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Mass and angular-momentum inequalities for axi-symmetric initial data sets. II. Angular momentum

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Abstract

We extend the validity of Dain's angular-momentum inequality to maximal, asymptotically flat, initial data sets on a simply connected manifold with several asymptotically flat ends which are invariant under a $U(1)$ action and which admit a twist potential.

Keywords: Angular momentum in general relativity; Axisymmetric general relativistic initial data

1. Introduction

In [3] Dain proved an upper bound for angular momentum in terms of the mass for a class of maximal, vacuum, initial data sets with a metric of the form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\varphi + \rho B_\rho d\rho + A_z dz)^2, \tag{1.1}$$

where the functions are assumed to be φ-independent. The existence of the global coordinate system (1.1) has been justified for asymptotically flat axi-symmetric initial data sets on a simply connected manifold in the first paper of this series [2].

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In this paper we extend the validity of Dain's inequality to all maximal, asymptotically flat, simply connected initial data sets (M, g, K) invariant under a $U(1)$ action, with several asymptotically flat ends and positive scalar curvature, and admitting a *twist potential* ω as defined by (2.6) below.

In order to give a detailed statement, some preliminary remarks are in order. We choose an asymptotic region, say M_1 and, following Dain, the remaining asymptotic regions are described by punctures on the z-axis in the (ρ, z) plane. The axial symmetry of the problem implies that the angular momentum $\vec{J}_i$ of each asymptotic region M_i is aligned along the rotation axis, and we shall write J_i for the relevant component of $\vec{J}_i$. As is well known, ω is constant on each connected component of the punctured axis, and (cf., e.g., [9, Section 6]) J_1 is proportional to the difference of the values of ω on the extreme segments of the axis, while for $i \neq 1$ the z-component J_i of the angular-momentum vector is proportional to the jump of ω at the i-th puncture. This implies

$$J_1 = \sum_{i=2}^{N} J_i, \tag{1.2}$$

so that J_1 is determined by the remaining angular momenta.

We denote by $8\pi f(J_2, \ldots, J_N)$ the numerical value of the action functional (2.13) of the harmonic map, from $\mathbb{R}^3 \setminus \{\rho = 0\}$ to the two-dimensional hyperbolic space, constructed in Proposition 2.1 below.

It follows from (1.2) that for $N = 1$ the angular momentum necessarily vanishes (so that no non-trivial inequality involving the angular momentum can be obtained in the $N = 1$ case), while for $N = 2$ we have

$$J_1 = J_2 \quad \text{and} \quad f(J_2) = f(J_1) = \sqrt{|\vec{J}_1|}.$$

We are ready now to present our version of Dain's inequality:

Theorem 1.1. *Let (M, g, K) be a three-dimensional $U(1)$—invariant initial data set, with positive matter density, on a simply connected manifold M which is the union of a compact set and of a finite number of asymptotic regions M_i, $i = 1, \ldots, N$, $N \geqslant 2$, and with (g, K)—asymptotically flat on each end in the sense of (2.1) and (2.2) with $k \geqslant 6$, together with (2.4). If (M, g, K) admits a twist potential[1] satisfying (2.12), then the ADM mass m_1 of M_1 satisfies*

$$m_1 \geqslant f(J_2, \ldots, J_N).$$

For $N = 2$ this is Dain's inequality $m_1 \geqslant \sqrt{|\vec{J}_1|}$.

Conceivably the case of main interest is $N = 2$ (already analyzed under different hypotheses by Dain [3]), since the function f is not known in general. (It would thus be of interest to obtain some lower estimates on f when $N \geqslant 3$.) Now, it is far from clear how large is the class of metrics considered by Dain in [3], in view of several *a priori* restrictive conditions imposed there; the analysis in [2] and here shows that the inequality applies in substantial generality.

A sketch of the argument seems appropriate: Following Dain, we show that the mass is bounded from below by the action of a map (U, ω), with values in the hyperbolic space,

[1] This will necessarily be true in vacuum.

determined by the norm of the rotational Killing vector and by the twist potential. This map is singular on the rotation axis, with further distinct singularities on punctures on that axis, which correspond to the remaining asymptotically flat regions. One then wishes to show that the action of the map is bounded from below by the action of the extreme Kerr solution when $N = 2$, or by the action of a harmonic map $(\widetilde{U}, \tilde{\omega})$ with singularities at the punctures resembling those of the extreme Kerr solution for $N > 2$. Such maps are constructed in Proposition 2.1; the result is essentially due to Weinstein [10]. The key element of the remainder of the argument is a result of Hildebrandt, Kaul and Widman [6] that, on compact domains with smooth boundary, harmonic maps with negatively curved target space are minimizers of the action. Since we are working on a non-compact set, with maps satisfying singular boundary conditions, some work is needed to apply this result. We start by showing that the action is, roughly speaking, decreased by deforming (U, ω) to a map $(\check{U}_\delta, \check{\omega}_\delta)$ with a singularity structure at the punctures resembling that of the extreme solutions. The maps (U_η, ω_η) of Lemma 2.5 are further deformations of $(\check{U}_\delta, \check{\omega}_\delta)$ which coincide with $(\widetilde{U}, \tilde{\omega})$ near the punctures, *and* for large r. Lemma 2.6 introduces a final deformation which takes into account the fact that $(\widetilde{U}, \tilde{\omega})$ satisfies the harmonic map equations away from the rotation axis only.

2. Angular-momentum inequalities

We will consider Riemannian manifolds (M, g) that are asymptotically flat, in the usual sense that there exists a region $M_{\rm ext} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R, such that in local coordinates on $M_{\rm ext}$ obtained from $\mathbb{R}^3 \setminus B(R)$ the metric satisfies the fall-off conditions, for some $k \geqslant 1$,

$$g_{ij} - \delta_{ij} = o_k(r^{-1/2}), \tag{2.1}$$

$$\partial_k g_{ij} \in L^2(M_{\rm ext}), \tag{2.2}$$

$$R^i_{jk\ell} = o(r^{-5/2}), \tag{2.3}$$

where we write $f = o_k(r^\lambda)$ if f satisfies

$$\partial_{k_1} \ldots \partial_{k_\ell} f = o(r^{\lambda-\ell}), \quad 0 \leqslant \ell \leqslant k.$$

Let (M, g, K) be a general relativistic, not necessarily vacuum, initial data set, with 2π-periodic Killing vector η. We impose asymptotic flatness on g as before and, in each asymptotic region, we assume the following asymptotic decay of K, for large r

$$|K|_g = O(r^{-\lambda}), \qquad \lambda > 5/2. \tag{2.4}$$

We will further suppose that $\lambda \leqslant 3$, as faster decay would necessarily lead to zero angular momentum; $\lambda = 3$ is the decay rate corresponding to the Kerr family of solutions. Note that (2.4) enforces the vanishing of the ADM momentum of the initial data set.

We also assume that the initial data set is maximal, $\mathrm{tr}_g K = 0$, and that the Einstein constraint equations hold with matter density μ satisfying a positivity condition,

$${}^{(3)}R = |K|_g^2 + 16\pi\mu \geqslant |K|_g^2, \tag{2.5}$$

A key restrictive hypothesis in what follows is the existence of a *twist potential* ω:

$$\epsilon_{ijk} K^j_\ell \eta^k \eta^\ell \mathrm{d}x^i = \mathrm{d}\omega. \tag{2.6}$$

As discussed in [3] this holds e.g., for vacuum initial data sets on simply connected manifolds.

It has been shown in [2] that, under the hypotheses of Theorem 1.1, there exist coordinates in which the metric can globally written in the form (1.1) (compare [1,5]). Consider an orthonormal frame e_i such that e_3 is proportional to η, let θ^i be the dual co-frame; for definiteness we take

$$\theta^1 = \mathrm{e}^{-U+\alpha}\mathrm{d}\rho, \quad \theta^2 = \mathrm{e}^{-U+\alpha}\mathrm{d}z, \quad \theta^3 = \rho\mathrm{e}^{-U}(\mathrm{d}\varphi + \rho B_\rho \mathrm{d}\rho + A_z \mathrm{d}z).$$

By (2.6),

$$\mathrm{d}\omega = \epsilon_{ijk} K^j_\ell \eta^k \eta^\ell \mathrm{d}x^i = \epsilon(\partial_A, \partial_B, \partial_\varphi) K(\mathrm{d}x^B, \partial_\varphi)\mathrm{d}x^A = g(\eta,\eta)\epsilon(e_a, e_b, e_3) K(\theta^b, e_3)\theta^a,$$

for some φ-independent function ω. Here, as before, the upper case indices $A, B = 1, 2$ correspond to the coordinates (ρ, z), while the lower case indices $a, b = 1, 2$ are frame indices. Thus, writing K_{b3} for $K(e_b, e_3) = K(\theta^b, e_3)$, we have

$$\rho^2 \mathrm{e}^{-2U}(K_{23}\theta^1 - K_{13}\theta^2) = \partial_\rho \omega \ \mathrm{d}\rho + \partial_z \omega \mathrm{d}z;$$

equivalently

$$K_{13} = -\frac{\mathrm{e}^{3U-\alpha}}{\rho^2}\partial_z\omega, \quad K_{23} = \frac{\mathrm{e}^{3U-\alpha}}{\rho^2}\partial_\rho\omega, \tag{2.7}$$

so that

$$\mathrm{e}^{2(\alpha-U)}|K|^2_g \geqslant 2\mathrm{e}^{2(\alpha-U)}(K^2_{13} + K^2_{23}) = 2\frac{\mathrm{e}^{4U}}{\rho^4}|\mathrm{d}\omega|^2_\delta. \tag{2.8}$$

In [2] (compare [1,3,5]) it has been shown that

$$m = \frac{1}{16\pi}\int \left[{}^{(3)}R + \frac{1}{2}\rho^2 \mathrm{e}^{-4\alpha+2U}(\rho B_{\rho,z} - A_{z,\rho})^2 \right] \mathrm{e}^{2(\alpha-U)}\mathrm{d}^3x + \frac{1}{8\pi}\int (DU)^2 \mathrm{d}^3 x. \tag{2.9}$$

Inserting (2.8) into (2.9) we obtain

$$m \geqslant \frac{1}{16\pi}\int \left[{}^{(3)}R + 2(DU)^2 \right] \mathrm{e}^{2(\alpha-U)}\mathrm{d}^3x \geqslant \frac{1}{8\pi}\int \left[(DU)^2 + \frac{\mathrm{e}^{4U}}{\rho^4}(D\omega)^2 \right]\mathrm{d}^3x. \tag{2.10}$$

It immediately follows from (2.7) that the twist potential ω is constant on each connected component $\mathcal{A}_j, j = 1, \ldots, N$, of the axis $\mathcal{A} = \{\rho = 0\}$; in this section we assume that $N \geqslant 2$. We set

$$\omega_j := \omega|_{\mathcal{A}_j}. \tag{2.11}$$

As in [2,3], in the coordinate system of (1.1) each asymptotically flat end, except the chosen one, is represented by a point $\vec{a}_i$ lying on the symmetry axis $\mathcal{A}$.

To gain some insight into the problem at hand the following comments are in order: Roughly speaking, asymptotic flatness implies that the twist potential ω approaches some smooth function of the angles $\mathring{\omega}_i$, typically different in each asymptotic region, as one recedes to infinity there:

$$\omega - \mathring{\omega}_i = o(1). \tag{2.12}$$

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The exact form of $\mathring{\omega}_i$ is actually irrelevant for our purposes[2]; the essential point is that, after mapping infinity to a neighborhood of a puncture $\vec{a}_i$, $\partial\mathring{\omega}$ behaves as $1/r_i$ in the local coordinates there. This is at the origin of Dain's mass inequality: Indeed, the first term in (2.10) is minimized by $U \equiv 0$. However, a constant U would lead to an infinite mass integral because of the second term in (2.10). So the second term, plus the requirement that the integral converges, forces U to approach minus infinity as one approaches each puncture, enforcing thus a non-zero lower bound on the mass.

Note that the explosion rate is faster for non-extreme solutions, compare (2.14) and (2.15), giving a larger contribution from the first term, as compared to an extreme solution. However, there is a tension between the two terms, as the decrease of energy of the first term appears to be comparable to the increase of the second, which makes the comparison delicate. In fact, it is not at all clear *a priori* which term will win in the balance. In the argument below (closely related to, but not identical to Dain's [3]) we follow Dain's insight that cylindrical ends have less energy than asymptotically flat ones:

Proof of theorem 2.1. If the mass is infinite there is nothing to prove, otherwise by (2.10) we need to find a lower bound on

$$I := \int \left[(DU)^2 + \frac{e^{4U}}{\rho^4} (D\omega)^2 \right] d^3x. \tag{2.13}$$

In a coordinate system where a Kelvin inversion has been performed in all remaining asymptotically flat regions, as in [2], near each puncture the function U has, for small r_i, the asymptotic behavior as in [2, Theorem 2.9],

$$U = 2\ln r_i + O(1), \quad dU = 2d(\ln r_i) + O(1), \tag{2.14}$$

while ω approaches a non-trivial angle-dependent function $\mathring{\omega}_i$ as r_i approaches zero.

We note the following result; when $N = 2$ the solution $(\widetilde{U}, \tilde{\omega})$ below is the pair of functions (U, ω) corresponding to an extreme Kerr metric with angular momentum along the z–axis equal to $(\omega_2 - \omega_1)/8$.

Proposition 2.1. *For any set of aligned punctures $\vec{a}_i$ and of axis values ω_j there exists a solution $(\widetilde{U}, \tilde{\omega})$ of the variational equations associated with the action (2.13), with finite value of I, satisfying (2.11), with the asymptotic behavior near each puncture*

$$\widetilde{U} = \ln r_i + O(1), \tag{2.15}$$

$$\tilde{\omega} = O(1). \tag{2.16}$$

Remark 2.2. It would be of interest to study in detail the regularity of the solution near the punctures, compare [8] for some related results.

Remark 2.3. As discussed in Appendix C, an identical proof gives existence of harmonic maps with any prescribed number of non-degenerate (as in [9]) and degenerate horizons, with several asymptotically flat regions. We also prove there uniqueness of solutions.

[2] An explicit form of ω for Kerr metrics can be found in Appendix A.

Proof. This is a rather straightforward consequence of the results in [10], we give some details to justify the estimates. Let the reference map $(\bar{U}, \bar{\omega})$ be any map from $\mathbb{R}^3 \setminus \{\vec{a}_i\}$ such that

1. $\bar{\omega}$ takes the prescribed values ω_j on the relevant connected components of the punctured axis of symmetry;
2. $(\bar{U}, \bar{\omega})$ coincides, for all large values of $\rho^2 + z^2$, with the extreme Kerr solution with the z–component of the angular-momentum vector equal to $(\omega_N - \omega_1)/8$;
3. in a small neighborhood of the i-th puncture, after perhaps a constant shift in $\bar{\omega}$, $(\bar{U}, \bar{\omega})$ coincides with the extreme Kerr solution with angular momentum $(\omega_{i+1} - \omega_i)/8$ near its cylindrical end;
4. from the explicit form (A.2) of the twist potential $\omega = \omega_{\mathrm{Kerr}}$ for the Kerr metrics one has, for small ρ, uniformly in z, away from the plane $\{z = 0\}$ (see (A.11) and (A.12))
$$|\bar{\omega} - \omega_i| \leqslant C\frac{\rho^4}{r_i^4} \quad \text{and} \quad |D\bar{\omega}|_\delta \leqslant C\frac{\rho^3}{r_i^4}, \tag{2.17}$$
when $\bar{\omega} = \omega_{\mathrm{Kerr}}$ and $r_i = r$; one can therefore also arrange that (2.17) be satisfied by the reference map $\bar{\omega}$ away from the planes $\{z = a_i\}$.
5. To make things precise, near the axis $\rho = 0$ and away from small neighborhoods of the punctures we let $(\bar{U}, \bar{\omega})$ be defined by the usual convex linear combination of two solutions using a smooth cut-off function which depends only upon z for small ρ.

For i large let $(\bar{U}_i, \bar{\omega}_i)$ be the map which coincides with $(\bar{U}, \bar{\omega})$ for $\rho < 1/i$ and for $r \geqslant i$, and such that $(\bar{x} := \bar{U} - \ln \rho, \bar{\omega})$ solves the harmonic map equations, with target manifold metric
$$b := \mathrm{d}x^2 + \mathrm{e}^{4x}\mathrm{d}\omega^2, \tag{2.18}$$
away from the union of those last two sets; such a map exists by, e.g., [6].

By construction the tension map associated to $(\bar{U} - \ln \rho, \bar{\omega})$, as defined in [10], has compact support on $\mathbb{R}^3 \setminus \{\vec{a}_i\}$, and is uniformly bounded in the norm defined by the metric b. Indeed, the last property is clear away from the axis. In the interpolation region near the axis the map $(\bar{x}, \bar{\omega})$ is of the form
$$\bar{x} = -\ln \rho + \alpha_0(z) + \alpha_2(z)\rho^2 + O(\rho^4), \quad \bar{\omega} = \omega_i + \beta_4(z)\rho^4 + O(\rho^6), \tag{2.19}$$
for some smooth functions $\alpha_0(z), \alpha_2(z), \beta_4(z)$, with the obvious associated behavior of the derivatives. The non-vanishing Christoffel symbols of the metric b are
$$\Gamma^x_{\omega\omega} = -2\mathrm{e}^{4x}, \quad \Gamma^\omega_{x\omega} = \Gamma^\omega_{\omega x} = 2.$$
This leads to the following formula for the norm squared of the tension,
$$|T|_b^2 = (\Delta\bar{x} - 2\mathrm{e}^{4\bar{x}}|D\bar{\omega}|^2)^2 + \mathrm{e}^{4\bar{x}}(\Delta\bar{\omega} + 4D\bar{x} \cdot D\bar{\omega})^2,$$
where Δ is the flat Laplacian on $\mathbb{R}^3$, with the scalar product, and norm of D, taken with respect to the flat metric on $\mathbb{R}^3$. A uniform bound on $|T|_b$ readily follows from (2.19).

Note that $(\bar{U}_i, \bar{\omega}_i)$ has finite action I which is smaller than or equal to the action of $(\bar{U}, \bar{\omega})$, as the action of $(\bar{U}_i, \bar{\omega}_i)$ is strictly smaller than that of $(\bar{U}, \bar{\omega})$ on the region where they differ by [6].

As outlined in [10, Section 3], an appropriately chosen diagonal subsequence of the sequence $(\bar{U}_i, \bar{\omega}_i)$ converges uniformly on compact subsets of $\mathbb{R}^3 \setminus \{\vec{a}_i\}$ to the desired harmonic map $(\widetilde{U} - \ln\rho, \tilde{\omega})$, with $(\widetilde{U} - \ln\rho, \tilde{\omega})$ lying a b–finite distance from $(\bar{U} - \ln\rho, \bar{\omega})$. The estimate (2.15) follows.

The action I of the limit is smaller than or equal to that of $(\bar{U}, \bar{\omega})$ by Fatou's Lemma, in particular it is finite. □

The arguments in [10], together with elementary scaling in coordinate balls of radius $\rho/2$ centered at (ρ, z), show that there is a uniform gradient estimate

$$|\mathrm{d}(\widetilde{U} - \ln\rho)|^2 + \mathrm{e}^{4(\widetilde{U} - \ln\rho)}|\mathrm{d}\tilde{\omega}|^2 = |\mathrm{d}(\widetilde{U} - \ln\rho)|^2 + \frac{\mathrm{e}^{4\widetilde{U}}}{\rho^4}|\mathrm{d}\tilde{\omega}|^2 \leqslant C\rho^{-2}. \tag{2.20}$$

An identical estimate (with possibly a different constant, independent of i) holds for the approximating sequence, which implies that $\mathrm{d}\tilde{\omega}$ vanishes on the punctured axis, and $\tilde{\omega}$ attains the desired values there. In fact, near $\vec{a}_i$ from (2.20) one obtains

$$|\mathrm{d}\widetilde{U}| \leqslant \frac{C'}{\rho}, \qquad |\mathrm{d}\tilde{\omega}| \leqslant \frac{C'\rho}{r^2}. \tag{2.21}$$

For further purposes we will need a stronger estimate, which we prove in integral form. We consider small, non-overlapping balls near each puncture. Since all the functions are invariant under rotations around the rotation axis $\mathcal{A}$ it suffices to work in a half-disc

$$D^+(C\sqrt{\delta}) := \{x^1 \geqslant 0,\ (x^1)^2 + (x^2)^2 \leqslant C^2\delta\} \subset \mathbb{R}^2$$

of radius $C\sqrt{\delta}$, with polar coordinates centered at $(0, a_i) : (x^1, x^2) = (\rho\sin\theta, a_i + \rho\cos\theta)$. The reader is warned that the polar coordinate ρ here corresponds to r_i in the applications that follow, and x^1 here is ρ in the applications below; this explains the weight x^1 in the measure in (2.22).

Proposition 2.4. *Let $\mu > 1/2$, and let $\bar{\omega}$ be as in the proof of Proposition 2.1, and let $\delta > 0$ be such that the half-disc $D^+(\sqrt{\delta})$ centered at the origin contains only one puncture. There exists a constant C_1, independent of δ, such that for any positive measurable function $g = g(|\vec{x}|)$, where $|\vec{x}| = \sqrt{(x^1)^2 + (x^2)^2}$, we have*

$$\int_{D^+(\sqrt{\delta})} (\tilde{\omega} - \bar{\omega})^2 \frac{1}{\sin^{2\mu+2}\theta}\frac{g(|\vec{x}|)}{|\vec{x}|^2} x^1 \mathrm{d}x^1 \mathrm{d}x^2$$
$$\leqslant C_1 \int_{D^+(\sqrt{\delta})} |D(\tilde{\omega} - \bar{\omega})|^2 \frac{1}{\sin^{2\mu}\theta} g(|\vec{x}|) x^1 \mathrm{d}x^1 \mathrm{d}x^2. \tag{2.22}$$

Proof. Let, first, u be any function which vanishes near the axis $x^1 = 0$, we claim that there exists a constant C_1, independent of δ, such that

$$\int_{D^+(\sqrt{\delta})} u^2 \frac{1}{\sin^{2\mu+2}\theta}\frac{g(|\vec{x}|)}{|\vec{x}|^2} x^1 \mathrm{d}x^1 \mathrm{d}x^2 \leqslant C_1 \int_{D^+(\sqrt{\delta})} |Du|^2 \frac{1}{\sin^{2\mu}\theta} g(|\vec{x}|) x^1 \mathrm{d}x^1 \mathrm{d}x^2. \tag{2.23}$$

In order to see that, we first prove that for $a \neq -1$, and for $f \in C^1(0, \pi)$, vanishing near 0 and π,

$$\int_0^\pi \theta^a f(\theta)^2 \mathrm{d}\theta \leqslant \frac{4}{(a+1)^2}\int_0^\pi \theta^{a+2} f'(\theta)^2 \mathrm{d}\theta. \tag{2.24}$$

Indeed,

$$\begin{aligned}\int_0^\pi \theta^a f(\theta)^2 \mathrm{d}\theta &= \frac{1}{a+1}\int_0^\pi f(\theta)^2 \mathrm{d}(\theta^{a+1}) = -\frac{2}{a+1}\int_0^\pi \theta^{a+1} f(\theta) f'(\theta)\mathrm{d}\theta \\ &\leqslant \frac{2}{a+1}\sqrt{\int_0^\pi \theta^a f(\theta)^2 \mathrm{d}\theta}\sqrt{\int_0^\pi \theta^{a+2} f'(\theta)^2 \mathrm{d}\theta}.\end{aligned}$$

The inequality (2.24) follows from the above.

Next, for $a < -2$ (we will apply this with $a = -2\mu - 1$, which then requires $\mu > -1/2$), by using (2.24) we obtain:

$$\begin{aligned}\int_0^\pi \theta^a(\pi-\theta)^a f(\theta)^2\mathrm{d}\theta &\leqslant (\pi/2)^a\left(\int_0^{\frac{\pi}{2}} \theta^a f(\theta)^2\mathrm{d}\theta + \int_{\frac{\pi}{2}}^\pi (\pi-\theta)^a f(\theta)^2 \mathrm{d}\theta\right) \\ &\leqslant (\pi/2)^a\left(\int_0^\pi \theta^a f(\theta)^2\mathrm{d}\theta + \int_0^\pi (\pi-\theta)^a f(\theta)^2\mathrm{d}\theta\right) \\ &= (\pi/2)^a\left(\int_0^\pi \theta^a f(\theta)^2\mathrm{d}\theta + \int_0^\pi \theta^a f(\pi-\theta)^2\mathrm{d}\theta\right) \\ &\leqslant C(a)\left(\int_0^\pi \theta^{a+2} f'(\theta)^2 + \int_0^\pi \theta^{a+2} f'(\pi-\theta)^2\right) \qquad (\text{by}(2.24)) \\ &\leqslant C(a)\left(\int_0^\pi \theta^{a+2} f'(\theta)^2 + \int_0^\pi (\pi-\theta)^{a+2} f'(\theta)^2\right) \\ &\leqslant C'(a)\int_0^\pi \theta^{a+2}(\pi-\theta)^{a+2} f'(\theta)^2 \qquad (\text{used } a+2<0).\end{aligned} \tag{2.25}$$

Now, for $x \in D^+(1)$ set $f(x) = u(\sqrt{\delta}x)$, we have

$$\begin{aligned}\int_{D^+(\sqrt{\delta})} u^2 \frac{1}{\sin^{2\mu+2}\theta}\frac{g(|\vec{x}|)}{|\vec{x}|^2}x^1\mathrm{d}x^1\mathrm{d}x^2 &= \sqrt{\delta}\int_{D^+(1)} f^2\frac{1}{\sin^{2\mu+2}\theta}\frac{g(|\vec{x}|)}{|\vec{x}|^2}x^1\mathrm{d}x^1\mathrm{d}x^2 \\ &= \sqrt{\delta}\int_{\rho=0}^1\int_{\theta=0}^\pi f^2\frac{1}{\rho^2\sin^{2\mu+2}\theta}g(\rho)\rho^2\sin\theta\mathrm{d}\rho\mathrm{d}\theta \\ &\underbrace{\leqslant}_{(*)} C_1\sqrt{\delta}\int_{\rho=0}^1\int_{\theta=0}^\pi\left(\frac{\partial f}{\partial\theta}\right)^2\frac{1}{\rho^2\sin^{2\mu}\theta}g(\rho)\rho^2\sin\theta\mathrm{d}\rho\mathrm{d}\theta \\ &\leqslant C_1\sqrt{\delta}\int_{\rho=0}^1\int_{\theta=0}^\pi |Df|^2\frac{1}{\sin^{2\mu}\theta}g(\rho)\rho^2\sin\theta\mathrm{d}\rho\mathrm{d}\theta \\ &= C_1\int_{D^+(\sqrt{\delta})}|Du|^2\frac{1}{\sin^{2\mu}\theta}g(|\vec{x}|)x^1\mathrm{d}x^1\mathrm{d}x^2,\end{aligned}$$

where in the step (*) we have used (2.25).

Since ω_i coincides with $\bar{\omega}$ for small x^1, we conclude that (2.23) holds with u replaced by $\omega_i - \bar{\omega}$. Passing to the limit $i \to \infty$, (2.22) follows. □

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By an abuse of terminology, the couple $(\widetilde{U}, \tilde{\omega})$ constructed in Proposition 2.1 will be referred to as an *extreme Kerr solution*; this is justified when there is only one singular puncture.

Let $(\widetilde{U}, \tilde{\omega})$ be the functions U and ω given by Proposition 2.1 with the same value of $\tilde{\omega}$ on the axis as the map (U, ω) under consideration, so that

$$(\omega - \tilde{\omega})|_{\mathcal{A}} = 0. \tag{2.26}$$

We will show that a lower bound on the action can be obtained by working in the class of U's of the form (2.15). For this let $\delta > 0$ be small, we start by deforming (U, ω) to a pair $(\check{U}_\delta, \check{\omega}_\delta)$ with the following properties:

1. Away from balls centered at the punctures $\vec{a}_i$ of radius $C\sqrt{\delta}$, for an appropriate constant C, the new pair of functions $(\check{U}_\delta, \check{\omega}_\delta)$ coincides with the original one (U, ω).
2. For $r_i < \delta$ the pair $(\check{U}_\delta, \check{\omega}_\delta)$ coincides with $(\ln r, \tilde{\omega})$, where $\tilde{\omega}$ is the function ω of Proposition 2.1.
3. The action I calculated for $(\check{U}_\delta, \check{\omega}_\delta)$, which we denote by $\check{I}_\delta$, is smaller than the action I calculated for the original solution, except perhaps for an error which tends to zero as δ tends to zero[3].

This can be done as follows: (2.14) shows that for all $0 < \delta < 1$ small enough the equation

$$U(r_i, \theta) = \ln \delta$$

has a solution $r_i = \mathring{r}_i(\theta, \delta) \approx \sqrt{\delta}$ satisfying

$$\delta \leqslant \mathring{r}_i \leqslant \epsilon := C\sqrt{\delta},$$

for a large constant C. We let $\check{U}_\delta$ to be equal to U away from a collection of non-overlapping balls centered at the punctures, where we set

$$\check{U}_\delta(r_i, \theta) := \begin{cases} \ln r_i, & r_i \leqslant \delta; \\ \ln \delta, & \delta \leqslant r_i \leqslant \mathring{r}_i(\theta, \delta); \\ U(r_i, \theta), & r_i \geqslant \mathring{r}_i(\theta, \delta). \end{cases} \tag{2.27}$$

Then $\check{U}_\delta$ is continuous, piecewise differentiable, hence in H^1_{loc}. Now,

$$\begin{aligned} \int_{\mathbb{R}^3} |D\check{U}_\delta|^2 &= \int_{\cup_i \{0 < r_i \leqslant \delta\}} \underbrace{|D\check{U}_\delta|^2}_{r_i^{-2}} + \underbrace{\int_{\cup_i \{\delta < r_i \leqslant \mathring{r}_i\}} |D\check{U}_\delta|^2}_{0} + \int_{\cap_i \{\mathring{r}_i < r_i\}} |D\check{U}_\delta|^2 \\ &= 4\pi(N-1)\delta + \int_{\cap_i \{\mathring{r}_i < r_i\}} |DU|^2, \end{aligned}$$

(recall that there are N asymptotically flat ends, hence $N - 1$ punctures). On the other hand

[3] We expect $\check{I}_\delta$ to be strictly smaller than I, but this is apparent from our proof for small angular momenta only.

$$
\begin{aligned}
\int_{\mathbb{R}^3} |DU|^2 &= \int_{\cup_i\{0<r_i\leqslant\delta\}} |DU|^2 + \int_{\cup_i\{\delta<r_i\leqslant\mathring{r}_i\}} |DU|^2 + \int_{\cap_i\{\mathring{r}_i<r_i\}} |DU|^2 \\
&\geqslant \int_{\cup_i\{0<r_i\leqslant\delta\}} \underbrace{|DU|^2}_{\geqslant(\partial_r U)^2} + \int_{\cap_i\{\mathring{r}_i<r_i\}} |DU|^2 \\
&\geqslant \int_{\cup_i\{0<r_i\leqslant\delta\}} r_i^{-2}(4+o(1)) + \int_{\cap_i\{\mathring{r}_i<r_i\}} |DU|^2 \\
&= 16\pi(N-1)\delta + o(\delta) + \int_{\cap_i\{\mathring{r}_i<r_i\}} |DU|^2 \\
&= 12\pi(N-1)\delta + o(\delta) + \int_{\mathbb{R}^3} |D\check{U}_\delta|^2.
\end{aligned} \tag{2.28}
$$

It clearly follows for all δ small enough that the first term in I will be decreased when U is replaced by $\check{U}_\delta$.

It remains to check that the possible increase of the second term in I can be controlled uniformly in δ. For this we need to understand the behavior of ω near the axis. It is convenient to rewrite (2.7) as

$$\partial_z\omega = -e^{-3U+\alpha}\rho^2 K_{13}, \quad \partial_\rho\omega = e^{-3U+\alpha}\rho^2 K_{23}. \tag{2.29}$$

Condition (2.4) implies that there exists a constant $\widehat{C}$ such that in each asymptotically flat region we have

$$|D\omega|_\delta \leqslant \widehat{C}\rho^2 r^{-\lambda}. \tag{2.30}$$

Let $(x^A) = (\rho, z)$ be the symmetry-adapted coordinates which extend to infinity in the i'th asymptotic region. Performing an inversion (compare [2])

$$y^A - a_i^A = x^A/|x|^2, \tag{2.31}$$

near each puncture $\vec{a}_i = (0, a_i) = (a_i^A)$ we obtain

$$|D\omega|_\delta \leqslant \widehat{C}\rho^2 r_i^{\lambda-6}. \tag{2.32}$$

This shows that

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{e^{4U}}{\rho^4}|D\omega|^2 &= \int_{\cup_i\{0\leqslant r_i\leqslant C\sqrt{\delta}\}} \underbrace{\frac{e^{4U}}{\rho^4}}_{\leqslant C' r_i^8\rho^{-4}} \underbrace{|D\omega|^2}_{\leqslant(\widehat{C}\rho^2 r_i^{\lambda-6})^2} + \int_{\cap_i\{r_i\geqslant C\sqrt{\delta}\}} \frac{e^{4U}}{\rho^4}|D\omega|^2 \\
&= O(\delta^{\lambda-1/2}) + \int_{\cap_i\{r_i\geqslant C\sqrt{\delta}\}} \frac{e^{4U}}{\rho^4}|D\omega|^2 \\
&= o(\delta^2) + \int_{\cap_i\{r_i\geqslant C\sqrt{\delta}\}} \frac{e^{4U}}{\rho^4}|D\omega|^2.
\end{aligned} \tag{2.33}
$$

Next, let $\check{\omega}_\delta$ be equal to ω away from a collection of non-overlapping balls centered at the punctures, while in those balls we set

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$$\check{\omega}_\delta(r_i,\theta) := \begin{cases} \omega(r_i,\theta), & r_i \geqslant C\sqrt{\delta}; \\ \omega(r_i,\theta)\dfrac{\ln\left(\frac{r_i}{\delta}\right)}{\ln\left(\frac{C\sqrt{\delta}}{\delta}\right)} + \tilde{\omega}(r_i,\theta)\dfrac{\ln\left(\frac{r_i}{C\sqrt{\delta}}\right)}{\ln\left(\frac{\delta}{C\sqrt{\delta}}\right)}, & \delta \leqslant r_i \leqslant C\sqrt{\delta}; \\ \tilde{\omega}(r_i,\theta), & r_i \leqslant \delta. \end{cases}$$

We have

$$\int_{\mathbb{R}^3} \frac{e^{4\check{U}_\delta}}{\rho^4}|D\check{\omega}_\delta|^2 = \int_{\cup_i\{0\leqslant r_i\leqslant\delta\}} \frac{r_i^4}{\rho^4}|D\tilde{\omega}|^2 + \underbrace{\int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{e^{4\check{U}_\delta}}{\rho^4}|D\check{\omega}_\delta|^2}_{=:A}$$
$$+ \int_{\cap_i\{r_i\geqslant C\sqrt{\delta}\}} \frac{e^{4U}}{\rho^4}|D\omega|^2.$$

The first term goes to zero as δ goes to zero because $(\widetilde{U},\tilde{\omega})$ has finite action. We claim that A goes to zero as δ goes to zero as well, this requires some work. For $\delta \leqslant r_i \leqslant C\sqrt{\delta}$ we rewrite $\check{\omega}_\delta$ as

$$\check{\omega}_\delta(r_i,\theta) = (\omega-\bar{\omega})(r_i,\theta)\frac{\ln(\frac{r_i}{\delta})}{\ln(\frac{C\sqrt{\delta}}{\delta})} + (\tilde{\omega}-\bar{\omega})(r_i,\theta)\frac{\ln(\frac{r_i}{C\sqrt{\delta}})}{\ln(\frac{\delta}{C\sqrt{\delta}})} + \bar{\omega}(r_i,\theta),$$

where $\bar{\omega}$ is as in the proof of Proposition 2.1. By (2.27) and (2.14)

$$e^{4\check{U}_\delta}(r_i,\theta) = \begin{cases} \delta^4, & \delta \leqslant r_i \leqslant \mathring{r}_i(\theta,\delta); \\ e^{4U(r_i,\theta)} \leqslant C' r_i^8 \leqslant C''\delta^4, & \mathring{r}_i(\theta,\delta) \leqslant r_i \leqslant C\sqrt{\delta}. \end{cases} \tag{2.34}$$

Hence, for some constant C_2,

$$C_2^{-1}A \leqslant \int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{\delta^4}{\rho^4}(|D\omega|^2 + |D\bar{\omega}|^2 + |D\tilde{\omega}|^2) + \int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{\delta^4}{\rho^4}\frac{(\omega-\bar{\omega})^2}{r_i^2}$$
$$+ \int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{\delta^4}{\rho^4}\frac{(\tilde{\omega}-\bar{\omega})^2}{r_i^2}. \tag{2.35}$$

The integral involving $D\bar{\omega}$ goes to zero as δ goes to zero because

$$\int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{\delta^4}{\rho^4}|D\bar{\omega}|^2 \leqslant C_3 \int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{e^{4\overline{U}}}{\rho^4}|D\bar{\omega}|^2, \tag{2.36}$$

while $(\bar{U},\bar{\omega})$ has finite action; similarly for that involving $D\tilde{\omega}$. The integral involving $D\omega$ goes to zero by direct estimation using (2.32). Next, using Proposition 2.4 with $g \equiv 1$ and $\mu = 2$, we can write

$$\int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{\delta^4}{\rho^4}\frac{(\tilde{\omega}-\bar{\omega})^2}{r_i^2} \leqslant \int_{\cup_i\{\delta\leqslant r_i\leqslant C\sqrt{\delta}\}} \frac{r_i^4}{\rho^6}(\tilde{\omega}-\bar{\omega})^2 \leqslant \int_{\cup_i\{r_i\leqslant C\sqrt{\delta}\}} \frac{r_i^4}{\rho^6}(\tilde{\omega}-\bar{\omega})^2$$
$$\leqslant \int_{\cup_i\{r_i\leqslant C\sqrt{\delta}\}} \frac{r_i^4}{\rho^4}|D(\tilde{\omega}-\bar{\omega})|^2.$$

The right-hand-side is integrable over the set $\cup_i\{r_i \leqslant \epsilon\}$ as in (2.36), and thus goes to zero as δ does, hence also the left-hand-side.

Consider, finally, the integral in the second line of (2.35). It is convenient to split the integration region into two, according to whether or not $|z-a_i| \leqslant \rho$. In the region

$$\mathcal{V}_2 := \{\delta \leqslant r_i \leqslant C\sqrt{\delta},\ |z - a_i| \leqslant \rho\},$$

the function ρ is equivalent to r_i, while both ω and $\bar{\omega}$ are bounded there. This gives the straightforward estimate

$$\int_{\mathcal{V}_2} \frac{\delta^4}{\rho^4} \frac{(\bar{\omega} - \omega)^2}{r_i^2} = O(\delta).$$

In the region

$$\mathcal{V}_1 := \{\delta \leqslant r_i \leqslant C\sqrt{\delta},\ |z - a_i| \geqslant \rho\}$$

the function $|z - a_i|$ is equivalent to r_i. Both ω and $\bar{\omega}$ satisfy (2.32). By integration along rays within the planes $z =$ const from each connected component $\mathcal{A}_j$ of the axis we obtain in $\mathcal{V}_1$, for $r_i \leqslant \epsilon$ small enough and $z \neq a_i$,

$$|\omega - \bar{\omega}| \leqslant \widehat{C}\rho^3 |z - a_i|^{\lambda - 6}. \tag{2.37}$$

Integration over $\mathcal{V}_1$ gives

$$\int_{\mathcal{V}_1} \frac{\delta^4}{\rho^4} \frac{(\bar{\omega} - \omega)^2}{r_i^2} = O(\delta^{2\lambda - 5}),$$

which goes to zero by our hypothesis that $\lambda > 5/2$.

Summarizing all this,

$$I \geqslant \check{I}_\delta - o(1), \tag{2.38}$$

where $o(1)$ goes to zero as δ does. Smoothing out $(\check{U}_\delta, \check{\omega}_\delta)$ at the corners, we can further assume that $(\check{U}_\delta, \check{\omega}_\delta)$ is smooth without affecting (2.38).

To continue, we show that $\check{I}_\delta \geqslant \widetilde{I}$, where $\widetilde{I}$ is the action corresponding to the map of Proposition 2.1. In order to prove this, let $\mathring{\varepsilon} > 0$ be such that the balls of radius $\mathring{\varepsilon}$ centered at the punctures do not overlap, and note that there exists a large constant $\mathring{C}$ such that both $(\check{U}_\delta, \check{\omega}_\delta)$ and $(\widetilde{U}, \tilde{\omega})$ belong to the class $\mathcal{F}_{\mathring{C}}$ of maps (U, ω) defined as follows:

1. $\int_{\mathbb{R}^3} \left(|DU|^2 + \frac{e^{4U}}{\rho^4} |D\omega|^2 \right) \leqslant \mathring{C}$;
2. For $0 < r_i < \mathring{\varepsilon}$ we have $|U - \ln r_i| \leqslant \mathring{C}$;
3. ω coincides with $\tilde{\omega}$ near the punctures;
4. $\omega = \omega_j$ on $\mathcal{A}_j$.

It also follows from [2] that the map (U, ω) of Theorem 1.1 satisfies the following

1. For $0 < r_i < \mathring{\varepsilon}$ we have $|DU|_\delta = o(r_i^{-1/2})$;
2. for $0 < \rho < \mathring{\varepsilon}$, we have,

$$|D\omega|_\delta \leqslant \mathring{C} \rho^2 \sum_i (r_i^{\lambda - 6} + r_i^{-\lambda}),$$

   for some $5/2 < \lambda \leqslant 3$ and for each i, where the first term $r_i^{\lambda-6}$ accounts for small r_i behavior, while the second one $r_i^{-\lambda}$ accounts for large r's;
3. We have $U = o(r^{-1/2})$ and $|DU| = o(r^{-3/2})$ as r tends to infinity.

(this is also true for $(\widetilde{U},\tilde{\omega})$ when $N=2$, and is expected to be true without this restriction on N, but such an estimate has not been established so far).

Dropping the subscript $(\check{U}_\delta,\check{\omega}_\delta)$ for notational simplicity, we wish to show that the corresponding action $\check{I}$ is larger than or equal to that of $(\widetilde{U},\tilde{\omega})$. In order to see that, let $\eta>0$ and let $\varphi_\eta\in C^\infty(\mathbb{R}^3)$ be any family of functions satisfying

- $0\leqslant\varphi_\eta\leqslant 1$;
- $\varphi_\eta=0$ for $0\leqslant r_i\leqslant\eta/2$ and for $r_i\geqslant 2/\eta$;
- $\varphi_\eta=1$ on the set $\mathcal{W}_\eta$, where

  $$\mathcal{W}_\lambda := \{r\leqslant 1/\lambda\}\cap_i\{r_i\geqslant\lambda\};$$
- $|D\varphi_\eta|\leqslant C/\eta$ for $\eta/2\leqslant r_i\leqslant\eta$; and
- $|D\varphi_\eta|\leqslant C\eta$ for $1/\eta\leqslant r\leqslant 2/\eta$.

Set

$$U_\eta=\varphi_\eta\check{U}+(1-\varphi_\eta)\widetilde{U},$$
$$\omega_\eta=\varphi_\eta\check{\omega}+(1-\varphi_\eta)\tilde{\omega}.$$

Note that, by definition of $\check{\omega}$, we have $\omega_\eta=\tilde{\omega}$ near the punctures for η small enough. Using again $\check{I}$ denote the value of the action I for $(\check{U},\check{\omega})$, we claim that the action I_η of (U_η,ω_η) satisfies

Lemma 2.5. $\lim_{\eta\to 0}I_\eta=\check{I}$.

Proof. Indeed, we have

$$\int_{\mathbb{R}^3}|DU_\eta|^2=\underbrace{\int_{\cup_i\{0\leqslant r_i\leqslant\eta/2\}}|D\widetilde{U}|^2}_{I}+\underbrace{\int_{\cup_i\{\eta/2\leqslant r_i\leqslant\eta\}}|DU_\eta|^2}_{II}$$
$$+\underbrace{\int_{\cap_i\{\eta\leqslant r_i,\ r\leqslant 1/\eta\}}|D\check{U}|^2}_{III}+\underbrace{\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}}|DU_\eta|^2}_{IV}+\underbrace{\int_{\{2/\eta\leqslant r\}}|D\widetilde{U}|^2}_{V}.$$

The integrals I and V converge to zero by the dominated convergence theorem, while III converges to the integral over $\mathbb{R}^3$ of $|D\check{U}|^2$ by, e.g., the monotone convergence theorem. The term IV can be handled as follows:

$$\begin{aligned}\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}}|DU_\eta|^2&=\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}}|(\check{U}-\widetilde{U})D\varphi_\eta+\varphi_\eta D\check{U}+(1-\varphi_\eta)D\widetilde{U}|^2\\&\leqslant 3\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}}(|(\check{U}-\widetilde{U})D\varphi_\eta|^2+|D\check{U}|^2+|D\widetilde{U}|^2)\\&\leqslant 3\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}}(C(\check{U}-\widetilde{U})^2r^{-2}+|D\check{U}|^2+|D\widetilde{U}|^2).\end{aligned}$$

The second and third term go to zero by the Lebesgue dominated convergence theorem. Letting $(\bar{U},\bar{\omega})$ be as in the proof of Proposition 2.1, for small η the first term can be estimated as follows:

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$$\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} r^{-2}(\check{U}-\widetilde{U})^2 = \int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} r^{-2}(U-\widetilde{U})^2$$
$$\leqslant 2\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} r^{-2}(2U^2+2\bar{U}^2+(\bar{U}-\widetilde{U})^2).$$

The first two integrals tend to zero by direct estimations. As $\bar{U}-\widetilde{U}$ is the limit of compactly supported functions the weighted Poincaré inequality applies to the third term, implying that the function $r^{-2}(\bar{U}-\widetilde{U})^2$ is in L^1. The vanishing of the limit of IV as η goes to zero follows now from the Lebesgue dominated convergence theorem.

The analysis of II is identical.

A similar analysis applies to the remaining integral in I_η. The only delicate term is

$$\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{e^{4U_\eta}}{\rho^4}|D\omega_\eta|^2 \leqslant \int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} |D\omega_\eta|^2$$
$$\leqslant \int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{C}{\rho^4}|(\omega-\tilde{\omega})D\varphi_\eta+\varphi_\eta D\omega+(1-\varphi_\eta)D\tilde{\omega}|^2$$
$$\leqslant 3\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{C}{\rho^4}(|(\omega-\tilde{\omega})D\varphi_\eta|^2+|D\omega|^2+|D\tilde{\omega}|^2)$$
$$\leqslant 3\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{C}{\rho^4}(C(\omega-\tilde{\omega})^2r^{-2}+|D\omega|^2+|D\tilde{\omega}|^2).$$

The last two terms go to zero as before. The first can be estimated as

$$\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{1}{\rho^4r^2}(\omega-\tilde{\omega})^2 \leqslant 2\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{1}{\rho^4r^2}((\omega-\bar{\omega})^2+(\bar{\omega}-\tilde{\omega})^2).$$

The first term goes to zero by direct estimations. The weighted Poincaré inequality (2.22) with $\mu=1$ and $g(r)=r^{-4}$ applies to the last term, giving

$$\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{1}{\rho^4r^2}(\bar{\omega}-\tilde{\omega})^2 \leqslant \int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{1}{r^4\sin^2\theta}|D(\bar{\omega}-\tilde{\omega})|^2$$
$$\leqslant \int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \frac{1}{\rho^4}|D(\bar{\omega}-\tilde{\omega})|^2$$
$$\leqslant C\int_{\{1/\eta\leqslant r\leqslant 2/\eta\}} \left(\frac{e^{4\bar{U}}}{\rho^4}|D\bar{\omega}|^2+\frac{e^{4\widetilde{U}}}{\rho^4}|D\tilde{\omega}|^2\right),$$

and the right-hand-side goes to zero in the limit. □

The next step is to prove

Lemma 2.6. *$I_\eta \geqslant \widetilde{I}$ for all η small enough.*

Proof. For all η small the maps (U_η,ω_η) and $(\widetilde{U},\tilde{\omega})$ coincide on balls of radius $\eta/2$ around the punctures, as well as on the complement of a ball of radius $2/\eta$. One would like to use a result of [6], that the action I is minimized by the solution of the Dirichlet problem, which is expected to be $(\widetilde{U},\tilde{\omega})$; however, that result does not apply directly because of the singularity of the equations at the axis $\rho=0$; moreover, we are working in an unbounded domain. To take care of that, for $\epsilon<1$ let

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$$\hat{\varphi}_\epsilon = \begin{cases} 0, & \rho \leqslant \epsilon; \\ \dfrac{\ln \frac{\rho}{\epsilon}}{\ln \frac{\sqrt{\epsilon}}{\epsilon}}, & \epsilon \leqslant \rho \leqslant \sqrt{\epsilon}; \\ 1, & \rho \geqslant \sqrt{\epsilon}. \end{cases}$$

Set

$$U_{\eta,\epsilon} = \hat{\varphi}_\epsilon U_\eta + (1-\hat{\varphi}_\epsilon)\widetilde{U}, \qquad \omega_{\eta,\epsilon} = \hat{\varphi}_\epsilon \omega_\eta + (1-\hat{\varphi}_\epsilon)\tilde{\omega}.$$

Let $I_{\eta,\epsilon}$ denote the action of $(U_{\eta,\epsilon}, \omega_{\eta,\epsilon})$. We claim that

$$\int_{\{\rho\leqslant\sqrt{\epsilon}\}} \left[(DU_{\eta,\epsilon})^2 + \frac{e^{4U_{\eta,\epsilon}}}{\rho^4} (D\omega_{\eta,\epsilon})^2 \right] d^3x \to_{\epsilon\to 0} 0. \tag{2.39}$$

Equivalently,

$$I_{\eta,\epsilon} \to_{\epsilon\to 0} I_\eta. \tag{2.40}$$

In order to see this, note that, for $\epsilon \leqslant \eta/2$, the integrand of (2.39) is non-zero only away from balls of radius $\eta/2$ centered at the punctures, with moreover $r \leqslant 2/\eta$. Next, the integral over the set $\{0 \leqslant \rho \leqslant \epsilon\}$ approaches zero as ϵ tends to zero by the Lebesgue dominated convergence theorem. So it remains to consider the integral over

$$\mathcal{W}_{\eta,\epsilon} := \{\epsilon \leqslant \rho \leqslant \sqrt{\epsilon}\} \cap \{r \leqslant 2/\eta\} \cap_i \{r_i \geqslant \eta/2\},$$

which can be handled as follows:

$$\begin{aligned} \int_{\mathcal{W}_{\eta,\epsilon}} (DU_{\eta,\epsilon})^2 &= \int_{\mathcal{W}_{\eta,\epsilon}} \Bigg(\underbrace{(U_\eta - \widetilde{U}) D\hat{\varphi}_\epsilon}_{\leqslant C(\rho|\ln\epsilon|)^{-1}} + \hat{\varphi}_\epsilon DU_\eta + (1-\hat{\varphi}_\epsilon) D\widetilde{U} \Bigg)^2 \\ &\leqslant 3 \int_{\mathcal{W}_{\eta,\epsilon}} \left(\frac{C^2}{\rho^2 |\ln\epsilon|^2} + |DU_\eta|^2 + |D\widetilde{U}|^2 \right) \\ &\leqslant O(\frac{1}{|\ln\epsilon|}) + 3 \int_{0\leqslant\rho\leqslant\epsilon, r\leqslant 2/\eta} (|DU_\eta|^2 + |D\widetilde{U}|^2). \end{aligned}$$

The integral in the last line goes to zero by the dominated convergence theorem.

The analysis of the term containing the derivatives of $\omega_{\eta,\epsilon}$ is similar, using (2.32), (2.37), and Proposition 2.4 with $\mu = 4$ and $g(r) = r^2$:

$$\begin{aligned} \int_{\mathcal{W}_{\eta,\epsilon}} \frac{e^{4U_{\eta,\epsilon}}}{\rho^4} (D\omega_{\eta,\epsilon})^2 &\leqslant \int_{\mathcal{W}_{\eta,\epsilon}} \frac{C(\eta)}{\rho^4} ((\omega_\eta - \tilde{\omega}) D\hat{\varphi}_\epsilon + \hat{\varphi}_\epsilon D(\omega_\eta - \tilde{\omega}) + D\tilde{\omega})^2 \\ &\leqslant C'(\eta) \int_{\mathcal{W}_{\eta,\epsilon}} \frac{1}{\sin^6\theta |\ln\epsilon|} ((\omega - \bar{\omega})^2 + (\bar{\omega} - \tilde{\omega})^2) \\ &\quad + \frac{1}{\sin^4\theta} (|D\omega|^2 + |D\tilde{\omega}|^2) \leqslant o(\epsilon) + CC'(\eta) \\ &\quad \times \int_{\mathcal{W}_{\eta,\epsilon}} \frac{1}{\sin^4\theta} (|D\omega|^2 + |D\bar{\omega}|^2 + |D\tilde{\omega}|^2) \to_{\epsilon\to 0} 0. \end{aligned}$$

This ends the proof of (2.39).

Now, $(U_{\eta,\epsilon}, \omega_{\eta,\epsilon})$ coincides with $(\widetilde{U}, \tilde{\omega})$ on the set $\{\rho \leqslant \epsilon\}$, *and* on balls of radius $\eta/2$ around the punctures, *and* on the complement of a ball of radius $2/\eta$. Further, after shifting U by $\ln\rho$, the variational equations associated with the action I are the harmonic map equations, with target space — the two-dimensional hyperbolic space. Hence the target manifold satisfies the convexity conditions of [6] (see Remark (i), p. 5 there). We can thus conclude from [6] that action minimizers with Dirichlet boundary conditions exist, are smooth, and satisfy the variational equations. It is also well known that solutions of the Dirichlet boundary value problem are unique when the target manifold has negative sectional curvatures. All this implies that $(\widetilde{U}, \tilde{\omega})$, with its own boundary data, minimizes the action integral over the set

$$\{\rho \geqslant \epsilon\} \cap \{r \leqslant 2/\eta\} \cap_i \{r_i \geqslant \eta/2\}.$$

Hence

$$\int_{\{\rho\geqslant\epsilon\}} \left[(DU_{\eta,\epsilon})^2 + \frac{e^{4U_{\eta,\epsilon}}}{\rho^4} (D\omega_{\eta,\epsilon})^2 \right] d^3x \geqslant \int_{\{\rho\geqslant\epsilon\}} \left[(D\widetilde{U})^2 + \frac{e^{4\widetilde{U}}}{\rho^4} (D\tilde{\omega})^2 \right] d^3x.$$

By the monotone convergence theorem we have

$$\int_{\{\rho\geqslant\epsilon\}} \left[(D\widetilde{U})^2 + \frac{e^{4\widetilde{U}}}{\rho^4} (D\tilde{\omega})^2 \right] d^3x \rightarrow_{\epsilon\rightarrow 0} \widetilde{I},$$

so that

$$I_\eta = \lim_{\epsilon\rightarrow 0} \int_{\{\rho\geqslant\epsilon\}} \left[(DU_{\eta,\epsilon})^2 + \frac{e^{4U_{\eta,\epsilon}}}{\rho^4} (D\omega_{\eta,\epsilon})^2 \right] d^3x \geqslant \lim_{\epsilon\rightarrow 0} \int_{\{\rho\geqslant\epsilon\}} \left[(D\widetilde{U})^2 + \frac{e^{4\widetilde{U}}}{\rho^4} (D\tilde{\omega})^2 \right] d^3x$$
$$= \widetilde{I}.$$

Returning to the proof of Theorem 1.1, Lemmas 2.5 and 2.6 applied to $(\check{U}, \check{\omega}) = (\check{U}_\delta, \check{\omega}_\delta)$ give $\check{I}_\delta \geqslant \widetilde{I}$. Passing to the limit $\delta \rightarrow 0$ we obtain, by (2.38), $I \geqslant \widetilde{I}$. In the case of two asymptotic regions one concludes by noting, following [3], that

$$\widetilde{I} = \sqrt{|\vec{J}|}. \qquad \square$$

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Appendix A. Kerr solutions

The Kerr black holes provide an explicit family of solutions of the singular harmonic map equations, as follows: In Boyer–Lindquist coordinates, which are denoted by $(t, \tilde{r}, \theta, \varphi)$, the metrics take the form

$$g = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 + \frac{4ma\tilde{r}\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{(\tilde{r}^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}$$
$$\times \sin^2\theta\mathrm{d}\varphi^2 + \frac{\Sigma}{\Delta}\mathrm{d}\tilde{r}^2 + \Sigma\mathrm{d}\theta^2. \tag{A.1}$$

Here

$$\Sigma = \tilde{r}^2 + a^2\cos^2\theta, \qquad \Delta = \tilde{r}^2 + a^2 - 2m\tilde{r} = (\tilde{r} - r_+)(\tilde{r} - r_-),$$

and $r_+ < \tilde{r} < \infty$, where

$$r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}}.$$

The function ω reads [4]

$$\omega = J(\cos^3\theta - 3\cos\theta) - \frac{ma^3\cos\theta\sin^4\theta}{\Sigma}. \tag{A.2}$$

The constant m is of course the total mass and $a = J/m$, where J is the angular momentum. Note that the leading order term in ω is uniquely determined by J.

We relate now $\tilde{r}$ and θ to ρ and z. If $|a| \leqslant m$ let $r_+ = m + \sqrt{m^2 - a^2}$ be the largest root of Δ, and let $r_+ = 0$ otherwise. For

$$\tilde{r} > \tilde{r}_+,$$

so that $\Delta > 0$, define a new radial coordinate r by

$$r = \frac{1}{2}\left(\tilde{r} - m + \sqrt{\Delta}\right); \tag{A.3}$$

this has been tailored [4] so that after setting

$$\rho = r\sin\theta, \quad z = r\cos\theta, \tag{A.4}$$

the space-part of the space-time metric takes the form

$$g = \mathrm{e}^{-2\widetilde{U}+2\alpha}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{e}^{-2\widetilde{U}}(\mathrm{d}\varphi + \rho B_\rho\mathrm{d}\rho + A_z\mathrm{d}z)^2. \tag{A.5}$$

We have

$$\tilde{r} = r + m + \frac{m^2 - a^2}{4r}. \tag{A.6}$$

We emphasize that while those coordinates bring the metric to the form (A.5), familiar in the context of the reduction of the stationary axi-symmetric vacuum Einstein equations to a harmonic map problem, the coordinate ρ in (A.4) is *not* the area coordinate needed for that reduction[4] *except* when $m = a$.

To analyze the behavior near $r = 0$ we have to distinguish between the extreme and non-extreme cases. Let us first assume that $m^2 \neq a^2$, then we have

$$\widetilde{U} = 2\ln\left(\frac{2r}{m}\right) - \ln\left|1 - \frac{a^2}{m^2}\right| + O(r). \tag{A.7}$$

On the other hand, in the extreme case $m^2 = a^2$ it holds that

[4] The correct (ρ, z) coordinates for the harmonic map reduction are $\rho = \sqrt{\Delta}\sin\theta$, $z = (\tilde{r} - m)\cos\theta$.

$$\widetilde{U} = \ln\left(\frac{r}{2m}\right) + \frac{1}{2}\ln(1+\cos^2\theta) + O(r). \tag{A.8}$$

Furthermore, for $m = \pm a$, for small r,

$$\omega = \mp 4m^2 \frac{\cos\theta}{1+\cos^2\theta} + O(r). \tag{A.9}$$

We will also need derivative estimates for ω: from (A.2) we obtain the uniform estimates

$$|\partial_\theta \omega| \leqslant C\sin^3\theta = C\frac{\rho^3}{r^3}, \quad |\partial_r \omega| \leqslant C\frac{r\sin^4\theta}{1+r^4} = C\frac{\rho^4}{r^3(1+r^4)}, \tag{A.10}$$

so that, for $r \leqslant 1$,

$$|D\omega|_\delta = \sqrt{(\partial_r\omega)^2 + r^{-2}(\partial_\theta\omega)^2} \leqslant C\frac{\rho^3}{r^4}. \tag{A.11}$$

We claim that away from the plane $z = 0$ we have the uniform estimate

$$|\omega - \omega_i| \leqslant C\sin^4\theta. \tag{A.12}$$

where $\omega_1 = -2J$ is the value appropriate in the half-space $\{z > 0\}$, while $\omega_2 = 2J$ is the one which should be used for $z < 0$. This is clear for the last term in (A.2); for the first one this follows immediately from the following identity, obtained by expanding $\cos^3\theta = (\cos\theta - \cos(n\pi) + \cos(n\pi))^3$ with $n = 0, 1$:

$$\cos^3\theta - 3\cos\theta = -2\cos(n\pi) + (\cos\theta - \cos(n\pi))^2(2\cos(n\pi) + \cos\theta),$$

Appendix B. Asymptotic equations near a degenerate horizon

The maps $(\widetilde{U}, \tilde{\omega})$ given by Proposition 2.1 are axially symmetric (i.e., invariant under rotations around the z-axis) solutions of the variational equations for the action

$$\widetilde{I} := \frac{1}{8\pi}\int_{\mathbb{R}^3}\left[(D\widetilde{U})^2 + \frac{e^{4\widetilde{U}}}{\rho^4}(D\tilde{\omega})^2\right]d^3x. \tag{B.1}$$

where D is the usual gradient of the flat Euclidean metric on $\mathbb{R}^3$. Thus the equations read

$$\Delta\widetilde{U} = 2\frac{e^{4\widetilde{U}}}{\rho^4}(D\tilde{\omega})^2, \tag{B.2}$$

$$D_i\left(\frac{e^{4\widetilde{U}}}{\rho^4}D^i\tilde{\omega}\right) = 0. \tag{B.3}$$

One expects that

$$\widetilde{U} = \ln r + \mathring{U}(\theta) + O(r), \quad \tilde{\omega} = \mathring{\omega}(\theta) + O(r),$$

where the remainder terms are bounded under r—or θ— differentiation.[5] Inserting this into the Eqs. (B.2) and (B.3) it must hold

[5] In fact, assuming the weaker condition that any error term, say $\epsilon(r,\theta)$, has the property that $\partial_\theta\epsilon = O(r), \partial_r\epsilon = O(1), \partial_r^2\epsilon = O(1/r), \partial_\theta^2\epsilon = O(r)$, leads to error terms $O(r)$ in (B.4) and (B.5), and to errors $O(1/r)$ in (B.2) and (B.3) when (B.4) and (B.5) hold.

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta \mathring{U}) = -1 + 2\sin^{-4}\theta e^{4\mathring{U}}(\partial_\theta \mathring{\omega})^2, \tag{B.4}$$

$$\partial_\theta\left(\sin^{-3}\theta e^{4\mathring{U}}\partial_\theta \mathring{\omega}\right) = 0. \tag{B.5}$$

Elementary ODE theory shows that these equations admit a four-parameter family of solutions, some of which might fail to be regular everywhere. Note that at this stage the question of differentiality of the solutions at the axis is open, and any such regularity in our context would need a careful justification. (Recall, however, that for the solutions we are considering the functions $\mathring{U}$ and $\mathring{\omega}$ are bounded.)

Solving (B.5) for $\mathring{\omega}$ one finds that there exists a constant c such that

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta \mathring{U}) = -1 + 2c^2\sin^2\theta e^{-4\mathring{U}}. \tag{B.6}$$

All solutions with $c = 0$ have a constant $\mathring{\omega}$ and, for some constants $\alpha, \beta \in \mathbb{R}$,

$$\mathring{U} = \alpha + (1-\beta)\ln\sin\theta + \beta\ln(1-\cos\theta),$$

so that

$$\widetilde{U} = \ln\rho + \alpha + \beta(\ln(1-\cos\theta) - \ln\sin\theta).$$

This is not of the form we are looking for as the correction $\mathring{U}$ to $\ln r$ is never bounded.

Next, the addition of a constant to $\mathring{U}$ can be used to rescale the constant c to an arbitrary value; e.g., one can choose $2c^2 = 1$.

Finally, let $\alpha, \beta \in \mathbb{R}$; one readily checks that the couples $(\mathring{U}, \mathring{\omega})$ given by

$$\mathring{U} = \alpha + \frac{1}{2}\ln(1+\cos^2\theta), \tag{B.7}$$

$$\mathring{\omega} = \pm e^{-2\alpha}\frac{\cos\theta}{1+\cos^2\theta} + \beta, \tag{B.8}$$

satisfy (B.4) and (B.5). Note that

$$\partial_\theta \mathring{\omega} = \mp\frac{e^{-2\alpha}\sin^3\theta}{(1+\cos^2\theta)^2},$$

which exhibits clearly the high order of vanishing of $d\mathring{\omega}$ at the axis. This agrees with the functions arising from (A.8) to (A.9) if

$$\alpha = -\ln(2m).$$

According to Jezierski [7], the general solution of (B.6) with $c \neq 0$ can be parameterized by two real constants β and γ, with $\beta\gamma \neq 0$, as follows:

$$\mathring{U} = \frac{1}{2}\ln\left[\frac{c\left((1-\cos\theta)^\beta \pm \gamma^2(1+\cos\theta)^\beta\right)\sin^{2-\beta}\theta}{\beta\gamma}\right]. \tag{B.9}$$

In fact, it is straightforward though tedious to show that these functions do indeed solve (B.6). Further, within this family we have

$$\mathring{U}\left(\frac{\pi}{2}\right) = \frac{1}{2}\ln\left[\frac{c(1\pm\gamma^2)}{\beta\gamma}\right], \quad \partial_\theta \mathring{U}\left(\frac{\pi}{2}\right) = \frac{\beta(1\mp\gamma^2)}{2(1\pm\gamma^2)}.$$

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One checks that this can always be solved for β and γ in terms of $\mathring{U}(\frac{\pi}{2})$ and $\partial_\theta \mathring{U}(\frac{\pi}{2})$, showing that all solutions of (B.6) which are bounded near $\pi/2$ are given by (B.9). Note that the case $\beta < 0$ in (B.9) can be reduced to $\beta > 0$ by simple redefinitions. So, assuming $\beta \geqslant 0$, one checks that the only solutions which are uniformly bounded are the ones with $\beta = 2, \gamma > 0$, and with the plus sign chosen

$$\mathring{U} = \frac{1}{2} \ln \left[\frac{c((1-\cos\theta)^2 + \gamma^2(1+\cos\theta)^2)}{2\gamma} \right]. \tag{B.10}$$

One can then integrate (B.5) to obtain $\mathring{\omega}$. We thus conclude that there exists a two-parameter family of bounded solutions of (B.4) and (B.5), and that the extreme Kerr solutions provide only a one-parameter family thereof.

If one approximates $(\widetilde{U}, \tilde{\omega})$ by $(\ln r + \mathring{U}, \mathring{\omega})$, then (B.2) and (B.3) will be satisfied up to terms $O(1/r)$. Clear to the reader how to push the expansion one order higher to obtain a bounded tension map, but the details of this calculation have no interest.

In the notation of [9], the target space metric takes the form

$$h = X^{-2}(\mathrm{d}X^2 + \mathrm{d}Y^2). \tag{B.11}$$

In this parameterization, (A.8) and (A.9) can be rewritten as

$$(X, Y) = 4m^2\left(\frac{\sin^2\theta}{1+\cos^2\theta}, \frac{\mp 2\cos\theta}{1+\cos^2\theta}\right) + O(r). \tag{B.12}$$

In this context one could also look for solutions which depend only upon θ; the equations then read

$$\partial_\theta(\sin\theta X^{-2}\partial_\theta Y) = 0, \quad \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta X) = X^{-1}((\partial_\theta X)^2 - (\partial_\theta Y)^2).$$

A solution is given by

$$(X, Y) = (\sin\theta, \cos\theta). \tag{B.13}$$

Thus $|\mathrm{d}Y|_\delta = |\partial_\theta Y/r|$ equals ρ/r^2, as in the scaling estimate

$$|\mathrm{d}Y| \leqslant C\frac{\rho}{r^2},$$

and not better. However, the solution (B.13) *does not* behave like the extreme solutions we are looking for: U here equals $\ln r$ plus an angle-dependent correction as desired, but the latter blows-up badly at the axis.

Appendix C. On uniqueness of harmonic maps associated to black holes

It is expected that to a stationary "multi-black-hole" vacuum space-time one can associate a harmonic map which lies to a finite distance, in the hyperbolic target space, from a map with the following properties, modelled on a Kerr solution:

1. There exists $N_{\mathrm{dh}} \geqslant 0$ degenerate event horizons, which are represented by punctures $(\rho = 0,\ z = b_i)$, each of them labeled by a mass parameter $m_i > 0$ and angular momentum parameter $a_i = \pm m_i$, with the following behavior for small $r_i := \sqrt{\rho^2 + (z-b_i)^2}$,

$$U = \ln\left(\frac{r_i}{2m_i}\right) + \frac{1}{2}\ln\left(1 + \frac{(z-b_i)^2}{r_i^2}\right) + O(r_i). \tag{C.1}$$

The twist potential ω is a bounded, angle-dependent function which jumps by $4J_i$ when crossing b_i from $z < b_i$ to $z > b_i$, where J_i is the "angular momentum of the puncture".

2. There exists $N_{\rm ndh} \geqslant 0$ non-degenerate horizons, which are represented by bounded open intervals $I_i \subset \mathcal{A}$, with none of the previous b_j's belonging to the union of the closures of the I_i. The functions $U - 2\ln\rho$ and ω extend smoothly across each interval I_i, with the following behavior near the ends points, for some constant C:

$$|U - \frac{1}{2}\ln(\sqrt{\rho^2 + (z-c_i)^2} + z - c_i)| \leqslant C \quad \text{near} \quad (0, c_i). \tag{C.2}$$

The function ω is assumed to be constant near the c_i's. [6]

3. The functions U and ω are smooth across $\mathcal{A} \setminus (\cup_i\{b_i\}\cup_j I_j)$, with ω locally constant there.

As pointed out by Dain, and used in our work above, an alternative way of representing a non-degenerate Kerr black hole is provided by a map into the hyperbolic space, which is *not* harmonic, with a puncture on the symmetry axis corresponding to the second asymptotically flat region. This generalises naturally as follows:

4. There exists $N_{\rm AF}+1$ asymptotically flat regions, for some $N_{\rm AF} \geqslant 0$. A set of explicitly asymptotically Euclidean coordinates for the first asymptotic region is provided by $\vec{x} = (\rho\cos\varphi, \rho\sin\varphi, z)$, with $|\vec{x}|$ taking large values. The remaining asymptotic regions are represented by punctures $\vec{b}_i = (0, 0, b_i) \in \mathcal{A}$. If we set

$$r_i = \sqrt{\rho^2 + (z-b_i)^2},$$

then we have the following asymptotic behavior near each of the punctures, which corresponds to a Kelvin inversion of asymptotically flat coordinates of a Kerr solution with mass m_i and angular momentum parameter a_i,

$$U = 2\ln\left(\frac{2r_i}{m_i}\right) - \ln\left|1 - \frac{a_i^2}{m_i^2}\right| + O(r_i). \tag{C.3}$$

(see [2, Theorem 2.9]; compare Appendix A). The twist potential ω jumps by $4J_i$ when crossing b_i from $z < b_i$ to $z > b_i$.

The structure described in points 1-4 above will be referred to as the *axis data*. Thus, some of the punctures b_i correspond to degenerate horizons, while the remaining ones correspond to further asymptotically flat regions. Defining the distance between two maps Φ_1 and Φ_2 as

$$d(\Phi_1, \Phi_2) = \sup\nolimits_{p\in\mathbb{R}^3\setminus\mathcal{A}} d_b(\Phi_1(p), \Phi_2(p)),$$

where the distance d_b is taken with respect to the hyperbolic metric (2.18), we have the following generalisation of Proposition 2.1:

[6] For the Kerr solution the twist potential ω is of course not constant near the end points, but this simple condition is good enough for the purposes of Theorem C.1.

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Theorem C.1. *For any set of axis data there exists a unique harmonic map* $\Phi : \mathbb{R}^3 \setminus \mathcal{A} \to \mathcal{H}^2$ *which lies a finite distance from a solution with the singularity structure above, such that* $\omega = 0$ *on* $\mathcal{A}$ *for large positive z.*

Remark C.2. There does not seem to be any obvious relationship between the harmonic maps here with $N_{\mathrm{AF}} \neq 1$ and stationary vacuum black holes: we emphasise that the map Φ corresponding to *non-degenerate* Kerr black holes is *not* harmonic in conformal coordinates (ρ,z) in which the second asymptotically flat region is represented by a puncture.

Remark C.3. For $N_{\mathrm{AF}} = N_{\mathrm{dh}} = 0$ existence, and uniqueness under a supplementary H^1 condition, have been previously proved by Weinstein [10]. Similarly, uniqueness under again an additional H^1 condition for $N_{\mathrm{AF}} = N_{\mathrm{ndh}} = 0$, $N_{\mathrm{dh}} = 1$ has been proved by Dain [3].

Proof. Existence can be established by repeating the proof of Proposition 2.1. For uniqueness, a simple proof can be given as follows: Because of the negative sectional curvature of the target, the distance function $f(p) = d(\Phi_1(p),\Phi_2(p)) \geqslant 0$ is subharmonic on $\mathbb{R}^3 \setminus \mathcal{A}$. The vanishing of f follows then from Proposition C.4 below. □

Recall that $\mathcal{A}$ denotes the $z-$ axis. We have:

Proposition C.4. *Let* $f \in C^0(\mathbb{R}^3 \setminus \mathcal{A})$ *satisfy*

$$\Delta f \geqslant 0 \text{ in } \mathbb{R}^3 \setminus \mathcal{A}, \text{ in the distribution sense,} \tag{C.4}$$

$$0 \leqslant f \leqslant 1, \text{ on } \mathbb{R}^3 \setminus \mathcal{A}, \tag{C.5}$$

and

$$\lim_{(x,y,z)\in\mathbb{R}^3\setminus\mathcal{A},|(x,y,z)|\to\infty} f(x,y,z) = 0. \tag{C.6}$$

Then

$$f \equiv 0, \text{ on } \mathbb{R}^3 \setminus \mathcal{A}.$$

Proof of proposition 1. Given any $\epsilon > 0$, there exists, because of (C.6), some positive constant $R > 0$, such that

$$f(x,y,z) \leqslant \epsilon, \quad \forall\ (x,y,z) \in \mathbb{R}^3 \setminus \mathcal{A}, |(x,y,z)| \geqslant R. \tag{C.7}$$

For $0 < \delta < R$, let

$$D_\delta := \{(x,y,z) \mid |(x,y,z)| < R, |(x,y)| > \delta\}.$$

Define, on D_δ,

$$g_\delta(x,y,z) := \epsilon + \frac{\log(|(x,y)|/R)}{\log(\delta/R)}.$$

Clearly

$$g_\delta \geqslant \epsilon \text{ on } D_\delta.$$

In particular, in view of (C.7),

$$g_\delta(x,y,z) \geqslant \epsilon \geqslant f(x,y,z), \ \forall\ (x,y,z) \in \partial D_\delta \cap \{(x,y,z)\ |(x,y,z)| = R\}.$$

Using (C.5), we also have

$$g_\delta(x,y,z) = 1+\epsilon \geqslant f(x,y,z), \ \forall\ (x,y,z) \in \partial D_\delta \cap \{(x,y,z) \mid |(x,y)| = \delta\}.$$

Thus we have proved

$$g_\delta \geqslant f, \text{ on } \partial D_\delta.$$

Since g_δ is harmonic in D_δ, and f is subharmonic in D_δ, we have, in view of the above,

$$g_\delta \geqslant f, \quad \text{on } D_\delta.$$

Namely, for the ϵ and R,

$$\epsilon + \frac{\log(|(x,y)|/R)}{\log(\delta/R)} \geqslant f(x,y,z), \quad \forall\ |(x,y,z)| \leqslant R, |(x,y)| \geqslant \delta, R > \delta > 0.$$

Sending δ to 0 in the above leads to, for the ϵ and R,

$$\epsilon \geqslant f(x,y,z), \quad \forall |(x,y,z)| \leqslant R, |(x,y)| > 0. \tag{C.8}$$

This, together with (C.7), implies

$$\epsilon \geqslant f, \quad \text{on } \mathbb{R}^3 \setminus \mathcal{A}.$$

Sending ϵ to zero leads to

$$0 \geqslant f, \quad \text{on } \mathbb{R}^3 \setminus \mathcal{A}.$$

We have thus proved

$$f \equiv 0 \quad \text{on } \mathbb{R}^3 \setminus \mathcal{A}$$

and the proposition is established. □

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