $u \geq v$ in Ω , is established.

5.3

The following two theorems are not used in this paper.

Theorem 9 For $n = 2$, let Ω be in (4), and let u and v satisfy (5), (7), (8), (18), (9), (13), (75) and

$$\begin{cases} \text{if } u(t, y) = v(s, y), \quad 0 < s < 1, \quad |y| < 1, \quad \text{then there} \\ \Delta u(t, y) = \Delta v(s, y). \end{cases} \tag{102}$$

We also assume that if $u_t(0, \bar{y}) = 0$ for some $|\bar{y}| < 1$, then for some integer $\bar{k} \geq 2$, which may depend on \bar{y} ,

$$\frac{\partial^{\bar{k}} u}{\partial t^{\bar{k}}}(0, \bar{y}) \neq 0. \tag{103}$$

Then $u \equiv v$ in Ω .

Proof It is easy to see that we only need to consider the following two cases:

Case 1 There exist $-1 < \alpha < \beta < 1$ such that $u_t(0, y) = 0$ for all $y \in (\alpha, \beta)$.

Case 2 There exist $-1 < y^- < 0 < y^+ < 1$ such that $u_t(0, y^\pm) > 0$.

In Case 1, we can find some point $\bar{y} \in (\alpha, \beta)$ and some even integer $k \geq 2$ such that $\partial_t^k u(0, \bar{y}) = \partial_t^k v(0, \bar{y}) > 0$, and $\partial_t^i u(0, y) = \partial_t^i v(0, y) = 0$ for all $1 \leq i \leq k - 1$ and all y in some neighborhood of \bar{y} . Without loss of generality, $\bar{y} = 0$. By subtracting $u(0, y)$ from both u and v , we may assume without loss of generality that (19) holds. Now (84) holds with $k!a_k(y) = \partial_t^k u(0, y) = \partial_t^k v(0, y)$. Thus, for some $\delta > 0$,

$$u \geq v > 0 \quad \text{in } \delta\Omega. \tag{104}$$

By (9), the map $(t, y) \rightarrow (u(t, y), y)$ is a local diffeomorphism and, by the implicit function theorem, t is locally a smooth function of u and y in Ω . Thus, in view of (9) and (19),

$$\Delta u = f(y, u) \quad \text{in } \delta\Omega,$$

where f is some unknown smooth function in $\{(y, u) \mid u > 0, |y| < 1\}$ and continuous in $\{(y, u) \mid u \geq 0, |y| < 1\}$. By (104), for every $(s, y) \in \delta\Omega$, there exists some (t, y) , with $0 < t < s$, such that $u(t, y) = v(s, y)$. Thus, by (102),

$$\Delta v = f(y, v) \quad \text{in } \delta\Omega.$$

An application of Theorem 6 yields $u \equiv v$ near $(0, \bar{y})$. Theorem 9 follows in this case in view of Theorem 4.

In Case 2, we still have (78) and (79) for small $\epsilon > 0$. As explained in the proof of Theorem 5, we still have $\tau_s < 1$ in Ω^+ . Thus we still have $L\tau \leq 0 < L(s + s^{\frac{3}{2}})$ in Ω_ϵ^+ and, for some $c > 0$, $\tau \geq c(s + s^{\frac{3}{2}})$ on $\partial\Omega_\epsilon^+$, where

$$L = \partial_{ss} + \Delta_y + (2 - \tau_s) \frac{u_{tt}}{u_t} \partial_s - \eta \cdot \nabla_y \tau.$$

Theorem 9 in this case follows as in the proof of Theorem 5.

Finally we include the following result.

Theorem 10 *Let u be a C^∞ function in the unit ball B_1 in \mathbb{R}^n , $n \geq 1$, satisfying*

$$\Delta u(x) = V(x)u(x), \quad x \in B_1, \tag{105}$$

where $V \in C^1(B_1 \setminus \{0\})$ satisfies, in polar coordinates (r, θ) , $\theta \in \mathbb{S}^{n-1}$,

$$(r^2 V)_r := \frac{\partial}{\partial r}(r^2 V) \geq 0 \quad \text{in } B_1 \setminus \{0\}. \tag{106}$$

Assume that u vanishes of infinite order at the origin, i.e.,

$$\partial^\alpha u(0) = 0 \quad \text{for all multi-index } \alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \geq 0. \tag{107}$$

Then $u \equiv 0$ in B_1 .

Proof We make use of ideas in Agmon and Nirenberg [1]. Using polar coordinates (r, θ) , equation (105) takes the form

$$r^2 u_{rr} + (n - 1) r u_r + \Delta_\theta u = r^2 V u. \tag{108}$$

Set $r = e^s$. Then

$$u_s = u_r e^s, \quad u_{ss} = u_{rr} e^{2s} + u_r e^s = r^2 u_{rr} + r u_r,$$

and (108) takes the form

$$u_{ss} + (n - 2) u_s + \Delta_\theta u = r^2 V u, \quad (s, \theta) \in (-\infty, 0) \times \mathbb{S}^{n-1}.$$

Because of (107),

$$\lim_{s \rightarrow -\infty} \max_{\theta \in \mathbb{S}^{n-1}} \sum_{i=0}^2 (|\partial_s^i u(s, \theta)| + |\partial_\theta^i u(s, \theta)|) e^{bs} = 0, \quad \forall b < 0. \tag{109}$$

Set

$$u = e^{as}v \quad \text{with } a = -\frac{n-2}{2}.$$

Since $u_s = e^{as}(v_s + av)$, $u_{ss} = e^{as}(v_{ss} + 2av_s + a^2v)$, v satisfies

$$v_{ss} + \Delta_\theta v = mv, \quad \text{in } (-\infty, 0) \times \mathbb{S}^{n-1}, \tag{110}$$

where

$$m := \left(\frac{n-2}{2}\right)^2 + r^2V.$$

Consider

$$\rho(s) := \int_{\mathbb{S}^{n-1}} v^2(s, \theta) d\theta.$$

We will prove

Lemma 5

$$\frac{d^2}{ds^2} \log \rho(s) \geq 0 \quad \text{whenever } \rho(s) > 0. \tag{111}$$

Proof By computation,

$$\rho_s = 2 \int_{\mathbb{S}^{n-1}} vv_s d\theta, \quad \rho_{ss} = 2 \int_{\mathbb{S}^{n-1}} (v_s^2 + vv_{ss}) d\theta.$$

So

$$\rho_{ss} = 2 \int_{\mathbb{S}^{n-1}} v_s^2 d\theta + 2 \int_{\mathbb{S}^{n-1}} v(-\Delta_\theta v + mv) d\theta = 2 \int_{\mathbb{S}^{n-1}} [v_s^2 + |\nabla_\theta v|^2 + mv^2] d\theta. \tag{112}$$

Next, by the Schwartz inequality,

$$\frac{\rho_s^2}{\rho} = \frac{4(\int_{\mathbb{S}^{n-1}} vv_s d\theta)^2}{\int_{\mathbb{S}^{n-1}} v^2 d\theta} \leq 4 \int_{\mathbb{S}^{n-1}} v_s^2 d\theta. \tag{113}$$

Multiplying (110) by $2v_s$ and integrating in s from $-\infty$ to 0 , and integrating over \mathbb{S}^{n-1} , we find, using Green's theorem,

$$\int_{\mathbb{S}^{n-1}} v_s^2 - 2 \int_{-\infty}^s \int_{\mathbb{S}^{n-1}} \nabla_\theta v_s \cdot \nabla_\theta v = \int_{-\infty}^s \int_{\mathbb{S}^{n-1}} 2m v v_s,$$

i.e.,

$$\int_{\mathbb{S}^{n-1}} v_s^2 = \int_{\mathbb{S}^{n-1}} |\nabla_\theta v|^2 + \int_{-\infty}^s \int_{\mathbb{S}^{n-1}} 2m v v_s = \int_{\mathbb{S}^{n-1}} |\nabla_\theta v|^2 + \int_{\mathbb{S}^{n-1}} m v^2 - \int_{-\infty}^s \int_{\mathbb{S}^{n-1}} m_s v^2.$$

We know from (106) that $m_s \geq 0$. Thus

$$\int_{\mathbb{S}^{n-1}} v_s^2 \leq \int_{\mathbb{S}^{n-1}} [|\nabla_\theta v|^2 + m v^2], \quad s \in (-\infty, 0). \tag{114}$$

We deduce from (112)–(114) that

$$\rho_{ss} \geq \frac{\rho_s^2}{\rho}, \quad \text{whenever } \rho(s) > 0,$$

which is equivalent to (111). Lemma 5 is established.

To prove Theorem 10, we only need to show that $\rho \equiv 0$. Suppose $\rho(s) > 0$ for some $\bar{s} \in (-\infty, 0)$. By (111), $\log \rho$ is convex in any open interval where $\rho > 0$. So, for any interval $(-T, \bar{s})$ where ρ is positive, we have

$$\log \rho(s) \geq \log \rho(\bar{s}) + \frac{d}{ds} \log \rho(\bar{s})(s - \bar{s}), \quad \forall -T < s < \bar{s}.$$

It follows from the above that $\rho(s) > 0$ for all $-\infty < s < \bar{s}$ and, for some constant $C_1, C_2 > 0$,

$$\rho(s) \geq C_1 e^{C_2 s}, \quad \forall -\infty < s < \bar{s},$$

which violates (109). Theorem 10 is established.

6 Appendix

Let $\mathcal{S}^{n \times n}$ denote the set of real $n \times n$ symmetric matrices, and let $O(n)$ denote the set of real $n \times n$ orthogonal matrices. For $N \in \mathcal{S}^{n \times n}$, we use $|N| := \sqrt{\sum_{0 \leq k, l \leq n-1} |N_{kl}|^2}$ to denote the norm of N .

Lemma 6 *Let G be a C^3 function defined on $\mathcal{S}^{n \times n}$ satisfying*

$$G(O^{-1}NO) = G(N), \quad \forall N \in \mathcal{S}^{n \times n}, \forall O \in O(n).$$

Then, for some constant C depending only on n and G ,

$$\left| \sum_{\alpha=1}^{n-1} \frac{\partial G}{\partial N_{0\alpha}}(N) N_{0\alpha} \right| \leq C \sum_{\beta=1}^{n-1} |N_{0\beta}|^2, \quad \forall N \in \mathcal{S}^{n \times n}, |N| \leq 1.$$

Proof Let \bar{N} denote elements in $\mathcal{S}^{n \times n}$ satisfying

$$\bar{N}_{0\alpha} = \bar{N}_{\alpha 0} = 0, \quad 1 \leq \alpha \leq n-1,$$

and let e denote elements in $\mathcal{S}^{n \times n}$ satisfying

$$e_{00} = e_{\alpha\beta} = 0, \quad 1 \leq \alpha, \beta \leq n-1.$$

Consider the following function of e :

$$h(e) = \frac{d}{dt} G(\bar{N} + te) \Big|_{t=1}.$$

For $N = \bar{N} + e$,

$$h(e) = 2 \sum_{\alpha=1}^{n-1} \frac{\partial G}{\partial N_{0\alpha}}(N) N_{0\alpha}.$$

Clearly, $h(0) = 0$. For $O = \text{diag}(-1, 1, \dots, 1)$,

$$O(\bar{N} + te)O = \bar{N} - te.$$

So $h(e) \equiv h(-e)$. Consequently, $\nabla h(0) = 0$. Since h is a C^2 function, we obtain

$$|h(e)| \leq C|e|^2.$$

Lemma 6 is established.

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