

Local Gradient Estimates of Solutions to Some Conformally Invariant Fully Nonlinear Equations

YANYAN LI
Rutgers University

1 Introduction

A classical theorem of Liouville says:

$$(1.1) \quad u \in C^2, \Delta u = 0, \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{const.}$$

The Laplacian operator Δ is invariant under rigid motions: For any function u on \mathbb{R}^n and for any rigid motion $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\Delta(u \circ T) = (\Delta u) \circ T.$$

T is called a rigid motion if $Tx \equiv Ox + b$ for some $n \times n$ orthogonal matrix O and some vector $b \in \mathbb{R}^n$. It is clear that a linear second-order partial differential operator

$$Lu := a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u$$

is invariant under rigid motion, i.e.,

$$L(u \circ T) = (Lu) \circ T \text{ for any function } u \text{ and any rigid motion } T$$

if and only if $L = a\Delta + c$ for some constants a and c .

Instead of rigid motions, we look at Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$ and *nonlinear* operators that are invariant under Möbius transformations. A map $\varphi : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}$ is called a Möbius transformation if it is a composition of finitely many of the following three types of transformations:

A translation : $x \rightarrow x + \bar{x}$ where \bar{x} is a given point in \mathbb{R}^n ,

A dilation : $x \rightarrow ax$, where a is a positive number,

A Kelvin transformation : $x \rightarrow \frac{x}{|x|^2}$.

For a function u on \mathbb{R}^n , let

$$u_\varphi := |J_\varphi|^{\frac{n-2}{2n}} (u \circ \varphi)$$

where J_φ denotes the Jacobian of φ .

Let $H(x, s, p, M)$ be a smooth function in its variables, where $s > 0$, $x, p \in \mathbb{R}^n$, and $M \in \mathcal{S}^{n \times n}$, the set of $n \times n$ real symmetric matrices. We say that a

second-order fully nonlinear operator $H(\cdot, u, \nabla u, \nabla^2 u)$ is conformally invariant if

$$H(\cdot, u_\varphi, \nabla u_\varphi, \nabla^2 u_\varphi) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \varphi$$

holds for all positive smooth functions u and all Möbius transformations φ .

For a positive C^2 function u , set

$$(1.2) \quad \begin{aligned} A^u &:= -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u \\ &\quad - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I \\ &\equiv A_w := w \nabla^2 w - \frac{|\nabla w|^2}{2} I \quad \text{with } w = u^{-\frac{2}{n-2}}, \end{aligned}$$

where I denotes the $n \times n$ identity matrix.

Let φ be a Möbius transformation; then for some $n \times n$ orthogonal matrix functions $O(x)$ (i.e., $O(x)O(x)^\top = I$), depending on φ ,

$$A^{u_\varphi}(x) \equiv O(x)A^u(\varphi(x))O^\top(x).$$

Thus it is clear that $f(\lambda(A^u))$ is a conformally invariant operator for all symmetric functions f , where $\lambda(A^u)$ denotes the eigenvalues of A^u .

It was proved in [17] that an operator $H(\cdot, u, \nabla u, \nabla^2 u)$ is conformally invariant if and only if it is of the form

$$H(\cdot, u, \nabla u, \nabla^2 u) \equiv f(\lambda(A^u)),$$

where $f(\lambda)$ is some symmetric function in $\lambda = (\lambda_1, \dots, \lambda_n)$. Due to the above characterizing conformal invariance property, the operator A_w is called the conformal Hessian of w .

Taking $f(\lambda) = \sigma_1(\lambda) := \lambda_1 + \dots + \lambda_n$, we have a simple expression:

$$(1.3) \quad \sigma_1(\lambda(A^u)) \equiv -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u.$$

In general, $f(\lambda(A^u))$ is a fully nonlinear operator and is rather complex even for $f(\lambda) = \sigma_k(\lambda)$, $k \geq 3$, where

$$\sigma_k(\lambda) := \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k^{th} elementary symmetric function. The expression for σ_2 is still quite pleasant:

$$\sigma_2(\lambda(A^u)) \equiv \frac{1}{2} (\sigma_1(\lambda(A^u)))^2 - (A^u)^\top A^u.$$

Let

(1.4) $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying

$$(1.5) \quad \Gamma_n := \{\lambda \mid \lambda_i > 0, 1 \leq i \leq n\} \subset \Gamma \subset \left\{ \lambda \mid \sum_{i=1}^n \lambda_i > 0 \right\} =: \Gamma_1.$$

Γ being symmetric means that $(\lambda_1, \dots, \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \dots, \lambda_{i_n}) \in \Gamma$ for any permutation (i_1, \dots, i_n) of $(1, \dots, n)$.

For $1 \leq k \leq n$, let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone Γ_n . It is known (see, for instance, [2]) that Γ_k satisfies (1.4) and (1.5). In fact,

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \dots, \sigma_k(\lambda) > 0\}.$$

Let Ω be an open subset of \mathbb{R}^n ; we consider

$$(1.6) \quad \lambda(A^u) \in \partial\Gamma \quad \text{in } \Omega$$

or

$$(1.7) \quad \lambda(A_w) \in \partial\Gamma \quad \text{in } \Omega.$$

It is easy to see that in dimension $n \geq 3$

$$(1.8) \quad A^u \equiv A_w \quad \text{for any positive } C^2 \text{ function } w \text{ and } u = w^{-\frac{2}{n-2}}.$$

Equations (1.6) and (1.7) are fully nonlinear second-order degenerate elliptic equations. Fully nonlinear second-order elliptic equations with $\lambda(\nabla^2 u)$ in such general Γ were first studied by Caffarelli, Nirenberg, and Spruck in [2].

Equations (1.6) and (1.7) have obvious meaning if u and w are C^2 functions. If they are in $C_{\text{loc}}^{1,1}(\Omega)$, the equations are naturally understood to be satisfied almost everywhere. We give the notion of viscosity solutions of (1.6) and (1.7).

DEFINITION 1.1 A positive continuous function w in Ω is a *viscosity supersolution* (respectively, *subsolution*) of (1.7) when the following holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(w - \varphi)(x_0) = 0$, and $w - \varphi \geq 0$ near x_0 , then

$$\lambda(A_\varphi(x_0)) \in \mathbb{R}^n \setminus \Gamma$$

(respectively, if $(w - \varphi)(x_0) = 0$ and $w - \varphi \leq 0$ near x_0 , then $\lambda(A_\varphi(x_0)) \in \bar{\Gamma}$).

We say that w is a viscosity solution of (1.7) if it is both a supersolution and a subsolution.

Similarly, we have the following:

DEFINITION 1.1' A positive continuous function u in an open subset Ω of \mathbb{R}^n , $n \geq 3$, is a *viscosity subsolution* (respectively, *supersolution*) of (1.6) when the following holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(u - \varphi)(x_0) = 0$, and $u - \varphi \leq 0$ near x_0 , then

$$\lambda(A^\varphi(x_0)) \in \mathbb{R}^n \setminus \Gamma.$$

(respectively, if $(u - \varphi)(x_0) = 0$ and $u - \varphi \geq 0$ near x_0 then $\lambda(A^\varphi(x_0)) \in \bar{\Gamma}$).

We say that u is a viscosity solution of (1.6) if it is both a supersolution and a subsolution.

Remark 1.2. In dimension $n \geq 3$, a positive continuous function u is a viscosity subsolution (supersolution) of (1.6) if and only if $w := u^{-2/(n-2)}$ is a viscosity supersolution (subsolution) of (1.7). This is clear in view of (1.8).

Remark 1.3. Viscosity solutions of (1.7) are invariant under conformal transformations and multiplication by positive constants. Namely, if w is a viscosity supersolution (subsolution) of (1.7) then, for any constants $b, \lambda > 0$ and for any $x \in \mathbb{R}^n$, bw is a viscosity supersolution (subsolution) of (1.7), $\xi(y) := \frac{1}{b}w(x + by)$ is a viscosity supersolution (subsolution) of $\lambda(A_\xi) \in \partial\Gamma$ in $\{y \mid x + by \in \Omega\}$, and

$$\eta(y) := \left(\frac{|y - x|^2}{\lambda} \right)^2 w \left(x + \frac{\lambda^2(y - x)}{|y - x|^2} \right)$$

is a viscosity supersolution (subsolution) of $\lambda(A_\eta) \in \partial\Gamma$ in

$$\left\{ y \mid x + \frac{\lambda^2(y - x)}{|y - x|^2} \in \Omega \right\}.$$

One of the two main theorems in this paper is the following Liouville theorem for positive locally Lipschitz viscosity solutions of

$$(1.9) \quad \lambda(A_w) \in \partial\Gamma \quad \text{in } \mathbb{R}^n.$$

THEOREM 1.4 For $n \geq 3$, let Γ satisfy (1.4) and (1.5), and let w be a positive locally Lipschitz viscosity solution of (1.9). Then $w \equiv w(0)$ in \mathbb{R}^n .

Remark 1.5. For $n = 2$, $\Gamma = \Gamma_1$, the conclusion does not hold. Indeed, $w = e^{x_1}$ satisfies $\lambda(A_w) \in \partial\Gamma_1$. In fact, $\lambda(A_w) \in \partial\Gamma_1$ is equivalent to $\Delta \log w = 0$ in dimension $n = 2$.

Theorem 1.4 can be viewed as a nonlinear extension of the classical Liouville theorem (1.1). Indeed, in view of (1.3), the Liouville theorem (1.1) is equivalent to

$$u \in C^2, \lambda(A^u) \in \partial\Gamma_1, \text{ and } u > 0 \text{ in } \mathbb{R}^n \text{ imply that } u \equiv \text{const.}$$

Such a Liouville theorem was proved by Chang, Gursky, and Yang in [3] for $u \in C_{\text{loc}}^{1,1}$, $\Gamma = \Gamma_2$, and $n = 4$; by Aobing Li in [16] for $u \in C_{\text{loc}}^{1,1}$, $\Gamma = \Gamma_2$, and $n = 3$; independently by Aobing Li in [16] and by Sheng, Trudinger, and Wang in [30] for

$u \in C^3$, $\Gamma = \Gamma_k$, $k \leq n$, and $n \geq 3$. By entirely different methods we established in [26, 27] the following theorems:

Consider

$$(1.10) \quad f \in C^1(\Gamma) \cap C^0(\bar{\Gamma}) \text{ is symmetric in } \lambda_i,$$

$$(1.11) \quad f \text{ is homogeneous of degree 1,}$$

$$(1.12) \quad f > 0, \quad f_{\lambda_i} := \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad f|_{\partial\Gamma} = 0,$$

$$(1.13) \quad \sum_{i=1}^n f_{\lambda_i} \geq \delta \quad \text{in } \Gamma \text{ for some } \delta > 0.$$

Examples of such (f, Γ) include those given by elementary symmetric functions: For $1 \leq k \leq n$, $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies all the above properties; see, for instance, [2].

THEOREM A ([26]) *For $n \geq 3$, let (f, Γ) satisfy (1.4), (1.5), (1.10), and (1.12), and let u be a positive $C_{\text{loc}}^{1,1}$ solution of*

$$(1.14) \quad f(\lambda(A^u)) = 0 \quad \text{in } \mathbb{R}^n.$$

Then $u \equiv u(0)$ in \mathbb{R}^n .

THEOREM B ([27]) *For $n \geq 3$, let (f, Γ) satisfy (1.4), (1.5), (1.10), and (1.12), and let u be a positive locally Lipschitz weak solution of (1.14). Then $u \equiv u(0)$ in \mathbb{R}^n .*

Throughout this paper, by a weak solution of (1.14) we mean in the sense of definition 1.1 in [26, 27], with $F(M) := f(\lambda(M))$ and $U := \{M \mid \lambda(M) \in \Gamma\}$. Our proof of Theorem 1.4 is along the line of [26, 27], which makes use of ideas developed in [19, 23] in treating the isolated singularity of u at ∞ .

Remark 1.6. Let (f, Γ) satisfy (1.4), (1.5),

$$f \in C^0(\bar{\Gamma}) \text{ is symmetric in } \lambda_i, \quad f > 0 \text{ in } \Gamma, \quad f|_{\partial\Gamma} = 0, \\ f(\lambda + \mu) \geq f(\lambda) \quad \forall \lambda \in \Gamma, \mu \in \Gamma_n,$$

and let Ω be an open subset of \mathbb{R}^n . If u is a $C_{\text{loc}}^{1,1}$ solution of

$$(1.15) \quad f(\lambda(A^u)) = 0 \quad \text{in } \Omega,$$

then it is a weak solution of (1.15). The proof is standard in view of lemma 3.7 in [26, 27]. If u is a weak solution of (1.15), then it is clearly a viscosity solution of (1.6).

The motivation of our study of such Liouville properties of entire solutions of $\lambda(A^u) \in \partial\Gamma$ is to answer the following questions concerning local gradient estimates of solutions to general second-order, conformally invariant, fully nonlinear elliptic equations.

Let $B_3 \subset \mathbb{R}^n$ be a ball of radius 3 and centered at the origin.

Question 1. Let $n \geq 3$, (f, Γ) satisfy (1.4), (1.5), and (1.10)–(1.13). For constants $0 < b < \infty$ and $0 < h \leq 1$, let $u \in C^3(B_3)$ satisfy

$$(1.16) \quad f(\lambda(A^u)) = h, \quad 0 < u \leq b, \quad \lambda(A^u) \in \Gamma \quad \text{in } B_3.$$

Is it true that

$$|\nabla \log u| \leq C \quad \text{in } B_1$$

for some constant C depending only on b and (f, Γ) ?

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$. We use i_0 and R_{ijkl} to denote, respectively, the injectivity radius and the curvature tensor. Consider the Schouten tensor

$$A_g = \frac{1}{n-2} \left(\text{Ric}_g - \frac{R_g}{2(n-1)} g \right),$$

where Ric_g and R_g denote, respectively, the Ricci tensor and the scalar curvature. We use $\lambda(A_g) = (\lambda_1(A_g), \dots, \lambda_n(A_g))$ to denote the eigenvalues of A_g with respect to g .

Let $\hat{g} = u^{4/(n-2)}g$ be a conformal change of metrics; then (see, for example, [35]),

$$\begin{aligned} A_{\hat{g}} &= -\frac{2}{n-2} u^{-1} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-2} \nabla u \otimes \nabla u \\ &\quad - \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g + A_g, \end{aligned}$$

where covariant derivatives on the right-hand side are with respect to g .

For $g_1 = u^{4/(n-2)}g_{\text{flat}}$, with g_{flat} denoting the Euclidean metric on \mathbb{R}^n ,

$$A_{g_1} = u^{\frac{4}{n-2}} A_{ij}^u dx^i dx^j$$

where A^u is defined in (1.2). In this case, $\lambda(A_{g_1}) = \lambda(A^u)$.

A more general question on Riemannian manifolds is the following:

Question 2. Let g be a smooth Riemannian metric on $B_3 \subset \mathbb{R}^n$, $n \geq 3$, and let (f, Γ) satisfy (1.4), (1.5), and (1.10)–(1.13). For a positive number b and a positive function $h \in C^1(B_3)$, let $u \in C^3(B_3)$ satisfy, with $\tilde{g} := u^{4/(n-2)}g$,

$$(1.17) \quad f(\lambda(A_{\tilde{g}})) = h, \quad 0 < u \leq b, \quad \lambda(A_{\tilde{g}}) \in \Gamma, \quad \text{in } B_3.$$

Is it true that

$$(1.18) \quad \|\nabla \log u\|_g \leq C \quad \text{in } B_1$$

for some constant C depending only on b , g , $\|h\|_{C^1(B_3)}$, and (f, Γ) ?

For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, the local gradient estimate (1.18) on Riemannian manifolds was established by Guan and Wang in [12]; see a related work by Chang, Gursky, and Yang [3] where global a priori C^0 and C^1 estimates for $f = \sigma_2^{1/2}$ and $n = 4$ were derived. Efforts at achieving further generality were made in [11, 13, 17, 30]. On locally conformally flat manifolds, “semilocal” gradient estimates were established and used in [17, 19] for (f, Γ) satisfying (1.4), (1.5), (1.10), and (1.12) via the method of moving spheres (or planes). A consequence of the semilocal gradient estimates is the following (see also lemma 0.5 and its proof in [18]):

THEOREM C *Under an additional assumption $u \geq a > 0$ in B_3 , the answer to Question 1 is yes, but with the constant C also depending on a .*

Remark 1.7. In Theorem C, assumptions (1.11) and (1.13) are not needed.

Equations (1.16) and (1.17) are fully nonlinear elliptic equations of u . Extensive studies have been given to fully nonlinear equations involving $f(\lambda(\nabla^2 u))$ by Caffarelli, Nirenberg, and Spruck [2], Guan and Spruck [10], Trudinger [31], Trudinger and Wang [33], and many others.

Fully nonlinear equations involving $f(\lambda(\nabla_g^2 u + g))$ on Riemannian manifolds are studied by Li [21], Urbas [34], and others. Fully nonlinear equations on Riemannian manifolds involving the Schouten tensor have been studied by Viaclovsky in [35, 36], by Chang, Gursky, and Yang in [3, 4], and by many others; see, for example, [5, 22, 32, 37] and the references therein. Here we study, on Riemannian manifolds (M, g) , local gradient estimates to solutions of

$$(1.19) \quad f(\lambda(A_{u^{4/(n-2)}})) = h, \quad \lambda(A_{u^{4/(n-2)}}) \in \Gamma.$$

If we make an additional concavity assumption

$$(1.20) \quad f \in C^2(\Gamma) \cap C^0(\bar{\Gamma}) \text{ is symmetric in } \lambda_i \text{ and is concave in } \Gamma,$$

then we have the following corollary of Theorem A and the proof of (1.39) in [17].

THEOREM 1.8 *Let (M, g) be as above and let (f, Γ) satisfy (1.4), (1.5), (1.11), (1.12), and (1.20). For a geodesic ball B_{3r} in M of radius $3r \leq \frac{1}{2}i_0$, let u be a C^4 positive solution of (1.19) in B_{3r} . Then*

$$(1.21) \quad \|\nabla(\log u)\|_g \leq C \quad \text{in } B_r,$$

where C is some positive constant depending only on (f, Γ) , upper bounds of $1/i_0$, $\sup_{B_{9r}} u$, $\|h\|_{C^2(B_{9r})}$, and a bound of R_{ijkl} together with their covariant derivatives up to second order.

It has been observed independently by Wang in [38] that Theorem 1.8 follows from Theorem A. The theorem is proved by Chen in [6] using a different method. It is well-known, see, e.g., [2], that $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies the hypotheses of the theorem.

Remark 1.9. It is easy to see from Section 3 that Theorem 1.8 holds under slightly weaker hypotheses on (f, Γ) : Assuming that it satisfies (1.4), (1.5), (1.12), and (1.20),

$$(1.22) \quad \lim_{s \rightarrow \infty} \inf_{\lambda \in K} f(s\lambda) = \infty \quad \text{for any compact subset } K \text{ of } \Gamma$$

and

$$(1.23) \quad \inf_{\lambda \in \Gamma, |\lambda| \geq \frac{1}{8}} \left(|\lambda| \sum_i f_{\lambda_i}(\lambda) \right) \geq \delta \quad \text{for some } \delta > 0.$$

The second main result in this paper is following:

THEOREM 1.10 *Let (M, g) be as above and let (f, Γ) satisfy (1.4), (1.5), and (1.10)–(1.13). For a geodesic ball B_{9r} in M of radius $9r \leq \frac{1}{2}i_0$, let u be a C^3 positive solution of (1.19) in B_{9r} . Then (1.21) holds, where C is some positive constant depending only on (f, Γ) , the upper bounds of $1/i_0$, $\sup_{B_{9r}} u$, $\|h\|_{C^1(B_{9r})}$, and a bound of R_{ijkl} together with their first covariant derivatives.*

Remark 1.11. If (f, Γ) satisfies (1.4), (1.5), and (1.10)–(1.12) and f is concave in Γ , then (1.13) is automatically satisfied; see [34]. Thus Theorem 1.10 implies Theorem 1.8. The main point of Theorem 1.10 is that no concavity assumption is made on f .

Remark 1.12. Replacing the function h in (1.19) by $h(\cdot, u)$ with

$$s^{\frac{4}{n-2}} h(x, s) \in C^1(B_{9r} \times (0, \infty)) \cap L^\infty(B_{9r} \times (0, b))$$

for all $b > 1$, estimate (1.21) still holds, with the constant C depending also on the function h . This is easy to see from the proof of the theorem.

Remark 1.13. Once (1.21) is established, it follows from the proof of (1.39) in [17], under the hypotheses of Theorem 1.8, that

$$\|\nabla_g^2(\log u)\|_g \leq C \quad \text{in } B_r,$$

where C is some positive constant depending only on an upper bound of $1/i_0$, $\sup_{B_{9r}} u$, $\sup_{B_{3r}} \|\nabla u\|_g$, $\|h\|_{C^2(B_{9r})}$, and a bound of R_{ijkl} together with their covariant derivatives up to second order.

A subtlety of the local gradient estimate (1.21) is that the bound depends on an upper bound of u , but not on upper bounds of u^{-1} . Global estimates of $|\nabla u|$ allowing the dependence of an upper bound of both u and u^{-1} were given by Viaclovsky in [36]; see a related work [21]. One application of the local gradient estimate is for a rescaled sequence of solutions in the following situation: For solutions $\{u_i\}$ of (1.19) in a unit ball B_1 satisfying, for some constant $b > 0$ independent of i ,

$$\sup_{B_1} u_i \leq b u_i(0) \rightarrow \infty,$$

consider

$$v_i(y) := \frac{1}{u_i(0)} v_i \left(\frac{y}{u_i(0)^{\frac{2}{n-2}}} \right).$$

One knows that

$$(1.24) \quad v_i(0) = 1 \text{ and } v_i(y) \leq b \quad \forall |y| \leq u_i(0)^{\frac{2}{n-2}},$$

and v_i satisfies the same equation with g replaced by the rescaled metric $g^{(i)}$. One would like to derive a bound of $|\nabla v_i|$ on $\{y \mid |y| < \beta\}$ for any fixed $\beta > 1$.

Some time ago the author arrived at the following idea: Try to establish the estimate of $|\nabla v_i|$ in two steps:

- (1) To establish, for solutions u of (1.19) for general (f, Γ) , local gradient estimates that depend on an upper bound of both u and u^{-1} .
- (2) To establish, for solutions u of (1.19) in B_1 satisfying $u(0) = 1$, an estimate on B_δ of u^{-1} from above, which depends on an upper bound of u .

Once these two steps were achieved, the needed gradient bound for solutions $\{v_i\}$ satisfying (1.24) would follow. The reason is that we know from step 2 that $v_i \geq a$ in B_δ for some $a, \delta > 0$ independent of i . Since $-L_{g^{(i)}} v_i \geq 0$ where $L_{g^{(i)}}$ denotes the conformal Laplacian of $g^{(i)}$, and since $g^{(i)}$ tends to the Euclidean metric in $C_{\text{loc}}^2(\mathbb{R}^n)$, we have, for any $\beta > 2$,

$$v_i \geq \xi_i \quad \text{on } B_\beta \setminus B_\delta,$$

where ξ_i is the solution of

$$L_{g^{(i)}} \xi_i = 0 \text{ in } B_\beta \setminus B_\delta, \quad \xi_i = a \text{ on } \partial B_\delta, \quad \xi_i = 0 \text{ on } \partial B_\beta.$$

Clearly,

$$\xi_i \rightarrow \frac{a\beta^{n-2}\delta^{n-2}}{\beta^{n-2} - \delta^{n-2}} \left(\frac{1}{|x|^{n-2}} - \frac{1}{\beta^{n-2}} \right) \quad \text{uniformly in } B_\beta \setminus B_\delta.$$

This provides an upper bound of v_i^{-1} on $B_{\beta/2}$, and the desired estimate follows from step 1.

Aobing Li and the author then started to implement this idea. Step 1 for locally conformally flat manifolds was known to us; see Theorem C. We established step 1 on general manifolds and for general (f, Γ) :

THEOREM D ([20]) *Let (M, g) be as above and let (f, Γ) satisfy (1.4), (1.5), and (1.10)–(1.13). For a geodesic ball B_{9r} in M of radius $9r \leq \frac{1}{2}i_0$, let u be a C^3 positive solution of (1.19) in B_{9r} satisfying, for some positive constants $0 < a < b < \infty$,*

$$a \leq u \leq b \quad \text{on } B_{9r}.$$

Then (1.21) holds, where C is some positive constant depending only on a, b, δ , the upper bounds of $1/i_0, \|h\|_{C^1(B_{9r})}$, and a bound of R_{ijkl} together with their first covariant derivatives.

This result was extended to manifolds with boundary under prescribed mean curvature boundary conditions in [15]; see theorem 1.3 there. The proof of Theorem D uses Bernstein-type arguments. The choice of the auxiliary function ϕ in the proof is similar in spirit to that in [21, 36]: finding a ϕ that satisfies on a *finite interval* some second-order ordinary differential inequalities (see (A.3)). If the differential inequalities (A.3) had a bounded solution ϕ on a half-line (α, ∞) , then Theorem 1.10, without the assumption $u \geq a > 0$, would have been proved by the same method. However, the differential inequalities do not have any bounded solution on any half-line.

The method the author had in mind for step 2 was to obtain, via Bernstein-type arguments, a bound on $|\nabla\Phi(u)| = |\Phi'(u)\nabla u|$ for an appropriate Φ . For instance, $|\nabla(u^\alpha)| \leq C$ for $\alpha < 0$ is weaker than $|\nabla \log u| \leq C$, and it becomes weaker when α is smaller. On the other hand, an estimate of $|\nabla(u^\alpha)|$ for any $\alpha < 0$ would yield an upper bound of u^{-1} near the origin. In principal, estimating $|\nabla(u^\alpha)|$ for very negative α should be easier than estimating $|\nabla \log u|$. However, we encountered some difficulties in completing this step.

The author then took another path that requires establishing appropriate Liouville theorems for general degenerate, conformally invariant equations (1.14). What is needed is to prove that any positive locally Lipschitz function u satisfying (1.14) in an appropriate weak sense must be a constant. In [26], a notion of weak solutions, tailored for the application to local gradient estimates, was introduced. Such a Liouville theorem for C_{loc}^1 weak solutions of (1.14) is established there. My first impression was that weakening the regularity assumption from C_{loc}^1 to $C_{\text{loc}}^{0,1}$ (locally Lipschitz) is perhaps a subtle borderline issue whose solution would require some new ideas beyond those used in [26]. It turns out, to our surprise, that this only requires some modification of our proof of the Liouville theorem for C_{loc}^1 weak solutions. The improvement, Theorem B, is given in [27]. Theorem B, together with Theorem C, is enough to answer Question 1 affirmatively; this can be seen in the proof of Theorem 1.10.

With the help of the Jensen approximations (see [1, 14]), we can further extend Theorem B for positive, locally Lipschitz viscosity solutions. The theory of viscosity solutions for nonlinear partial differential equations was developed by Crandall and Lions in [7]. Its basic idea also appears in earlier papers by Evans [8, 9].

Theorem 1.4 allows us to, by using Theorem D, first establish a local Hölder estimate of $\log u$ instead of the local gradient estimate of $\log u$. With the Hölder estimate of $\log u$, which yields the Harnack inequality of u , we then obtain the local gradient estimate of $\log u$ by another application of Theorem D.

The following problem looks reasonable and worthwhile to the author: using the Bernstein-type arguments to complete the above-mentioned step 2, without any concavity assumption on f , by choosing an appropriate Φ .

One important ingredient in our proof of Theorem 1.4 is a new proof of the classical Liouville theorem (1.1) that uses only the following two properties of harmonic functions.

Conformal invariance of harmonic functions: For any harmonic function u , and for any Möbius transformation φ , u_φ is harmonic.

Comparison principle for harmonic functions on balls: Let $B \subset \mathbb{R}^n$, $n \geq 2$, be a ball centered at the origin. Assume that $u \in C_{\text{loc}}^2(\overline{B} \setminus \{0\})$ and $v \in C^2(\overline{B})$ satisfy

$$\Delta u = 0, \quad u > 0, \quad \text{in } B \setminus \{0\}, \quad \Delta v = 0 \quad \text{in } B,$$

and

$$u \geq v \quad \text{on } \partial B.$$

Then

$$u \geq v \quad \text{in } \overline{B} \setminus \{0\}.$$

It is easy to see from this proof of the Liouville theorem (1.1) that the following comparison principle is sufficient for a proof of Theorem 1.4.

PROPOSITION 1.14 (Comparison Principle) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set containing m points $S_m := \{P_1, \dots, P_m\}$, $m \geq 0$, $v \in C^{0,1}(\overline{\Omega} \setminus S_m)$, and $w \in C^{0,1}(\overline{\Omega})$. Assume that w is a viscosity supersolution of $\lambda(A_w) \in \partial\Gamma$ in $\Omega \setminus S_m$, v is a viscosity subsolution of $\lambda(A_v) \in \partial\Gamma$ in $\Omega \setminus S_m$, and*

$$w > 0 \quad \text{in } \overline{\Omega}, \quad v > 0 \quad \text{in } \overline{\Omega} \setminus S_m, \quad w > v \quad \text{on } \partial\Omega.$$

Then

$$(1.25) \quad \inf_{\Omega \setminus S_m} (w - v) > 0.$$

Remark 1.15. The proposition was proved in [26, 27] under stronger hypotheses: Instead of $C^{0,1}$ viscosity super- or subsolutions, they were assumed to be $C^{0,1}$ weak super- or subsolutions that include $C^{1,1}$ super- or subsolutions.

Remark 1.16. Our equation $\lambda(A_w) \in \partial\Gamma$, or (1.14), does not satisfy the usual requirement of the dependence on w or u in the literature on viscosity solutions.

Remark 1.17. The proof of Proposition 1.14 for $m \geq 1$, which makes use of the method developed in [19] (proof of theorem 1.3), [23] (theorems 1.6–1.10) and [27] (theorem 1.6 and remark 1.8) in treating isolated singularities, is much more delicate than that for $m = 0$, $S_0 = \emptyset$. For $m = 0$, the conclusion of the above theorem still holds in dimension $n = 2$. On the other hand, the conclusion does not hold in dimension $n = 2$ for $\Gamma = \Gamma_1$ if $m \geq 1$. See the example below.

Example. Let $w(x) = (1 + \epsilon)e^{-(1/2)x_1}$, $v(x) = e^{-(1/2)x_1|x|^{-2}}$, $\epsilon > 0$. Clearly $w \in C^\infty(\overline{B}_1)$, $v \in C^\infty(\overline{B}_1 \setminus \{0\})$, $w > v$ on ∂B_1 , and they are positive functions. Since x_1 is harmonic in B_1 and $x_1|x|^{-2}$ is harmonic in $B_1 \setminus \{0\}$, we know that $w\Delta w - |\nabla w|^2 = 0$ in B_1 and $v\Delta v - |\nabla v|^2 = 0$ in $B_1 \setminus \{0\}$, i.e., $\lambda(A_w) \in \partial\Gamma_1$ in B_1 and $\lambda(A_v) \in \partial\Gamma_1$ in $B_1 \setminus \{0\}$. However, $\inf_{B_1 \setminus \{0\}} (w - v) < 0$ for small ϵ .

To prove Theorem 1.4, we only need Proposition 1.14 for $m = 1$ and $w \in C^{0,1}(\overline{\Omega})$ a viscosity supersolution of $\lambda(A_w) \in \partial\Gamma$ in Ω . In fact, we only need a *weak comparison principle* that assumes a priori $w \geq v$ in $\Omega \setminus \{0\}$; see [28].

THEOREM 1.18 *For $n \geq 3$, let Γ satisfy (1.4) and (1.5), and let u be a positive, locally Lipschitz viscosity solution of*

$$(1.26) \quad \lambda(A^u) \in \partial\Gamma \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Then

$$(1.27) \quad u_{x,\lambda}(y) \leq u(y) \quad \forall 0 < \lambda < |x|, |y - x| \geq \lambda, y \neq 0.$$

Consequently, u is radially symmetric about the origin and $u'(r) \leq 0$ for almost all $0 < r < \infty$.

The result was proved in [26, 27] under stronger hypotheses: assuming u is a $C_{\text{loc}}^{1,1}$ or a $C_{\text{loc}}^{0,1}$ weak solution of (1.26).

In the rest of this introduction we assume that (M, g) , $n \geq 3$, is a smooth compact Riemannian manifold with nonempty smooth boundary ∂M . Let h_g denote the mean curvature of ∂M with respect to the outer normal (a Euclidean ball has positive mean curvature). For a conformal metric $\hat{g} = u^{4/(n-2)}$, it is known that

$$h_{\hat{g}} = u^{-\frac{n-2}{2}} \left(-\frac{\partial u}{\partial \nu_g} + \frac{n-2}{2} h_g u \right),$$

where ν_g denotes the unit outer normal. We study

$$(1.28) \quad \begin{cases} f(\lambda(A_{u^{4/(n-2)}} g)) = \psi, \\ \lambda(A_{u^{4/(n-2)}} g) \in \Gamma & \text{on } O_1 \setminus \partial M, \\ -\frac{\partial u}{\partial \nu_g} + \frac{n-2}{2} h_g u = \eta(x) u^{n/(n-2)} & \text{on } O_1 \cap \partial M, \end{cases}$$

where O_1 is an open set of M , $\psi \in C^2(O_1)$, and $\eta \in C^2(O_1 \cap \partial M)$.

THEOREM 1.19 *Assume that (M, g) is a smooth, compact, n -dimensional ($n \geq 3$), Riemannian manifold with smooth boundary ∂M , and that (f, Γ) satisfies (1.4), (1.5), and (1.10)–(1.13). Let O_1 be an open set of M , and let $u \in C^3(O_1)$ be a solution of (1.28). If*

$$0 < u \leq b \quad \text{on } O_1$$

for some constant b , then, for any open set O_2 of M satisfying $\overline{O_2} \subset O_1$,

$$(1.29) \quad |\nabla(\log u)|_g \leq C \quad \text{on } O_2$$

for some positive constant C depending only on n , (f, Γ) , (M, g) , ψ , η , b , O_1 , and O_2 .

Remark 1.20. For (f, Γ) satisfying a more restrictive condition (H_1) defined in [17], which includes all $(\sigma_k^{1/k}, \Gamma_k)$, estimate (1.29) was established in [15].

Remark 1.21. Replacing the function ψ and η in (1.28) by $\psi(\cdot, u)$ and $\eta(\cdot, u)$, respectively, satisfying

$$s^{\frac{4}{n-2}} \psi(x, s) \in C^2(O_1 \times (0, \infty)) \cap L^\infty(O_1 \times (0, b))$$

and

$$\eta \in C^2((O_1 \cap \partial M) \times (0, \infty)) \cap L^\infty((O_1 \cap \partial M) \times (0, b))$$

for all $b > 1$, estimate (1.29) still holds. This is easy to see from the proof of the theorem.

Let

$$\mathbb{R}_+^n := \{x = (x', x_n) \in \mathbb{R}^n \mid x' = (x_1, \dots, x_{n-1}), x_n > 0\}$$

denote the half Euclidean space, and let $\Omega^+ \subset \mathbb{R}_+^n$ be an open set. We use the notation

$$\partial''\Omega^+ = \overline{\partial\Omega^+ \cap \mathbb{R}_+^n}, \quad \partial'\Omega^+ = \partial\Omega^+ \setminus \partial''\Omega^+.$$

The following definition is standard:

DEFINITION 1.22 A function $u \in C^0(\overline{\Omega^+})$ is said to satisfy

$$\frac{\partial u}{\partial x_n} \leq 0 \text{ (resp., } \geq 0) \text{ on } \partial'\Omega^+$$

in the viscosity sense, if $\bar{x} \in \partial'\Omega^+$, $\psi \in C^1(\overline{\Omega^+})$, and $u - \psi$ has a local minimum (respectively, local maximum) at \bar{x} , then

$$(1.30) \quad \frac{\partial \psi}{\partial x_n}(\bar{x}) \leq 0 \text{ (resp., } \geq 0).$$

Similarly, we define

$$\frac{\partial u}{\partial x_n} < 0 \quad \text{or} \quad \frac{\partial u}{\partial x_n} > 0 \quad \text{on } \partial'\Omega^+ \text{ in the viscosity sense}$$

by making the inequalities in (1.30) strict. We say that $\frac{\partial u}{\partial x_n} = 0$ on $\partial'\Omega^+$ in the viscosity sense if both $\frac{\partial u}{\partial x_n} \leq 0$ and $\frac{\partial u}{\partial x_n} \geq 0$ on $\partial'\Omega^+$ in the viscosity sense.

THEOREM 1.23 *Let Γ satisfy (1.4) and (1.5), and let $u \in C^{0,1}(\overline{\mathbb{R}_+^n})$ be a positive viscosity solution of*

$$(1.31) \quad \lambda(A^u) \in \partial\Gamma \quad \text{in } \mathbb{R}_+^n$$

satisfying, in the viscosity sense,

$$(1.32) \quad \frac{\partial u}{\partial x_n} = 0 \quad \text{on } \partial\mathbb{R}_+^n.$$

Then $u \equiv u(0)$ in $\overline{\mathbb{R}_+^n}$.

Theorem 1.4, Theorem 1.8, and Theorem 1.10 were announced in [25], and the proofs were given in [24]. The proof of Theorem 1.4 in this revised version of [24] is improved in presentation.

The paper is organized as follows: In Section 2 we prove Proposition 1.14, Theorem 1.4, and Theorem 1.18. In Section 3 we prove Theorem 1.8. In Section 4 we prove Theorem 1.10. In Section 5 we prove Theorem 1.23 and Theorem 1.19. In the Appendix we give, for the reader's convenience, the proof of Theorem D.

We end this introduction with a question related to Theorem 1.4. Let Γ satisfy (1.4) and (1.5), and let $E \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n)$ satisfy

$$E(x, \alpha s, \alpha p) = \alpha E(x, s, p) \quad \forall \alpha, s > 0, x, p \in \mathbb{R}^n,$$

and

$$E(x, 1, 0) \equiv 0 \quad \forall x \in \mathbb{R}^n.$$

Assume that w is a positive function in $C^\infty(\mathbb{R}^n)$ satisfying

$$(1.33) \quad \lambda(\nabla^2 w + E(x, w, \nabla w)) \in \partial\Gamma \quad \forall x \in \mathbb{R}^n.$$

Question 3. Under what additional hypothesis on E does the above imply that $w \equiv w(0)$ on \mathbb{R}^n ? What if w has less regularity, e.g., in $C_{\text{loc}}^{1,1}(\mathbb{R}^n)$, or a locally Lipschitz viscosity solution of (1.33)?

We know from Theorem 1.4 and Remark 1.5 that for

$$E(x, w, \nabla w) \equiv -\frac{|\nabla w|^2}{2w} I$$

the answer is yes in dimension $n \geq 3$ and no in dimension $n = 2$. What about

$$E(x, w, \nabla w) \equiv b \frac{|\nabla w|^2}{w} I$$

for other constants b ?

2 Proofs of Proposition 1.14, Theorem 1.4, and Theorem 1.18

We first give a new proof of the classical Liouville theorem (1.1). For every $x \in \mathbb{R}^n$ and for every $\lambda > 0$, let

$$u_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

We know that $u_{x,\lambda} = u$ on $\partial B_\lambda(x)$, $u \in C^0(B_\lambda(x))$, $u_{x,\lambda} \in C^0(B_\lambda(x) \setminus \{x\})$, and u and $u_{x,\lambda} > 0$ are positive harmonic functions in $B_\lambda(x)$ and $B_\lambda(x) \setminus \{x\}$, respectively. Note that we have used the conformal invariance of harmonic functions to obtain the harmonicity of $u_{x,\lambda}$. By the comparison principle for harmonic functions on balls, $u_{x,\lambda} \geq u$ in $B_\lambda(x) \setminus \{x\}$, which is equivalent to $u_{x,\lambda} \leq u$ in $\mathbb{R}^n \setminus B_\lambda(x)$. It follows that $u \equiv u(0)$; see, e.g., lemma 11.2 in [29] or lemma A.1 in [19].

Now we have the following:

PROOF OF THEOREM 1.4 USING PROPOSITION 1.14: For every $x \in \mathbb{R}^n$ and for every $\lambda > 0$, applying Proposition 1.14 to u and $u_{x,\lambda}$ on $B_\lambda(x)$ yields $u_{x,\lambda} \geq u$ in $B_\lambda(x) \setminus \{x\}$. This implies $u \equiv u(0)$. \square

PROOF OF THEOREM 1.18 USING PROPOSITION 1.14: For every $x \in \mathbb{R}^n \setminus \{0\}$ and for every $0 < \lambda < |x|$, applying Proposition 1.14 to u and $u_{x,\lambda}$ on $B_{x,\lambda}(x)$ yields $u_{x,\lambda} \geq u$ in $B_\lambda(x) \setminus \{x, |x|^{-2}(|x|^2 - \lambda^2)x\}$, i.e., (1.27) holds. It follows that u is radially symmetric about the origin and $u'(r) \leq 0$ for almost all $0 < r < \infty$; see, e.g., [23]. \square

In the rest of this section we give the following proof:

PROOF OF PROPOSITION 1.14: We prove by induction on the number of points m . We start from $m = 0$ with $S_0 = \emptyset$.

Step 1. Proposition 1.14 holds for $m = 0$.

Because of Remark 1.3, we only need to show that $w \geq v$ in Ω . We prove it by contradiction. Suppose the contrary: for some $\gamma > 0$,

$$\max_{\Omega} (v - w) \geq \gamma, \quad v - w \leq -\gamma \text{ on } \Omega \setminus \Omega_\gamma,$$

where $\Omega_\gamma := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \gamma\}$.

For small positive constants $0 < \epsilon \ll \mu \ll \delta \ll 1$ that we specify later, let

$$\begin{aligned} \hat{v}(y) &:= v(y) + \mu\varphi(y), \quad \varphi(y) := e^{\delta|y|^2}, \\ (2.1) \quad \hat{v}^\epsilon(y) &:= \sup_{x \in \Omega} \left\{ \hat{v}(x) + \epsilon - \frac{1}{\epsilon}|x - y|^2 \right\}, \\ w_\epsilon(y) &:= \inf_{x \in \Omega} \left\{ w(x) - \epsilon + \frac{1}{\epsilon}|x - y|^2 \right\}. \end{aligned}$$

\hat{v} has been used in [26, 27]. \hat{v}^ϵ and w_ϵ are Jensen approximations whose useful properties can be found in theorem 1.5 and lemma 5.2 in [1]. We list below some properties that we need.

$$(2.2) \quad \hat{v}^\epsilon \text{ and } w_\epsilon \text{ are punctually second-order differentiable a.e. in } \Omega_\gamma,$$

$$(2.3) \quad \nabla^2 \hat{v}^\epsilon \geq -\frac{C}{\epsilon} I, \quad \nabla^2 w_\epsilon \leq \frac{C}{\epsilon} I, \quad \text{in } \Omega_\gamma.$$

For any $x \in \Omega_\gamma$, there exist $x^* = x^*(x)$ and $x_* = x_*(x)$ in Ω such that

$$(2.4) \quad \hat{v}^\epsilon(x) = \hat{v}(x^*) + \epsilon - \frac{1}{\epsilon}|x^* - x|^2,$$

$$(2.5) \quad w_\epsilon(x) = w(x_*) - \epsilon + \frac{1}{\epsilon}|x_* - x|^2,$$

$$(2.6) \quad |x^* - x| + |x_* - x| \leq C\epsilon,$$

$$(2.7) \quad |\nabla \hat{v}^\epsilon| + |\nabla w_\epsilon| \leq C \quad \text{in } \Omega_\gamma,$$

where C denotes various positive constants independent of μ , δ , and ϵ . We shall use this notation in what follows as well. The punctual second-order differentiability is defined as in definition 1.4 in [1]. Properties (2.2)–(2.5) can be found in [1] that hold for continuous w and v . Property (2.6) follows from the proof of (5) in lemma 5.2 in [1] by using the Lipschitz regularity of w and v . Property (2.7) can easily be deduced as follows from (2.4)–(2.6) by again using the Lipschitz regularity of w and v : For any y and z in Ω_γ , we have, by (2.4) and the definition of \hat{v}^ϵ ,

$$\begin{aligned}\hat{v}^\epsilon(y) &\geq \hat{v}(y - z + z^*) + \epsilon - \frac{1}{\epsilon}|z^* - z|^2 = \hat{v}(y - z + z^*) - \hat{v}(z^*) + \hat{v}^\epsilon(z) \\ &\geq \hat{v}^\epsilon(z) - C|y - z|.\end{aligned}$$

This gives the bound of $|\nabla \hat{v}^\epsilon|$ in (2.7). The bound of $|\nabla w_\epsilon|$ can be obtained similarly.

Using the Lipschitz regularity of v and w , it is easy to deduce from (2.4)–(2.6) that

$$|\hat{v}^\epsilon - \hat{v}| + |w_\epsilon - w| \leq C\epsilon \quad \text{in } \Omega_\gamma.$$

Thus, for small ϵ and μ , there exists $1 < b_\epsilon < C$ such that

$$\begin{aligned}\hat{v}^\epsilon - b_\epsilon w_\epsilon &\leq -\gamma \quad \text{in } \Omega \setminus \Omega_\gamma, \\ \max_{\Omega} \{\hat{v}^\epsilon - b_\epsilon w_\epsilon\} &= \epsilon.\end{aligned}$$

Let $\xi_\epsilon := \hat{v}^\epsilon - b_\epsilon w_\epsilon$, and let $\Gamma_{\xi_\epsilon^+}$ denote the concave envelope of $\xi_\epsilon^+ := \max\{\xi_\epsilon, 0\}$ on Ω . By (2.3),

$$\nabla^2 \xi_\epsilon \geq -\frac{C}{\epsilon} I \quad \text{in } \Omega_\gamma.$$

Thus, by lemma 3.5 of [1],

$$\int_{\{\xi_\epsilon = \Gamma_{\xi_\epsilon^+}\}} \det(-\nabla^2 \Gamma_{\xi_\epsilon^+}) > 0.$$

It follows that the Lebesgue measure of $\{\xi_\epsilon = \Gamma_{\xi_\epsilon^+}\}$ is positive. By (2.2), there exists $x_\epsilon \in \{\xi_\epsilon = \Gamma_{\xi_\epsilon^+}\}$ such that both \hat{v}^ϵ and w_ϵ are punctually second-order differentiable at x_ϵ . Clearly, for small ϵ , $x_\epsilon \in \Omega_\gamma$,

$$(2.8) \quad 0 < \xi_\epsilon(x_\epsilon) < \epsilon,$$

$$(2.9) \quad |\nabla \xi_\epsilon(x_\epsilon)| \leq C\epsilon,$$

$$(2.10) \quad \nabla^2 \xi_\epsilon(x_\epsilon) = \nabla^2 \hat{v}^\epsilon(x_\epsilon) - b_\epsilon \nabla^2 w_\epsilon(x_\epsilon) \leq 0,$$

$$(2.11) \quad w_\epsilon(x_\epsilon + z) \geq w_\epsilon(x_\epsilon) + \nabla w_\epsilon(x_\epsilon) \cdot z + \frac{1}{2} z^\top \nabla^2 w_\epsilon(x_\epsilon) z - o(|z|^2),$$

$$(2.12) \quad \hat{v}^\epsilon(x_\epsilon + z) \leq \hat{v}^\epsilon(x_\epsilon) + \nabla \hat{v}^\epsilon(x_\epsilon) \cdot z + \frac{1}{2} z^\top \nabla^2 \hat{v}^\epsilon(x_\epsilon) z + o(|z|^2).$$

By the definition of \hat{v}^ϵ , we have, with $(x_\epsilon)^* = (x_\epsilon)^*(x)$ as in (2.4),

$$\hat{v}^\epsilon(x_\epsilon + z) \geq \hat{v}((x_\epsilon)^* + z) + \epsilon - \frac{1}{\epsilon} |(x_\epsilon)^* - x_\epsilon|^2,$$

and therefore, in view of (2.12) and (2.1),

$$\begin{aligned} v((x_\epsilon)^* + z) &\leq \hat{v}^\epsilon(x_\epsilon + z) - \epsilon + \frac{1}{\epsilon} |(x_\epsilon)^* - x_\epsilon|^2 - \mu \varphi((x_\epsilon)^* + z) \\ &\leq Q_\epsilon(z) + o(|z|^2), \end{aligned}$$

where $Q_\epsilon(z)$ is the quadratic polynomial with

$$\begin{aligned} Q_\epsilon(0) &= \hat{v}^\epsilon(x_\epsilon) - \epsilon + \frac{1}{\epsilon} |(x_\epsilon)^* - x_\epsilon|^2 - \mu \varphi((x_\epsilon)^*) \\ &= \hat{v}^\epsilon(x_\epsilon) - \mu \varphi((x_\epsilon)^*) + O(\epsilon), \\ \nabla Q_\epsilon(0) &= \nabla \hat{v}^\epsilon(x_\epsilon) - \mu \nabla \varphi((x_\epsilon)^*), \\ \nabla^2 Q_\epsilon(0) &= \nabla^2 \hat{v}^\epsilon(x_\epsilon) - \mu \nabla^2 \varphi((x_\epsilon)^*), \end{aligned}$$

where $|O(\epsilon)| \leq C\epsilon$.

By (2.4) and (2.1), $Q_\epsilon(0) = v((x_\epsilon)^*)$. Since v is a viscosity subsolution of (1.7), we have

$$(2.13) \quad \lambda(A_{Q_\epsilon}(0)) \in \bar{\Gamma}.$$

For small $0 < \epsilon \ll \mu \ll \delta$, we have, as in the proof of lemma 3.7 in [26, 27], that

$$(2.14) \quad \begin{aligned} A_{Q_\epsilon}(0) &\leq \left(1 - \mu \frac{[\varphi((x_\epsilon)^*) + O(\frac{\epsilon}{\mu})]}{\hat{v}^\epsilon(x_\epsilon)} \right) A_{\hat{v}^\epsilon(x_\epsilon)} \\ &\quad - \frac{\mu \delta}{2} \left[\varphi((x_\epsilon)^*) + O\left(\frac{\epsilon}{\mu \delta} + \frac{\mu}{\delta}\right) \right] \hat{v}^\epsilon(x_\epsilon) I. \end{aligned}$$

Similarly, using (2.11) and the definition of w_ϵ , we have

$$w((x_\epsilon)_* + z) \geq w_\epsilon(x_\epsilon + z) + \epsilon - \frac{1}{\epsilon} |(x_\epsilon)_* - x_\epsilon|^2 \geq P_\epsilon(z) - o(|z|^2),$$

where $P_\epsilon(z)$ is the quadratic polynomial with

$$\begin{aligned} P_\epsilon(0) &= w_\epsilon(x_\epsilon) + \epsilon - \frac{1}{\epsilon} |(x_\epsilon)_* - x_\epsilon|^2 = w_\epsilon(x_\epsilon) + O(\epsilon), \\ \nabla P_\epsilon(0) &= \nabla w_\epsilon(x_\epsilon), \quad \nabla^2 P_\epsilon(0) = \nabla^2 w_\epsilon(x_\epsilon). \end{aligned}$$

By (2.5), $P_\epsilon(0) = w((x_\epsilon)_*)$. Since w is a viscosity supersolution of (1.7), we have

$$(2.15) \quad \lambda(A_{P_\epsilon}(0)) \in \mathbb{R}^n \setminus \Gamma.$$

By (2.7),

$$(2.16) \quad \begin{aligned} A_{P_\epsilon}(0) &= [w_\epsilon(x_\epsilon) + O(\epsilon)]\nabla^2 w_\epsilon(x_\epsilon) - \frac{1}{2}|\nabla w_\epsilon(x_\epsilon)|^2 I \\ &= (1 + O(\epsilon))A_{w_\epsilon}(x_\epsilon) + O(\epsilon). \end{aligned}$$

By (2.10),

$$\begin{aligned} A_{w_\epsilon}(x_\epsilon) &\geq \frac{w_\epsilon(x_\epsilon)}{b_\epsilon \hat{v}^\epsilon(x_\epsilon)} A_{\hat{v}^\epsilon}(x_\epsilon) \\ &\quad + \frac{1}{2b_\epsilon^2} \left(\frac{b_\epsilon w_\epsilon(x_\epsilon)}{\hat{v}^\epsilon(x_\epsilon)} |\nabla \hat{v}^\epsilon(x_\epsilon)|^2 - |\nabla(b_\epsilon w_\epsilon)(x_\epsilon)|^2 \right) I. \end{aligned}$$

By (2.8) and (2.9),

$$\left| \frac{b_\epsilon w_\epsilon(x_\epsilon)}{\hat{v}^\epsilon(x_\epsilon)} - 1 \right| \leq C\epsilon, \quad |\nabla \hat{v}^\epsilon(x_\epsilon) - \nabla(b_\epsilon w_\epsilon)(x_\epsilon)| \leq C\epsilon.$$

It follows, in view of (2.7),

$$\left| \frac{b_\epsilon w_\epsilon(x_\epsilon)}{\hat{v}^\epsilon(x_\epsilon)} |\nabla \hat{v}^\epsilon(x_\epsilon)|^2 - |\nabla(b_\epsilon w_\epsilon)(x_\epsilon)|^2 \right| \leq C\epsilon$$

and

$$(2.17) \quad A_{w_\epsilon}(x_\epsilon) \geq \frac{w_\epsilon(x_\epsilon)}{b_\epsilon \hat{v}^\epsilon(x_\epsilon)} A_{\hat{v}^\epsilon}(x_\epsilon) - C\epsilon I.$$

By (2.16), (2.17), and (2.14), we have, after fixing some small $0 < \mu \ll \delta$,

$$A_{P_\epsilon}(0) \geq a(\mu, \delta)A_Q(0) + b(\mu, \delta)I - C(\mu, \delta)\epsilon I,$$

where $a(\mu, \delta)$, $b(\mu, \delta)$, and $C(\mu, \delta)$ are some positive constants independent of ϵ . Now fix $\epsilon > 0$ such that $b(\mu, \delta) - C(\mu, \delta)\epsilon > 0$; we deduce from (2.13), using the properties of Γ , that $\lambda(A_{P_\epsilon}(0)) \in \Gamma$. This violates (2.15). Step 1 is established.

Step 2. Proposition 1.14 holds for m if it holds for $m - 1$.

Now we assume that the proposition holds for $m - 1$ points, $m - 1 \geq 0$, and we will prove that it holds for m points. We prove (1.25) by contradiction. Suppose it does not hold, then

$$\inf_{\Omega \setminus S_m} (w - v) \leq 0.$$

By shrinking Ω slightly and working with the smaller one, we may assume without loss of generality that w is $C^{0,1}$ in some open neighborhood of $\bar{\Omega}$. Let

$$u := v^{-\frac{n-2}{2}} \quad \text{and} \quad \xi := w^{-\frac{n-2}{2}}.$$

Then

$$\inf_{\Omega \setminus S_m} (u - \xi) \leq 0, \quad u > \xi \text{ on } \partial\Omega,$$

u is a viscosity supersolution of

$$(2.18) \quad \lambda(A^u) \in \partial\Gamma \quad \text{in } \Omega \setminus S_m,$$

and ξ is a viscosity subsolution of

$$\lambda(A^\xi) \in \partial\Gamma \quad \text{in } \Omega.$$

For a positive C^2 function ψ , $A^\psi(x_0) \in \bar{\Gamma}$ implies $\Delta\psi(x_0) \leq 0$. So by the definition of u being a viscosity supersolution of (2.18),

$$\Delta u \leq 0 \quad \text{in } \Omega \setminus S_m \text{ in the viscosity sense.}$$

It follows, from using also the positivity of u , that

$$\inf_{\Omega \setminus S_m} u \geq \inf_{\partial\Omega} u > 0.$$

Thus, for some $0 < a \leq 1$,

$$\inf_{\Omega \setminus S_m} (u - a\xi) = 0.$$

Since we can use $a^{-1}u$ instead of u , we may assume without loss of generality that $a = 1$. So we have, in addition,

$$\inf_{\Omega \setminus S_m} (u - \xi) = 0.$$

Let P_m be the origin, and let

$$\hat{\Omega} := \Omega \setminus S_{m-1}, \quad S_{m-1} := \{P_1, \dots, P_{m-1}\}.$$

LEMMA 2.1 *There exists $\epsilon > 0$ such that $u = \xi = \xi(0)$ in $B_\epsilon \setminus \{0\}$.*

PROOF OF LEMMA 2.1: We first claim that

$$(2.19) \quad \liminf_{|y| \rightarrow 0} (u - \xi)(y) = 0.$$

If (2.19) did not hold, there would be some $\epsilon > 0$ such that $\inf_{B_\epsilon \setminus \{0\}} (u - \xi) > 0$, i.e.,

$$(2.20) \quad \inf_{B_\epsilon \setminus \{0\}} (w - v) > 0.$$

Since the singular set of v in $\Omega \setminus \bar{B}_\epsilon$ is S_{m-1} , which contains only $m - 1$ points, we have, by the induction hypothesis,

$$\inf_{(\Omega \setminus B_\epsilon) \setminus S_{m-1}} (w - v) > 0.$$

This and (2.20) violate (2.19).

Now let

$$\Phi(\xi, x, \lambda; y) := \lambda\xi(x + y).$$

Since $\Phi(\xi, 0, 1; \cdot) = \xi$ and $u > \xi$ on $\partial\Omega$, we can fix some $\epsilon_4 > 0$ so that $|x| \leq \epsilon_4$ and $|\lambda - 1| \leq \epsilon_4$ guarantee

$$(2.21) \quad u > \Phi(\xi, x, \lambda; \cdot) \quad \text{on } \partial\Omega.$$

For such x and λ , if we assume both

$$(2.22) \quad \inf_{\widehat{\Omega} \setminus \{0\}} [u - \Phi(\xi, x, \lambda; \cdot)] = 0$$

and

$$\liminf_{|y| \rightarrow 0} [u(y) - \Phi(\xi, x, \lambda; y)] > 0,$$

we would have, for some $\epsilon, \epsilon' > 0$,

$$(2.23) \quad u(y) - \Phi(\xi, x, \lambda; y) > \epsilon' \quad \forall 0 < |y| \leq \epsilon.$$

Let

$$\tilde{u}(y) := \frac{1}{\lambda} u(y), \quad \tilde{\xi}(y) := \xi(x + y).$$

We know from (2.21) and (2.23) that $\tilde{u} > \tilde{\xi}$ on $\partial(\Omega \setminus \overline{B_\epsilon})$, i.e.,

$$v < \lambda^{-\frac{2}{n-2}} w(x + \cdot) \quad \text{on } \partial(\Omega \setminus \overline{B_\epsilon}).$$

Since $\lambda^{-2/(n-2)} w(x + \cdot)$ is still a viscosity supersolution of (1.7), while the singular set of v in $\Omega \setminus \overline{B_\epsilon}$ is S_{m-1} , which contains only $m - 1$ points, we have, by the induction hypothesis,

$$\inf_{(\Omega \setminus \overline{B_\epsilon}) \setminus S_{m-1}} [\lambda^{-\frac{2}{n-2}} w(x + \cdot) - v] > 0,$$

i.e.,

$$\inf_{(\Omega \setminus \overline{B_\epsilon}) \setminus S_{m-1}} [u - \Phi(\xi, x, \lambda; \cdot)] > 0.$$

This and (2.23) violate (2.22), which is impossible. We have proved that (2.22) implies

$$\liminf_{|y| \rightarrow 0} [u(y) - \Phi(\xi, x, \lambda; y)] = 0.$$

Therefore we can apply theorem 1.6 in [26, 27] to obtain, in view of (2.19), $u = \xi = \xi(0)$ near the origin. Lemma 2.1 is established. \square

Because of Lemma 2.1,

$$(2.24) \quad v = w = w(0) \quad \text{in } B_\epsilon$$

and therefore v is a viscosity solution of $\lambda(A_v) \in \Gamma$ in B_ϵ . Thus $v \in C_{\text{loc}}^{0,1}(\overline{\Omega} \setminus S_{m-1})$ is a viscosity subsolution of $\lambda(A_v) \in \partial\Gamma$ in $\Omega \setminus S_{m-1}$. By the induction hypothesis, we have

$$\inf_{\Omega \setminus S_{m-1}} (w - v) > 0.$$

This violates (2.24), which is impossible. Step 2 is established. We have therefore proved Proposition 1.14. \square

3 Proof of Theorem 1.8

PROOF OF THEOREM 1.8: Suppose the contrary of (1.21); then in B_2 , the ball in \mathbb{R}^n of radius 2 and centered at the origin, there exists a sequence of C^4 functions $\{u_i\}$, C^2 functions $\{h_i\}$, and $n \times n$ symmetric positive definite C^4 matrix functions $(a_{lm}^{(i)}(x))$ satisfying, for some $\bar{a} > 0$,

$$(3.1) \quad \frac{1}{\bar{a}}|\xi|^2 \leq a_{lm}^{(i)}(x)\xi^l\xi^m \leq \bar{a}|\xi|^2 \quad \forall x \in B_2, \xi \in \mathbb{R}^n,$$

$$(3.2) \quad \|a_{lm}^{(i)}\|_{C^4(B_2)}, \|h_i\|_{C^2(B_2)} \leq \bar{a}, \quad 0 < u_i \leq \bar{a} \text{ on } B_2,$$

and, for the Riemannian metric,

$$(3.3) \quad g_i := a_{lm}^{(i)}(x)dx^l dx^m,$$

$$(3.4) \quad \begin{aligned} f(\lambda(A_{u_i^{4/(n-2)}g_i})) &= h_i, \quad \lambda(A_{u_i^{4/(n-2)}g_i}) \in \Gamma \text{ in } B_2, \\ \sup_{B_{1/2}} |\nabla \log u_i| &\rightarrow \infty. \end{aligned}$$

It follows, for some $x_i \in B_1$, that

$$(1 - |x_i|)|\nabla \log u_i(x_i)| = \max_{|x| \leq 1} (1 - |x|)|\nabla \log u_i(x)| \rightarrow \infty,$$

where

$$|x| := \sqrt{\sum_{l=1}^n (x_l)^2}.$$

Let $\sigma_i := (1 - |x_i|)/2$ and $\epsilon_i := (2|\nabla \log u_i(x_i)|)^{-1}$. Then

$$(3.5) \quad \frac{\sigma_i}{\epsilon_i} \rightarrow \infty, \quad 2|\nabla \log u_i(x_i)| \geq |\nabla \log u_i(x)| \quad \forall |x - x_i| < \sigma_i.$$

Consider

$$(3.6) \quad v_i(y) := \frac{1}{u_i(x_i)} u_i(x_i + \epsilon_i y), \quad |y| < \frac{\sigma_i}{\epsilon_i}.$$

Then $v_i(0) = 1$ and, by (3.5) and the definition of ϵ_i ,

$$(3.7) \quad |\nabla \log v_i(y)| \leq 2|\nabla \log v_i(0)| = 1 \quad \forall |y| < \frac{\sigma_i}{\epsilon_i}.$$

Thus for any $\beta > 1$ there exists some positive constant $C(\beta)$, independent of i , such that

$$(3.8) \quad \frac{1}{C(\beta)} \leq v_i(y) \leq C(\beta) \quad \forall |y| < \beta.$$

For $g^{(i)} = a_{lm}^{(i)}(x_i + \epsilon_i y)dy^l dy^m$, $\gamma_i := u_i(x_i)^{-\frac{4}{n-2}}\epsilon_i^{-2} \rightarrow \infty$, and $x = x_i + \epsilon_i y$,

$$(3.9) \quad f(\gamma_i \lambda(A_{v_i(y)^{4/(n-2)}g^{(i)}})) = f(\lambda(A_{u_i(x)^{4/(n-2)}g_i})) = h_i, \quad |y| < \frac{\sigma_i}{\epsilon_i}.$$

By the proof of (1.39) in [17], applied to $f(\gamma_i \cdot)$, we have, for a possibly larger $C(\beta)$,

$$(3.10) \quad |\nabla^2 v_i(y)| \leq C(\beta) \quad \forall |y| \leq \beta.$$

Passing to a subsequence, $v_i \rightarrow v$ in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$ for all $0 < \alpha < 1$, where v is a positive function in $C_{\text{loc}}^{1,1}(\mathbb{R}^n)$ satisfying $|\nabla v(0)| = \frac{1}{2}$. In particular, v cannot be a constant.

By (3.8), (3.7), and (3.10), $|\lambda(A_{v_i(y)^{4/(n-2)}g^{(i)}})| \leq C(\beta) \forall |y| \leq \beta$. This and (3.9) imply, in view of (1.12) and (1.22), that $\lim_{i \rightarrow \infty} f(\lambda(A_{v_i(y)^{4/(n-2)}g^{(i)}})) = 0$. Therefore v is a $C_{\text{loc}}^{1,1}$ solution of $f(\lambda(A^v)) = 0$ in \mathbb{R}^n . By theorem 1.3 in [26, 27], v is identically a constant, which is a contradiction. Theorem 1.8 is established. \square

It is easy to see from the proof that in Remark 1.9 assumption (1.23) can be replaced by the following weaker one:

$$\inf_{\lambda \in \Gamma, |\lambda| \geq \frac{1}{8}} \left(|\lambda|^{2-\delta} \sum_i f_{\lambda_i}(\lambda) \right) \geq \delta \quad \text{for some } \delta > 0.$$

4 Proof of Theorem 1.10

In this section we prove Theorem 1.10. We first introduce some notation. Let v be a locally Lipschitz function in some open subset Ω of \mathbb{R}^n . For $0 < \alpha < 1$, $x \in \Omega$, and $0 < \delta < \text{dist}(x, \partial\Omega)$, let

$$[v]_{\alpha,\delta}(x) := \sup_{0 < |y-x| < \delta} \frac{|v(y) - v(x)|}{|y-x|^\alpha},$$

$$\delta(v, x; \Omega, \alpha) := \begin{cases} \infty & \text{if } [v]_{\alpha, \text{dist}(x, \partial\Omega)}(x) < 1, \\ \mu & \text{where } 0 < \mu \leq \text{dist}(x, \partial\Omega) \text{ and } \mu^\alpha [v]_{\alpha, \mu}(x) = 1 \\ & \text{if } [v]_{\alpha, \text{dist}(x, \partial\Omega)} \geq 1. \end{cases}$$

PROOF OF THEOREM 1.10: Before establishing the gradient estimate of $\log u$, we first prove the following Hölder estimates:

$$(4.1) \quad \sup_{|y|, |x| < r, |y-x| < 2r} \frac{|\log u(y) - \log u(x)|}{|y-x|^\alpha} \leq C(\alpha) \quad \forall 0 < \alpha < 1.$$

Suppose the contrary of (4.1), then for some $0 < \alpha < 1$, there exist, in B_2 , C^3 functions $\{u_i\}$, C^1 functions $\{h_i\}$, and $n \times n$ symmetric positive definite C^3 matrix functions $(a_{lm}^{(i)}(x))$ satisfying, for some $\bar{a} > 0$, (3.1), (3.2), and

$$\|a_{lm}^{(i)}\|_{C^3(B_2)}, \quad \|h_i\|_{C^1(B_2)} \leq \bar{a},$$

and (3.4) holds with g_i given by (3.3), but

$$\inf_{x \in B_{1/2}} \delta(\log u_i, x) \rightarrow 0,$$

where

$$\delta(\log u_i, x) := \delta(\log u_i, x; B_2, \alpha).$$

It follows, for some $x_i \in B_1$,

$$\frac{1 - |x_i|}{\delta(\log u_i, x_i)} = \max_{|x| \leq 1} \frac{1 - |x|}{\delta(\log u_i, x)} \rightarrow \infty.$$

Let

$$(4.2) \quad \sigma_i := \frac{1 - |x_i|}{2}, \quad \epsilon_i := \delta(\log u_i, x_i).$$

Then

$$(4.3) \quad \frac{\sigma_i}{\epsilon_i} \rightarrow \infty, \quad \epsilon_i \rightarrow 0,$$

and

$$(4.4) \quad \epsilon_i \leq 2\delta(\log u_i, z) \quad \forall |z - x_i| < \sigma_i.$$

Let v_i be defined as in (3.6) with the new ϵ_i as just defined. By the definition of $\delta(\log u_i, x_i)$,

$$\begin{aligned} [\log v_i]_{\alpha,1}(0) &= \epsilon_i^\alpha [\log u_i]_{\alpha,\epsilon_i}(x_i) \\ &= \delta(\log u_i, x_i)^\alpha [\log u_i]_{\alpha,\delta(\log u_i, x_i)}(x_i) = 1. \end{aligned}$$

For any $\beta > 1$ and $|x| < \beta$, we have, in view of (4.3), (4.4), and the triangle inequality, that for large i ,

$$\begin{aligned} &|\log u_i(z) - \log u_i(x_i + \epsilon_i x)| \\ &\leq \left| \log u_i(z) - \log u_i\left(\frac{1}{2}(z + x_i + \epsilon_i x)\right) \right| \\ &\quad + \left| \log u_i\left(\frac{1}{2}(z + x_i + \epsilon_i x)\right) - \log u_i(x_i + \epsilon_i x) \right|, \\ |z - (x_i + \epsilon_i x)| &= 2 \left| z - \frac{1}{2}(z + x_i + \epsilon_i x) \right| \\ &= 2 \left| \frac{1}{2}(z + x_i + \epsilon_i x) - (x_i + \epsilon_i x) \right|, \end{aligned}$$

$$\begin{aligned} [\log v_i]_{\alpha,1}(x) &= \epsilon_i^\alpha [\log u_i]_{\alpha,\epsilon_i}(x_i + \epsilon_i x) \\ &\leq 2^{-\alpha} \epsilon_i^\alpha \left(\sup_{|z - (x_i + \epsilon_i x)| < \epsilon_i} [\log u_i]_{\alpha,\frac{\epsilon_i}{2}}(z) + [\log u_i]_{\alpha,\frac{\epsilon_i}{2}}(x_i + \epsilon_i x) \right) \\ &\leq C(\beta) \left(\sup_{|z - (x_i + \epsilon_i x)| < \epsilon_i} \delta(\log u_i, z)^\alpha [\log u_i]_{\alpha,\delta(\log u_i, z)}(z) \right. \\ &\quad \left. + \delta(\log u_i, x_i + \epsilon_i x)^\alpha [\log u_i]_{\alpha,\delta(\log u_i, x_i + \epsilon_i x)}(x_i + \epsilon_i x) \right) \\ &\leq C(\beta). \end{aligned}$$

This implies (3.8) for any $\beta > 1$. By Theorem D, we have, for any $\beta > 1$,

$$|\nabla v_i(y)| \leq C(\beta) \quad \forall |y| < \beta.$$

Passing to a subsequence,

$$v_i \rightarrow v \quad \text{in } C_{\text{loc}}^\gamma(\mathbb{R}^n) \text{ for all } \alpha < \gamma < 1,$$

where v is a positive function in $C_{\text{loc}}^{0,1}(\mathbb{R}^n)$ satisfying $[\log v]_{\alpha,1}(0) = 1$. In particular, v cannot be a constant.

Clearly, (3.9) holds with the new ϵ_i given in (4.2). Thus, by (1.13) and (3.9),

$$\lim_{i \rightarrow \infty} f(\lambda(A_{v_i(y)^{4/(n-2)}g^{(i)}})) = 0.$$

CLAIM. $w := v^{-2/(n-2)}$, modulo a linear change of coordinates, is a positive locally Lipschitz viscosity solution of $\lambda(A_w) \in \partial\Gamma$ in \mathbb{R}^n .

Once the claim is proved, we have, by Theorem 1.4, that v is a constant. This leads to a contradiction. The Hölder estimate (4.1) is established.

PROOF: We may assume without loss of generality that

$$a_{lm}^{(i)}(x_i) = \delta_{lm},$$

since this can be achieved by making a change of variables

$$y^l = b_\alpha^l z^\alpha,$$

where (b_α^l) , depending on i , is the square root matrix of the inverse of the positive definite matrix $(a_{lm}^{(i)}(x_i))$.

We will only prove

(i) v is a viscosity supersolution of (1.6) in $\Omega = \mathbb{R}^n$.

We omit the very similar proof of

(ii) v is a viscosity subsolution of (1.6) in $\Omega = \mathbb{R}^n$.

The claim follows from (i) and (ii) in view of Remark 1.2.

Let $x_0 \in \mathbb{R}^n$, $\varphi \in C^2(\mathbb{R}^n)$, $(v - \varphi)(x_0) = 0$, and $v - \varphi \geq 0$ near x_0 . For small $\delta > 0$, consider

$$\varphi_\delta(x) = \varphi(x) - \delta|x - x_0|^2.$$

Then

$$\varphi_\delta(x) \leq \varphi(x) - \delta^3 \quad \forall |x - x_0| = \delta.$$

By the convergence of v_i to v ,

$$v_i(x) \geq \varphi_\delta(x) + \frac{\delta^3}{2} \quad \forall |x - x_0| = \delta$$

for large i .

Since $v_i(x_0) \rightarrow v(x_0) = \varphi_\delta(x_0)$, there exists some \hat{x}_i satisfying $|\hat{x}_i - x_0| < \frac{\delta}{2}$ such that

$$\beta_i := (v_i - \varphi_\delta)(\hat{x}_i) = \min_{|x - x_0| \leq \delta} (v_i - \varphi_\delta)(x) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Since

$$\beta_i = (v_i - \varphi)(\hat{x}_i) + \delta|\hat{x}_i - x_0|^2 \geq (v_i - v)(\hat{x}_i) + \delta|\hat{x}_i - x_0|^2,$$

we have, using $v_i \rightarrow v$ and $\beta_i \rightarrow 0$, that

$$\lim_{i \rightarrow \infty} \hat{x}_i = x_0.$$

Set

$$\hat{\varphi}_\delta^{(i)} = \varphi_\delta + \beta_i.$$

We know from the above that

$$v_i(\hat{x}_i) = \hat{\varphi}_\delta^{(i)}(\hat{x}_i), \quad v_i(x) \geq \hat{\varphi}_\delta^{(i)}(x), \quad \forall |x - \hat{x}_i| < \frac{\delta}{2},$$

$$\nabla v_i(\hat{x}_i) = \nabla \hat{\varphi}_\delta^{(i)}(\hat{x}_i), \quad \nabla^2 v_i(\hat{x}_i) \geq \nabla^2 \hat{\varphi}_\delta^{(i)}(\hat{x}_i).$$

It follows, at \hat{x}_i , that

$$\begin{aligned} \lambda(A_{v_i(y)^{4/(n-2)}g^{(i)}}) &= \lambda(v_i(y)^{-\frac{4}{n-2}} A_{v_i(y)^{4/(n-2)}g^{(i)}}; g^{(i)}) \\ &\leq \lambda(\hat{\varphi}_\delta^{(i)}(y)^{-\frac{4}{n-2}} A_{\hat{\varphi}_\delta^{(i)}(y)^{4/(n-2)}g^{(i)}}; g^{(i)}), \end{aligned}$$

where

$$\lambda(v_i(y)^{-\frac{4}{n-2}} A_{v_i(y)^{4/(n-2)}g^{(i)}}; g^{(i)})$$

and

$$\lambda(\hat{\varphi}_\delta^{(i)}(y)^{-\frac{4}{n-2}} A_{\hat{\varphi}_\delta^{(i)}(y)^{4/(n-2)}g^{(i)}}; g^{(i)})$$

denote, respectively, the eigenvalues of

$$v_i(y)^{-\frac{4}{n-2}} A_{v_i(y)^{4/(n-2)}g^{(i)}} \quad \text{and} \quad \hat{\varphi}_\delta^{(i)}(y)^{-\frac{4}{n-2}} A_{\hat{\varphi}_\delta^{(i)}(y)^{4/(n-2)}g^{(i)}}$$

with respect to $g^{(i)}$.

Since $\lambda(A_{v_i(\hat{x}_i)^{4/(n-2)}g^{(i)}})$ is in Γ , we deduce from the above and the property of Γ that

$$\lambda(\hat{\varphi}_\delta^{(i)}(\hat{x}_i)^{-\frac{4}{n-2}} A_{\hat{\varphi}_\delta^{(i)}(\hat{x}_i)^{4/(n-2)}g^{(i)}}; g^{(i)})$$

is in Γ . Sending i to infinity, we have

$$\lambda(\hat{\varphi}_\delta(x_0)^{-\frac{4}{n-2}} A_{\hat{\varphi}_\delta(x_0)^{4/(n-2)}\bar{g}}; \bar{g}) \in \bar{\Gamma},$$

where $\bar{g} = \sum_i (dz^i)^2$. Sending $\delta \rightarrow 0$ leads to $\lambda(A^\varphi(x_0)) \in \bar{\Gamma}$. (i) is established. \square

Now we establish the gradient estimate (1.21) based on the Hölder estimates. The Hölder estimate (4.1) yields the Harnack inequality:

$$\sup_{B_{2r}} u \leq C \inf_{B_{2r}} u.$$

Consider

$$w := \frac{1}{u(0)} u.$$

The equation of w on B_{3r} is

$$f(\lambda(A_{w^{4/(n-2)}g})) = u(0)^{\frac{4}{n-2}}h, \quad \lambda(A_{w^{4/(n-2)}g}) \in \Gamma,$$

and w satisfies

$$\frac{1}{C} \leq w \leq C \quad \text{in } B_{2r}.$$

Since $u(0)$ is bounded from above, we have, using Theorem D, that

$$|\nabla u| \leq C \quad \text{in } B_r.$$

Theorem 1.10 is established. \square

5 Proofs of Theorem 1.23 and Theorem 1.19

PROPOSITION 5.1 *Let $u^+ \in C^{0,1}(\overline{B_1^+})$ and $u^- \in C^{0,1}(\overline{B_1^-})$ be two positive functions satisfying $u^+ = u^-$ on $\partial' B_1^+$. We assume that*

$$(5.1) \quad u^+ \text{ is a viscosity supersolution (subsolution) of } \lambda(A^{u^+}) \in \partial\Gamma \text{ in } B_1^+,$$

$$(5.2) \quad \frac{\partial u^+}{\partial x_n} \leq (\geq) 0 \text{ on } \partial' B_1^+ \text{ in the viscosity sense,}$$

$$(5.3) \quad u^- \text{ is a viscosity supersolution (subsolution) of } \lambda(A^{u^-}) \in \partial\Gamma \text{ in } B_1^-,$$

$$(5.4) \quad \frac{\partial u^-}{\partial x_n} \geq (\leq) 0 \text{ on } \partial' B_1^- \text{ in the viscosity sense.}$$

Then

$$\tilde{u}(x', x_n) := \begin{cases} u^+(x', x_n) & \text{if } x_n \geq 0, \\ u^-(x', x_n) & \text{if } x_n < 0 \end{cases}$$

is a $C^{0,1}$ viscosity supersolution (subsolution) of $\lambda(A^{\tilde{u}}) \in \partial\Gamma$ in B_1 .

A consequence of Propositions 1.14 and 5.1 is the following:

COROLLARY 5.2 *Let $\Omega^+ \subset \mathbb{R}_+^n$ be a bounded open set. For m points $S_m := \{P_1, \dots, P_m\} \subset \Omega^+ \cup \partial'\Omega^+$, $m \geq 0$, let $u \in C^{0,1}(\overline{\Omega^+} \setminus S_m)$ and $v \in C^{0,1}(\overline{\Omega^+})$ be positive functions. Assume that*

u is a viscosity supersolution of $\lambda(A^u) \in \partial\Gamma$ in $\Omega^+ \setminus S_m$,

v is a viscosity subsolution of $\lambda(A^v) \in \partial\Gamma$ in Ω^+ ,

$\frac{\partial u}{\partial x_n} \leq 0 \leq \frac{\partial v}{\partial x_n}$ on $\partial'\Omega^+$ in the viscosity sense,

$u > v$ on $\partial''\Omega^+$.

Then

$$(5.5) \quad \inf_{\overline{\Omega^+} \setminus S_m} (u - v) > 0.$$

PROOF USING PROPOSITIONS 1.14 AND 5.1: Let $u^+ = u$, $u^-(x', x_n) := u(x', -x_n)$, $v^+ := v$, $v^-(x', x_n) := v(x', -x_n)$,

$$\tilde{u}(x', x_n) := \begin{cases} u^+(x', x_n) & \text{if } x_n \geq 0, \\ u^-(x', x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$\tilde{v}(x', x_n) := \begin{cases} v^+(x', x_n) & \text{if } x_n \geq 0, \\ v^-(x', x_n) & \text{if } x_n < 0. \end{cases}$$

An application of Proposition 1.14 to $v^{-2/(n-2)}$ and $u^{-2/(n-2)}$, in view of Proposition 5.1 and Remark 1.2, yields (5.5). \square

PROOF OF PROPOSITION 5.1: We first prove the proposition under, instead of (5.2) and (5.4),

$$(5.6) \quad \frac{\partial u^+}{\partial x_n} < 0 \text{ on } \partial' B_1^+ \quad \text{and} \quad \frac{\partial u^-}{\partial x_n} > 0 \text{ on } \partial' B_1^- \text{ in the viscosity sense.}$$

Let $\bar{x} \in B_1$, $\psi \in C^2(B_1)$, $\tilde{u}(\bar{x}) = \psi(\bar{x})$, and, for some $0 < \delta < 1 - |\bar{x}|$, $u(x) \geq \psi(x)$ for all $|x - \bar{x}| < \delta$. We need to show that

$$\lambda(A^\psi(\bar{x})) \in \mathbb{R}^n \setminus \Gamma.$$

If \bar{x} does not belong to $\partial' B_1^+$, this is obvious because of (5.1) and (5.3). So we only need to show that \bar{x} does not belong to $\partial' B_1^+$. Indeed, if $\bar{x} \in \partial' B_1^+$, then, since $\bar{x}_n = 0$,

$$u^+(\bar{x}) = u^-(\bar{x}) = \psi(\bar{x}), \quad u^+, u^- \geq \psi \text{ near } \bar{x}.$$

Thus, by (5.6),

$$\frac{\partial \psi}{\partial x_n}(\bar{x}) < 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x_n}(\bar{x}) > 0.$$

A contradiction.

Now we prove the proposition under (5.2) and (5.4). We will only give the proof when u^+ and u^- are viscosity supersolutions, since the proof is essentially the same when they are subsolutions. We start with a first variation of the operator A^u together with the Neumann boundary condition.

LEMMA 5.3 *Let $\Omega^+ \subset \mathbb{R}_+^n$ be a bounded open set, and let $w \in C^2(\Omega^+) \cap C^1(\Omega^+ \cup \partial'\Omega^+)$ satisfy, for some constant $c_1 > 0$,*

$$w \geq c_1 \quad \text{in } \Omega^+,$$

and let

$$\varphi^\pm(x) := e^{\delta|x|^2 \pm \delta^2 x_n}.$$

Then there exists some constant $\delta > 0$ depending only on $\sup\{|x| \mid x \in \Omega^+\}$, and there exists $\bar{\epsilon} > 0$ depending only on δ , c_1 , and $\sup\{|x| \mid x \in \Omega^+\}$ such that for any $0 < \epsilon < \bar{\epsilon}$,

$$\begin{aligned} A_{w+\epsilon\varphi^\pm} &\geq \left(1 + \epsilon \frac{\varphi^\pm}{w}\right) A_w + \frac{\epsilon\delta}{2} \varphi^\pm w I && \text{in } \Omega^+, \\ A_{w-\epsilon\varphi^\pm} &\leq \left(1 - \epsilon \frac{\varphi^\pm}{w}\right) A_w - \frac{\epsilon\delta}{2} \varphi^\pm w I && \text{in } \Omega^+, \\ \frac{\partial}{\partial x_n}(w + \epsilon\varphi^\pm) &= \frac{\partial w}{\partial x_n} \pm \epsilon\delta && \text{on } \partial'\Omega^+, \\ \frac{\partial}{\partial x_n}(w - \epsilon\varphi^\pm) &= \frac{\partial w}{\partial x_n} \mp \epsilon\delta && \text{on } \partial'\Omega^+. \end{aligned}$$

PROOF: The proof is very similar to that of lemma 3.7 in [26, 27]; we omit the details. \square

Let u^+ be the supersolution in Proposition 5.1; set

$$\xi^+ := (u^+)^{-\frac{2}{n-2}}, \quad \xi_\epsilon^+ := \xi^+ + \epsilon\varphi^+, \quad u_\epsilon^+ := (\xi_\epsilon^+)^{-\frac{n-2}{2}}.$$

We will prove that

$$(5.7) \quad u_\epsilon^+ \text{ is a viscosity supersolution of } \lambda(A^{u_\epsilon^+}) \in \partial\Gamma \text{ in } B_1^+$$

and

$$(5.8) \quad \frac{\partial u_\epsilon^+}{\partial x_n} < 0 \text{ on } \partial' B_1^+ \text{ in the viscosity sense.}$$

Let $\bar{x} \in B_1^+$, $\psi \in C^2(B_1^+)$, $u_\epsilon^+(\bar{x}) = \psi(\bar{x})$, and $u_\epsilon \geq \psi$ near \bar{x} . Then, with $\eta := \psi^{-\frac{2}{n-2}}$,

$$\xi^+ = \eta - \epsilon\varphi^+ \text{ at } \bar{x} \text{ and } \xi^+ \leq \eta - \epsilon\varphi^+ \text{ near } \bar{x}.$$

By Remark 1.2, ξ^+ is a viscosity subsolution of $\lambda(A_{\xi^+}) \in \partial\Gamma$, and therefore

$$\lambda(A_{\eta-\epsilon\varphi^+}(\bar{x})) \in \bar{\Gamma}.$$

By Lemma 5.3,

$$A_{\eta-\epsilon\varphi^+}(\bar{x}) < \left(1 - \epsilon \frac{\varphi^+}{\eta}\right)(\bar{x}) A_\eta(\bar{x}),$$

which implies, for small ϵ ,

$$A^\psi(\bar{x}) = A_\eta(\bar{x}) \in \Gamma.$$

We have proved (5.7).

To prove (5.8), let $\bar{x} \in \partial' B_1^+$, $\psi \in C^1(\overline{B_1^+})$, $u_\epsilon^+(\bar{x}) = \psi(\bar{x})$, and $u_\epsilon \geq \psi$ near \bar{x} . It follows that

$$u^+ = [\psi^{-\frac{2}{n-2}} - \epsilon\varphi^+]^{-\frac{n-2}{2}} \text{ at } \bar{x} \quad \text{and} \quad u^+ \geq [\psi^{-\frac{2}{n-2}} - \epsilon\varphi^+]^{-\frac{n-2}{2}} \text{ near } \bar{x}.$$

Since $\frac{\partial u^+}{\partial x_n} \leq 0$ on $\partial' B_1^+$ in the viscosity sense, we have

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial x_n} [\psi^{-\frac{2}{n-2}} - \epsilon \varphi^+]^{-\frac{n-2}{2}} \Big|_{x=\bar{x}} \\ &= \frac{\partial \psi}{\partial x_n} [1 + O(\epsilon)] + \frac{n-2}{2} \epsilon \delta^2 \psi^{\frac{n}{n-2}} + O(\epsilon^2). \end{aligned}$$

So, for small ϵ , we have $\frac{\partial \psi}{\partial x_n}(\bar{x}) < 0$. We have proved (5.8).

Similarly, we set for u^-

$$\xi^- = (u^-)^{-\frac{2}{n-2}}, \quad \xi_\epsilon^- := \xi^- + \epsilon \varphi^-, \quad u_\epsilon^- := (\xi_\epsilon^-)^{-\frac{n-2}{2}},$$

and prove

$$u_\epsilon^- \text{ is a viscosity supersolution of } \lambda(A^{u_\epsilon^-}) \in \partial\Gamma \text{ in } B_1^-$$

and

$$(5.9) \quad \frac{\partial u_\epsilon^-}{\partial x_n} > 0 \quad \text{on } \partial' B_1^- \text{ in the viscosity sense.}$$

It is clear that $u_\epsilon^+ = u_\epsilon^-$ on ∂B_1^+ .

Since we now have the strict inequalities (5.8) and (5.9),

$$\tilde{u}_\epsilon(x', x_n) := \begin{cases} u_\epsilon^+(x', x_n) & \text{if } x_n \geq 0, \\ u_\epsilon^-(x', x_n) & \text{if } x_n \leq 0 \end{cases}$$

is a viscosity supersolution of $\lambda(A^{\tilde{u}_\epsilon}) \in \partial\Gamma$ in B_1 . Since $\tilde{u}_\epsilon \rightarrow \tilde{u}$ in $C_{\text{loc}}^0(B_1)$, we have, by standard arguments, that \tilde{u} is a viscosity supersolution of $\lambda(A^{\tilde{u}}) \in \partial\Gamma$ in B_1 . Proposition 5.1 is established. \square

PROOF OF THEOREM 1.23: By Proposition 5.1,

$$\tilde{u}(x', x_n) := \begin{cases} u(x', x_n) & \text{if } x_n \geq 0, \\ u(x', -x_n) & \text{if } x_n \leq 0 \end{cases}$$

satisfies the hypothesis of Theorem 1.4 and therefore is a constant. \square

PROOF OF THEOREM 1.19: The proof is similar to that of Theorem 1.10. Let O_3 be an open set of M satisfying $\overline{O}_3 \subset O_3 \subset \overline{O}_3 \subset O_1$. We first establish

$$(5.10) \quad \sup_{\substack{y, x \in O_3 \\ \text{dist}(y, x) < 2r}} \frac{|\log u(y) - \log u(x)|}{\text{dist}(y, x)^\alpha} \leq C(\alpha) \quad \forall 0 < \alpha < 1.$$

Suppose the contrary of (5.10); then for some $0 < \alpha < 1$, there exist, in $\overline{B}_2^+ \subset \mathbb{R}^n$, C^3 functions $\{u_i\}$, C^1 functions $\{\psi_i\}$ and $\{\eta_i\}$, and $n \times n$ symmetric positive definite C^3 matrix functions $(a_{lm}^{(i)}(x))$ satisfying, for some $\bar{a} > 0$, (3.1) and (3.2) in \overline{B}_2^+ , and

$$\|a_{lm}^{(i)}\|_{C^3(B_2^+)}, \|\psi_i\|_{C^1(B_2^+)}, \|\eta_i\|_{C^1(B_2^+)} \leq \bar{a},$$

$$\begin{cases} f(\lambda(A_{u_i^{4/(n-2)} g_i})) = \psi_i & \text{for } \lambda(A_{u_i^{4/(n-2)} g_i}) \in \Gamma \text{ on } B_2^+, \\ -\frac{\partial u_i}{\partial v_{g_i}} + \frac{n-2}{2} h_{g_i} u_i = \eta_i u_i^{n/(n-2)} & \text{on } \partial' B_2^+, \end{cases}$$

where g_i is given by (3.3), but

$$\inf_{x \in B_{1/2}^+} \delta(\log u_i, x) \rightarrow 0,$$

where

$$\delta(\log u_i, x) := \delta(\log u_i, x; B_2^+, \alpha).$$

It follows, for some $x_i \in B_1^+ \cup \partial' B_1^+$, that

$$\frac{1 - |x_i|}{\delta(\log u_i, x_i)} = \max_{x \in B_1^+} \frac{1 - |x|}{\delta(\log u_i, x)} \rightarrow \infty.$$

Let σ_i and ϵ_i be defined as in (4.2). Then they satisfy (4.3) and

$$\epsilon_i \leq 2\delta(\log u_i, z) \quad \forall z \in B_{\sigma_i}(x_i) \cap B_1^+.$$

Let

$$v_i(y) := \frac{1}{u_i(x_i)} u_i(x_i + \epsilon_i y), \quad |y| < \frac{\sigma_i}{\epsilon_i}, \quad y_n > -T_i := -\frac{1}{\epsilon_i}(x_i)_n.$$

After passing to a subsequence, either

$$\lim_{i \rightarrow \infty} (-T_i) = -T > -\infty \quad \text{or} \quad \lim_{i \rightarrow \infty} (-T_i) = -\infty.$$

Following, with obvious modification, the arguments in the proof of Theorem 1.10, we see, passing to another subsequence, that either

$$v_i(\cdot + (0', -T_i)) \rightarrow v \quad \text{in } C_{\text{loc}}^\gamma(\overline{\mathbb{R}_+^n}) \text{ for all } 0 < \gamma < 1$$

for some positive locally Lipschitz viscosity solution v of (1.31) and (1.32) satisfying $[\log v]_{\alpha,1}(0', T) = 1$, or

$$v_i \rightarrow v \quad \text{in } C_{\text{loc}}^\gamma(R^n) \text{ for all } 0 < \gamma < 1$$

for some positive locally Lipschitz viscosity solution of $\lambda(A^v) \in \partial\Gamma$ in \mathbb{R}^n satisfying $[\log v]_{\alpha,1}(0) = 1$. By our Liouville theorems v must be a constant. But $[\log v]_{\alpha,1}(0', T) = 1$ or $[\log v]_{\alpha,1}(0) = 1$ does not allow v to be a constant. A contradiction. Theorem 1.19 is established. \square

Appendix

We give in this appendix the proof of Theorem D in [20]. For simplicity we present the proof on locally conformally flat manifolds. Namely, we give another proof of Theorem C, which can easily be extended to general Riemannian manifolds.

ANOTHER PROOF OF THEOREM C: We write $v = -\frac{2}{n-2} \log u$. Then v satisfies, with $\alpha = -\frac{2}{n-2} \log b$ and $\beta = -\frac{2}{n-2} \log a$,

$$(A.1) \quad f(\lambda(W)) = h, \quad \lambda(W) \in \Gamma, \quad \text{in } B_3,$$

and

$$\alpha \leq v \leq \beta \quad \text{on } B_3,$$

where

$$W := (W_{ij}) = e^{2v} \left(v_{ij} + v_i v_j - \frac{|\nabla v|^2}{2} \delta_{ij} \right).$$

We only need to prove, for some constant C depending on α, β , and (f, Γ) that

$$(A.2) \quad |\nabla v| \leq C \quad \text{on } B_1.$$

Fix some small constants $\epsilon, c_1 > 0$, depending only on α and β , such that the function $\phi(s) := \epsilon e^{-2s}$ satisfies

$$(A.3) \quad -\frac{1}{2}\phi' \geq c_1, \quad \phi'' + \phi' - (\phi')^2 \geq 0, \quad \text{on } [\alpha, \beta].$$

Let $\rho \geq 0$ be a smooth function taking value 1 in B_1 and 0 outside B_2 . It is known that ρ satisfies $|\nabla \rho|^2 \leq C_1$. Consider

$$G = \rho e^{\phi(v)} |\nabla v|^2.$$

Estimate (A.2) is established if we can show that $G \leq C$ on \bar{B}_2 . Let $G(x_0) = \max_{\bar{B}_2} G$ for some $x_0 \in \bar{B}_2$. Clearly $x_0 \in B_2$. After a rotation of the axis if necessary, we may assume that $W(x_0)$ is a diagonal matrix. In the following, we use subscripts of a function to denote derivatives. For example, $G_i = \partial_{x_i} G$, $G_{ij} = \partial_{x_i x_j} G$, and so on. We also use the notation $f^i := \frac{\partial f}{\partial \lambda_i}$.

Applying ∂_{x_k} to (A.1) leads to

$$(A.4) \quad f^i W_{iik} = 0.$$

By calculation,

$$\begin{aligned} G_i &= 2\rho e^{\phi} v_{ki} v_k + \rho \phi' e^{\phi} |\nabla v|^2 v_i + e^{\phi} |\nabla v|^2 \rho_i \\ &= 2\rho e^{\phi} v_{ki} v_k + \left(\phi' v_i + \frac{\rho_i}{\rho} \right) G. \end{aligned}$$

At x_0 , we have $G_i = 0$. Equivalently, we have

$$(A.5) \quad 2v_{ki} v_k = -\phi' |\nabla v|^2 v_i - \frac{\rho_i}{\rho} |\nabla v|^2 \quad \forall 1 \leq i \leq n.$$

Take the second covariant derivative of G and evaluate at x_0 ,

$$\begin{aligned} 0 \geq (G_{ij}) &= 2v_{kij} v_k e^{\phi} \rho + 2v_{ki} v_{kj} e^{\phi} \rho + 2v_{ki} v_k e^{\phi} \phi' v_j \rho + 2v_{ki} v_k e^{\phi} \rho_j \\ &\quad + \left(\phi'' v_i v_j + \phi' v_{ij} + \frac{\rho \rho_{ij} - \rho_i \rho_j}{\rho^2} \right) \rho e^{\phi} |\nabla v|^2. \end{aligned}$$

Therefore, at x_0 ,

$$\begin{aligned}
(A.6) \quad 0 &\geq e^{-\phi} f^i G_{ii} \\
&= 2\rho f^i v_{ik} v_k + 2\rho f^i v_{ki}^2 + 2\rho \phi' f^i v_{ki} v_k v_i + 2f^i v_{ki} v_k \rho_i \\
&\quad + f^i (\phi'' v_i^2 + \phi' v_{ii} + \frac{\rho \rho_{ii} - \rho_i^2}{\rho^2}) \rho |\nabla v|^2 \\
&= 2\rho f^i v_k \left\{ e^{-2v} W_{ii} - v_i^2 + \frac{|\nabla v|^2}{2} \delta_{ii} \right\}_k + 2\rho f^i v_{ki}^2 \\
&\quad - \rho \phi' f^i \left(|\nabla v|^2 \phi' v_i^2 + \frac{|\nabla v|^2 \rho_i v_i}{\rho} \right) - f^i \left(|\nabla v|^2 \phi' v_i + \frac{|\nabla v|^2 \rho_i}{\rho} \right) \rho_i \\
&\quad + \rho \phi'' |\nabla v|^2 f^i v_i^2 + \rho \phi' |\nabla v|^2 f^i \left(e^{-2v} W_{ii} - v_i^2 + \frac{|\nabla v|^2}{2} \delta_{ii} \right) \\
&\quad + |\nabla v|^2 f^i \frac{\rho \rho_{ii} - \rho_i^2}{\rho} \\
&= 2\rho f^i \left\{ e^{-2v} W_{iik} v_k - 2e^{-2v} |\nabla v|^2 W_{ii} + \phi' |\nabla v|^2 v_i^2 + \frac{\rho_i v_i}{\rho} |\nabla v|^2 \right. \\
&\quad \left. - \frac{1}{2} |\nabla v|^4 \phi' - \frac{1}{2} |\nabla v|^2 \frac{v_k \rho_k}{\rho} \right\} + 2\rho f^i v_{ki}^2 \\
&\quad - \rho \phi' f^i \left(|\nabla v|^2 \phi' v_i^2 + \frac{|\nabla v|^2 \rho_i v_i}{\rho} \right) - f^i \left(|\nabla v|^2 \phi' v_i + \frac{|\nabla v|^2 \rho_i}{\rho} \right) \rho_i \\
&\quad + \rho \phi'' |\nabla v|^2 f^i v_i^2 + \rho \phi' |\nabla v|^2 f^i \left(e^{-2v} W_{ii} - v_i^2 + \frac{|\nabla v|^2}{2} \delta_{ii} \right) \\
&\quad + |\nabla v|^2 f^i \frac{\rho \rho_{ii} - \rho_i^2}{\rho} \\
&= \left\{ -4\rho e^{-2v} |\nabla v|^2 f - |\nabla v|^2 v_k \rho_k \sum_i f^i - 2\phi' |\nabla v|^2 f^i \rho_i v_i \right. \\
&\quad \left. + \rho e^{-2v} \phi' |\nabla v|^2 f + |\nabla v|^2 f^i \frac{\rho \rho_{ii} - 2\rho_i^2}{\rho} + 2|\nabla v|^2 f^i \rho_i v_i \right\} \\
&\quad + 2\rho f^i v_{ki}^2 - \frac{1}{2} \rho \phi' |\nabla v|^4 \sum_i f^i + (\phi'' + \phi' - (\phi')^2) \rho |\nabla v|^2 f^i v_i^2.
\end{aligned}$$

In the following, we use C_2 to denote some positive constant depending only on α , β , and (f, Γ) that may vary from line to line. By (A.3), we derive from (A.6) that

$$\begin{aligned}
0 &\geq e^{-\phi} f^i G_{ii} \geq (-C_2 \sqrt{\rho} |\nabla v|^3 - C_2 |\nabla v|^2 + 2c_1 \rho |\nabla v|^4) \sum_i f^i \\
&= |\nabla v|^2 (-C_2 \sqrt{\rho} |\nabla v|^2 - C_2 + c_1 \rho |\nabla v|^2) \sum_i f^i,
\end{aligned}$$

which implies $\rho |\nabla v|^2(x_0) \leq C_2$, so is $G(x_0)$. Estimate (A.2) is established. \square

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YANYAN LI

Rutgers University

Department of Mathematics

110 Frelinghuysen Road

Piscataway, NJ 08854

E-mail: yyli@math.rutgers.edu

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