

A miscellany

YanYan Li*

Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854

Louis Nirenberg
Courant Institute
251 Mercer Street
New York, NY 10012

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Dedicated to the memory of Vittorio Cafagna

Vittorio and I, the second named author, were friends for many years. We first met at a conference in Bari in 1975 where we discussed some work he was doing. It was only several years later that we really became friends — through Henri Berestycki who was staying with Vittorio and his wife in New York.

After that we would meet in Italy or in New York, and it was always a great pleasure to spend time together. We would discuss problems on which he was working — they were always interesting, and I was infected by his enthusiasm — as well as everything under the sun: politics, music, etc. I well recall one evening we spent at a performance of tango.

Vittorio was full of life and seemed to enjoy whatever came his way. The last time we met was in Napoli, several years ago. He came from Salerno; the evening was delightful.

That's how my memory is of him: full of fun, and interest in everything.

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In recent years he turned to use of mathematics in music, with his usual contagious enthusiasm.

His life was cut short too soon.

In this paper we present several results, unrelated to each other.

1 Extension of a simple inequality

In [15] we gave some extensions of the following well known inequality:

Proposition 1 *Let $u \in C^2(-R, R)$, $u \geq 0$ and assume*

$$|\ddot{u}| \leq M. \quad (1)$$

Then

$$|\dot{u}(0)| \leq \sqrt{2u(0)M} \quad \text{if } M \geq \frac{2u(0)}{R^2}, \quad (2)$$

$$|\dot{u}(0)| \leq \frac{u(0)}{R} + \frac{R}{2}M \quad \text{if } M < \frac{2u(0)}{R^2}. \quad (3)$$

In [15] this was extended to functions $u \geq 0$ in higher dimensions, with (1) replaced by

$$|\Delta u| \leq M. \quad (4)$$

In [5], I. Capuzzo Delcetta and A. Vitola give far reaching extensions of the result to fully nonlinear elliptic operators.

Here are some further remarks.

Proposition 1'. *Proposition 1 still holds if (1) is replaced by*

$$\ddot{u} \leq M. \quad (5)$$

Indeed, if $M = 0$, (3) is a standard inequality for concave functions.

The proof of Proposition 1' is just the same as that of Proposition 1 — see for example [15] and [22].

Remark 1 *In the result of [15] the condition (4) above may not be replaced by $\Delta u \leq M$.*

We now give an extension of Proposition 1 to the heat operator.

In the interesting papers [20] and [21] various other extensions of Proposition 1 have been proved.

We consider $u \geq 0$, $u(t, x)$, in a parabolic cylinder

$$B_R = \{(x, t) \mid -R^2 \leq t \leq 0, x \in \mathbb{R}^n, |x| < R\},$$

satisfying

$$|u_t - \Delta u| \leq M. \quad (6)$$

Here Δ is the Laplace operator in the x -variables. The origin, $(0, 0)$, plays the role of the center of the cylinder.

Remark 2 *In case $M = 0$, i.e. u satisfies the heat equation, but it is not true that*

$$|\nabla_x u(0, 0)| \leq \frac{C}{R} u(0, 0). \quad (7)$$

Indeed, for $R = 1$ and $n = 1$, take

$$u = \frac{1}{(t+1)^{1/2}} e^{-\frac{(x+\xi)^2}{4(t+1)}}.$$

Then

$$u_x(0, 0) = -\frac{\xi}{2} u(0, 0).$$

ξ can be arbitrarily large, (7) cannot hold. Instead of (7) we can bound $|\nabla_x u|$ in a suitable subregion of B_R : For $r \leq R$ consider

$$A_r = \{(t, x) \mid -\frac{2}{3}r^2 < t < -\frac{1}{3}r^2, |x| < \frac{r}{\sqrt{3}}\} \subset B_{\sqrt{\frac{2}{3}}R}.$$

Theorem 1 *Suppose $u \geq 0$ and*

$$|u_t - \Delta u| \leq M \quad \text{in } B_R. \quad (8)$$

Then,

$$|\nabla_x u| + \frac{u}{r} \leq \overline{C} \sqrt{u(0, 0)M} \quad \text{in } A_r \text{ if } r = \sqrt{\frac{u(0, 0)}{M}} \leq R, \quad (9)$$

$$|\nabla_x u| + \frac{u}{R} \leq \overline{C} \left(\frac{u(0, 0)}{R} + MR \right) \quad \text{in } A_R \text{ if } R < \sqrt{\frac{u(0, 0)}{M}}. \quad (10)$$

Here \overline{C} depends only on n .

In the elliptic case in [15], the proof was based on standard elliptic estimates and the Harnack inequality. Here we make use of the corresponding Harnack inequality for nonnegative solutions v of the heat equation. It was proved independently in 1954 by Hadamard [8], Pini [23]; see Evans [7]. We formulate it in a form convenient for our application.

Parabolic Harnack Inequality. Assume

$$v_t - \Delta v = 0, \quad v > 0 \quad \text{in } B_\rho.$$

Then

$$\max_{V_\rho} v \leq C \min_{|x| \leq \frac{8}{9}\rho} v(x, 0),$$

where

$$V_\rho = \{(x, t) \mid |x| \leq \frac{8}{9}\rho, -\frac{8}{9}\rho^2 \leq t \leq -\frac{\rho^2}{9}\}$$

and C is independent of ρ .

In particular, it follows that

$$\max_{V_\rho} v \leq C v(0, 0).$$

Using the parabolic Harnack inequality and some standard estimates for the heat operator, we now present the

Proof of Theorem 1. For $0 < r < R$ let v be the solution of

$$\begin{aligned} v_t - \Delta v &= 0 & \text{in } B_r, \\ v &= u & \text{on } \partial_p B_r. \end{aligned}$$

where $\partial_p B_r$ is the parabolic boundary of B_r :

$$\partial_p B_r = \partial B_r \setminus \{t = 0\}.$$

We use the standard inequality:

$$v + r|\nabla_x v| \leq C_1 \max_{D_r} v \quad \text{in } A_r. \tag{11}$$

Here

$$D_r = \{(t, x) \mid -\frac{5}{6}r^2 < t < -\frac{r^2}{3}, |x| < \frac{r}{\sqrt{2}}\}$$

and C_1 is independent of r .

Now applying the parabolic Harnack inequality above, with $\rho = r$, and we find, using $D_r \subset V_r$, that

$$\max_{D_r} v \leq C v(0, 0).$$

Combining this with (11) we obtain:

$$v + r |\nabla_x v| \leq C C_1 v(0, 0) \quad \text{in } A_r. \quad (12)$$

Next, the function

$$w = u - v \quad \text{in } B_r$$

satisfies

$$|w_t - \Delta w| \leq M, \quad w = 0 \quad \text{on } \partial_p B_r.$$

Using another standard inequality for the heat operator we have

$$r |\nabla_x w| + |w| \leq C_2 M r^2 \quad \text{in } B_{3r/4} \text{ (containing } A_r). \quad (13)$$

This, together with (12), yield: in A_r ,

$$\begin{aligned} u + r |\nabla_x u| &\leq v + w + r |\nabla_x w| + r |\nabla_x v| \\ &\leq C_2 M r^2 + C C_1 v(0, 0) \\ &\leq C_2 M r^2 + C C_1 u(0, 0) + C C_1 |w(0, 0)| \\ &\leq C_2 M r^2 + C C_1 u(0, 0) + C C_1 C_2 M r^2; \end{aligned}$$

the last, by (13) again. Thus, in A_r we have

$$\frac{u}{r} + |\nabla_x u| \leq C_3 \left(\frac{u(0, 0)}{r} + M r \right). \quad (14)$$

Next,

Case 1. $R \geq \sqrt{\frac{u(0, 0)}{M}}.$

In this case, with

$$r = \sqrt{\frac{u(0, 0)}{M}},$$

(14) yields (9).

Case 2. $R < \sqrt{\frac{u(0, 0)}{M}}.$

In this case, (14) yields (10) if $r = R$.

□

Remark 3 *Theorem 1 also holds if (6) is replaced by*

$$|u_t - \partial_i(a_{ij}(t, x)u_j)| \leq M$$

if $\{a_{ij}\}$ is uniformly positive definite, and smooth.

2 On the Hopf Lemma

In this section we present an extension of the Hopf Lemma.

A standard form of the lemma for nonlinear second order elliptic operators:

$$F(x, u, \nabla u, \nabla^2 u)$$

is the following. Here we assume $F(x, s, p, A)$ is C^1 , $s \in \mathbb{R}$, $p \in \mathbb{R}^n$, $A \in \mathcal{S}^{n \times n}$, the set of $n \times n$ real symmetric matrices, and strongly elliptic:

$$\frac{\partial F}{\partial A_{ij}} \text{ is positive definite.}$$

Hopf Lemma: Suppose u, v are C^2 functions in a domain Ω in \mathbb{R}^n with C^2 boundary, both continuous in $\Omega \cup \{y\}$, y a boundary point, and that

$$u > v \quad \text{in } \Omega, \tag{15}$$

$$u(y) = v(y). \tag{16}$$

Assume that

$$F(x, u, \nabla u, \nabla^2 u) \leq F(x, v, \nabla v, \nabla^2 v) \quad \text{in } \Omega. \tag{17}$$

Then, if ν is the interior unit normal to $\partial\Omega$ at y ,

$$\liminf_{t \rightarrow 0^+} \frac{(u - v)(y + t\nu)}{t} > 0. \tag{18}$$

Remark 4 *The result holds if $\partial F / \partial u_{ij}$ is positive semidefinite provided*

$$\frac{\partial F}{\partial u_{ij}} \nu_i \nu_j > 0 \tag{19}$$

(say, for all values of the arguments in F).

Remark 5 *Since $\partial\Omega \in C^2$ is assumed in the result, one only needs to prove it for Ω being an open ball (working with a ball inside Ω and having y on its boundary).*

In an unpublished manuscript, [19], we extended the Hopf Lemma to domains with $C^{1,\alpha}$ boundary, $0 < \alpha < 1$. The paper was not published because we learned that the result had been proved earlier by Kamynin and Khimchenko in [10].

In the last 25 years or so many authors have studied various classes of nonlinear degenerate elliptic operators (see for example [1], [2], [3], [4], [9], [13], [14] where many references may be found). One way of looking at such operators, for a C^2 function u in Ω in \mathbb{R}^n is to consider the symmetric matrix

$$\tilde{A}_u := \nabla^2 u + L(\cdot, u, \nabla u),$$

where $L \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, is in $\mathcal{S}^{n \times n}$, the set of $n \times n$ real symmetric matrices.

For every x in Ω , the matrix is required to lie in a region \mathcal{G} in $\mathcal{S}^{n \times n}$.

One such matrix operator,

$$A_w = \nabla^2 w - \frac{|\nabla w|^2}{2w} I,$$

where I denotes the $n \times n$ identity matrix, has arisen in conformal geometry (see e.g. [12], [25] and the references therein). In particular some comparison principles for this matrix operator have been studied in [13] and [14].

In almost all the papers mentioned above, instead of supposing that \tilde{A}_u lies in some region in $\mathcal{S}^{n \times n}$ it was required that the eigenvalues of \tilde{A}_u lie in some region Γ in \mathbb{R}^n which is symmetric, i.e., if $\lambda = (\lambda_1, \dots, \lambda_n)$ lies in Γ then so does any permutation of the λ_i . In this section, however, we will follow the more general approach.

We consider an open set \mathcal{G} in $\mathcal{S}^{n \times n}$, satisfying

$$0 \in \partial\mathcal{G}, \quad \mathcal{G} + \mathcal{P} \subset \mathcal{G}, \quad t\mathcal{G} \subset \mathcal{G} \quad \forall t > 0. \quad (20)$$

Here \mathcal{P} is the set of nonnegative matrices.

Consider functions $u, v \in C^2(B_1) \cap C^0(\bar{B}_1)$, B_1 is the unit ball, satisfying (15) and (16). In place of (17) we require that

$$\tilde{A}_u \in \mathcal{S}^{n \times n} \setminus \mathcal{G} \quad \forall x \in B_1, \quad (21)$$

$$\tilde{A}_v \in \bar{\mathcal{G}} \quad \text{in } B_1. \quad (22)$$

The generalized Hopf Lemma would be to conclude, under possibly further conditions, that (18) holds.

In the first result below we will assume

$$u > 0 \quad \text{in } B_1 \quad (23)$$

and

$$L(x, \beta s, \beta p) \leq \beta L(x, s, p), \quad \forall \beta \geq 1, \quad x, p \in \mathbb{R}^n, \quad s > 0. \quad (24)$$

In order to conclude that (18) holds we impose, however, an additional condition on \mathcal{G} . It depends on the inner normal ν to ∂B_1 at y ; in our case $\nu = -y$; it is analogous to (19).

Condition \mathcal{G}_ν : Let ν be a unit vector in \mathbb{R}^n . \mathcal{G} is said to satisfy condition \mathcal{G}_ν if there exists some open half cone $C_\delta(\nu)$,

$$C_\delta(\nu) = \{t(\{\nu_i \nu_j\} + A) \mid 0 < t, A \in \mathcal{S}^{n \times n}, \|A\| < \delta\}, \quad \delta > 0$$

such that

$$\mathcal{G} + C_\delta(\nu) \subset \mathcal{G}. \quad (25)$$

Condition \mathcal{G}_ν cannot be simply dropped. In fact here is an example showing that if \mathcal{G} does not contain $\{\nu_i \nu_j\}$ then (18) fails in general.

Example 1 Assume that \mathcal{G} satisfies (20) and does not contain $\{\nu_i \nu_j\}$, with $\nu = e_1 = (1, 0, \dots)$. Suppose

$$L(x, s, p) \leq 0, \quad L(x, 1, 0) = 0.$$

Take

$$u(x) = 1 + (x_1 + 1)^2, \quad v(x) \equiv 1.$$

So $u > v$ in $\overline{B}_1 \setminus \{-e_1\}$. Here $y = -e_1$ and $\nu = e_1$.

Then

$$\tilde{A}_v = 0,$$

$$\tilde{A}_u \leq \text{diag}(2, 0, \dots, 0), \text{ therefore does not belong to } \mathcal{G},$$

and (18) does not hold.

Example 2 Assume that \mathcal{G} satisfies (20) and $\{\nu_i \nu_j\}$, with $\nu = e_1$, does not belong to $\overline{\mathcal{G}}$. Assume

$$L(x, 1, 0) \geq 0.$$

Take, for large $k > 1$,

$$u(x) = 1 + k(x_1 + 1)^2, \quad v(x) \equiv 1.$$

So $u > v$ in $\overline{B}_1 \setminus \{-e_1\}$. Here $y = -e_1$ and $\nu = e_1$.

Then

$$\begin{aligned} \tilde{A}_v &\geq 0, \\ \tilde{A}_u &= k[(2, 0, \dots, 0) + O(\frac{1}{k})], \quad \text{in } B_1 \cap \{x \mid |x_1 + 1|k < 1\}. \end{aligned}$$

So, for large k ,

$$\tilde{A}_u \text{ does not belong to } \overline{\mathcal{G}},$$

and (18) does not hold.

Our first extension of the Hopf Lemma in this setup is

Theorem 2 *Assume u and v satisfy (15), (16), (23), i.e. $u > 0$ in B_1 , and (21) and (22), and that L satisfies (24). Assume furthermore that \mathcal{G} satisfies (20) and condition \mathcal{G}_ν , with $\nu = -y$. Then (18) holds.*

Here is an easy consequence of the theorem.

Corollary 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, \mathcal{G} satisfy (20),

$$\text{diag}(1, 0, \dots, 0) \in \mathcal{G}, \quad \mathcal{G} \text{ is convex,}$$

and

$$O^t \mathcal{G} O \subset \mathcal{G}, \quad \forall O \subset O(n),$$

where $O(n)$ denotes the set of $n \times n$ orthogonal matrices, and let L satisfy (24). Suppose, for some $y \in \partial\Omega$, that $u, v \in C^2(\Omega) \cap C^0(\Omega \cup \{y\})$, and satisfy (15) and (16). Then (18) holds.

Proof of Theorem 2: Working with a smaller ball in B_1 which contains y on its boundary, we may assume without loss of generality that $u > 0$ in $\overline{B_1} \setminus \{y\}$. As usual, the proof makes use of a comparison function: Let

$$h = e^{-a|x|^2} - e^{-a},$$

where $a > 1$ is a large constant to be chosen.

For some small constant, $\mu > 0$, to be chosen, let

$$\Sigma_\mu := \{x \in B_1 \mid -1 < x \cdot (-y) < -1 + \mu\}.$$

Differentiating h we have

$$\begin{aligned} h_i &= -2ax_i e^{-a|x|^2} \\ h_{ij} &= (4a^2 x_i x_j - 2a\delta_{ij})e^{-a|x|^2} = 4a^2 e^{-a|x|^2} (x_i x_j + O(\frac{1}{a})). \end{aligned}$$

We will first fix the value of a small $\mu > 0$ and a large $a > 1$, and then fix the value of a small value $\epsilon > 0$.

It follows that

$$(v + \epsilon h)_{ij} = v_{ij} + 4a^2 e^{-a|x|^2} \epsilon (x_i x_j + O(\frac{1}{a})),$$

and

$$L(x, v + \epsilon h, \nabla(v + \epsilon h)) - L(x, v, \nabla v) = \epsilon O(h + a e^{-a|x|^2}) = \epsilon O(a e^{-a|x|^2}).$$

Hence

$$\tilde{A}_{v+\epsilon h} = \tilde{A}_v + 4a^2 e^{-a|x|^2} \epsilon (x_i x_j + O(\frac{1}{a})).$$

For small $\mu > 0$ and large $a > 1$, $x \in \Sigma_\mu$ is close to y and the matrix $\{x_i x_j\} + O(\frac{1}{a})$ is close to the matrix $\{\nu_i \nu_j\}$ and so lies in a cone $C_\delta(\nu)$ as above.

Fixing the values of μ and a , then for all $x \in \Sigma_\mu$,

$$4a^2 e^{-a|x|^2} (x_i x_j + O(\frac{1}{a}))$$

lies in the cone $C_\delta(\nu)$. By condition \mathcal{G}_ν , it follows that

$$\tilde{A}_{v+\epsilon h} \in \mathcal{G}.$$

Next, fix $0 < \epsilon$ small so that

$$u \geq v + \epsilon h \text{ on } \partial\Sigma_\mu.$$

Claim. $u \geq v + \epsilon h$ in Σ_μ .

Indeed if the claim does not hold there exists some constant $\beta > 1$ and some $\bar{x} \in \Sigma_\mu$ such that

$$\beta u \geq v + \epsilon h \text{ on } \Sigma_\mu, \quad \beta u(\bar{x}) = (v + \epsilon h)(\bar{x}). \quad (26)$$

Here we have used the fact that $u > 0$.

It follows, by (24) that

$$\beta \tilde{A}_u(\bar{x}) \geq \tilde{A}_{\beta u}(\bar{x}) \geq \tilde{A}_{v+\epsilon h}(\bar{x}).$$

But $\tilde{A}_{v+\epsilon h}(\bar{x}) \in \mathcal{G}$, hence

$$\beta \tilde{A}_u(\bar{x}) \in \mathcal{G}$$

and, by (20), $\tilde{A}_u(\bar{x}) \in \mathcal{G}$. Contradiction. The desired conclusion (18) follows from the claim. □

Here is a slight variant of Theorem 2. In place of (24) we assume

$$L(x, s, p) \text{ is nonincreasing in } s. \quad (27)$$

Theorem 3 *Let u, v and \mathcal{G} satisfy the conditions of Theorem 2 except that we do not assume $u > 0$, and we replace (24) by (27). Then (18) holds.*

Proof. The proof follows that of Theorem 2 except that in the proof of the claim, instead of choosing $\beta > 1$ so that (26) holds we choose a constant $\gamma > 0$ so that

$$u + \gamma \geq v + \epsilon h \text{ in } \Sigma_\mu \text{ and } u(\bar{x}) + \gamma = (v + \epsilon h)(\bar{x}).$$

The rest of the argument is the same. □

There is an open conjecture concerning a modified kind of Hopf Lemma that arose in [16] and [17], and to which we would like to call attention.

Consider $u \geq v$, positive functions of (t, y) , $y \in \mathbb{R}^n$, in

$$\Omega = \{(t, y) \mid 0 < t < 1, |y| < 1\}.$$

and smooth in $\bar{\Omega}$.

Assume that

$$u_t > 0 \quad \text{in } \Omega, \tag{28}$$

$$u(0, y) \equiv 0 \quad \text{for } |y| < 1, \tag{29}$$

$$u_t(0, 0) = 0, \tag{30}$$

and the main condition:

whenever $u(t, y) = v(s, y)$, for $0 \leq t \leq s < 1$, there

$$\Delta u(t, y) \leq \Delta v(s, y). \tag{31}$$

Conjecture 1. Then

$$u \equiv v. \tag{32}$$

A weaker conjecture is

Conjecture 2. Under the conditions above, (32) holds provided

$u(t, 0)$ and $v(t, 0)$ vanish of finite order at $t = 0$.

In [16] Conjecture 1 was proved in case u and v are functions of t alone; [18] contains a simpler proof. In the second paper, Conjecture 2 was proved in the very special case that, in addition,

- (a) the order of the first t -derivative of $u(t, 0)$ which does not vanish at the origin is ≤ 3 .
- (b) $\nabla_y u_{tt}(0, 0) = 0$.

3 A remark on the Hopf Lemma for the heat operator

We first recall the well known form; we describe it only for the classical heat operator but it holds, of course, for much more general ones, see for example [24].

Consider a domain G in (x, t) space: here $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, lying in $t < 0$, and whose boundary includes an open domain D on $\{t = 0\}$. Assume the origin lies in ∂D . $\partial G \setminus D =: P\partial G$ is called the parabolic boundary of G . For convenience we suppose that $(0, \dots, 0, 1, 0)$ is the inner normal to ∂D at $(0, 0)$, and denote x_n by y . Sometimes we use (x, y, t) to denote a point, with $x = (x_1, \dots, x_{n-1})$.

Consider a function u in $G \cup D \cup \{0, 0\}$, $u > 0$ in $G \cup D$, $u(0, 0) = 0$,

$$u \in C^2(G) \cap C^0(G \cup D \cup \{0, 0\}).$$

Suppose

$$(\partial_t - \Delta)u \geq 0 \quad \text{in } G. \quad (33)$$

Here Δ is the Laplacian in the space variables, (x, y) .

For $P\partial G$ in C^2 , the parabolic Hopf Lemma takes the form

Theorem 4 *Assume that the vector $(0, \dots, 0, 1, 0)$, the inner normal to ∂D at $(0, 0)$, is not tangent to $P\partial G$ at the origin. Then*

$$\liminf_{s \rightarrow 0^+} \frac{u(0, \dots, 0, s, 0)}{s} > 0. \quad (34)$$

But something more may be true. Suppose that near the origin D is given by

$$y > \alpha|x|^2, \quad \alpha > 0, \quad t = 0$$

and assume that for some constant $a > 0$, the domain

$$\Omega = \{(x, y, t) \mid t < 0, y > a|x|^2 + ct\}$$

near the origin, lies in G .

Theorem 5 *In this case we also have*

$$\liminf_{t \rightarrow 0^-} \frac{u(0, t)}{(-t)} > 0. \quad (35)$$

Proof. We may suppose u is continuous in $\overline{\Omega}$ near the origin. Indeed this may be achieved by shrinking Ω slightly. We use the comparison function

$$h = e^{a(y - \alpha|x|^2 - ct)}.$$

A calculation yields

$$\begin{aligned} h_t - \Delta u &= h[-ac - a^2 + 2a\alpha(n-1) - 4a^2\alpha^2|x|^2] \\ &< 0 \quad \text{near the origin for } a \text{ large.} \end{aligned}$$

For $0 < R$ small consider the region

$$\Omega_R = \Omega \cap \{|x|^2 + y^2 < R^2\}.$$

On the parabolic boundary $P\partial\Omega_R$ we have

$$u > 0.$$

Hence for $0 < \epsilon$ small,

$$u > \epsilon h \quad \text{on } P\partial\Omega_R.$$

From the parabolic maximum principle it follows that

$$u \geq \epsilon h \quad \text{in } \Omega_R. \tag{36}$$

□

Since $h_t = -ac$ at the origin, we infer from (36) that (35) holds.

What happen if $(0, \dots, 0, 1, 0)$ is tangent to $P\partial G$? Consider the following simple example. Suppose G is given by

$$G = \{(x, y, t) \mid t < 0, y - |x|^2 + \sqrt{-t} > 0\}.$$

(so $P\partial G$ is even analytic, namely, $-t = (y - |x|^2)^2$)

Theorem 6 *In this case, if u satisfies the conditions above, then*

$$\liminf_{t \rightarrow 0^-} \frac{u(0, t)}{\sqrt{-t}} > 0. \tag{37}$$

From this we see that $u(0, t)$ cannot be C^1 at the origin, or even C^α for $\alpha > 1/2$. In case of one space dimension there are known results on loss of regularity when $(1, 0)$ is tangent to $P\partial G$, see [11] and [6]. Our result is for higher dimension and seems to exhibit a phenomena not previously observed. The loss of regularity is not due to the boundary values of u near 0, for there, u could be $\equiv 0$, but still with u positive in G .

Proof of Theorem 6. For convenience we suppose u is continuous in \overline{G} near the origin (this may be achieved by replacing G by $y - |x|^2 + \sqrt{-\lambda t} > 0$ with $0 < 1 - \lambda$ small). We make use of the comparison function $h = y - |x|^2 + \sqrt{-t}$. We have

$$(\partial_t - \Delta)h = -\frac{1}{\sqrt{-t}} + 2(n-1) < 0 \quad \text{for } |t| \text{ small.}$$

Arguing as above, we find that for some $0 < \epsilon$ small,

$$u \geq \epsilon h$$

near the origin, and (37) then follows.

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