

Some remarks on singular solutions of nonlinear elliptic equations. I

Luis Caffarelli, YanYan Li and Louis Nirenberg

Dedicated to Felix Browder on his 80th birthday

Abstract. We study strong maximum principles for singular solutions of nonlinear elliptic and degenerate elliptic equations of second order. An application is given on symmetry of positive solutions in a punctured ball using the method of moving planes.

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1. Introduction

1.1.

Understanding the behavior of solutions of partial differential equations near an isolated singularity is of basic importance in the study of partial differential equations. A classical theorem of Bôcher (see e.g. [11]) states that a positive harmonic function u in a punctured ball $B_1 \setminus \{0\}$ in \mathbb{R}^n must be of the form

$$u(x) = \begin{cases} -a \log |x| + h(x) & \text{if } n = 2, \\ a|x|^{2-n} + h(x) & \text{if } n \geq 3, \end{cases}$$

where a is a nonnegative constant and h is a harmonic function in the ball B_1 .

For nonlinear partial differential equations or systems, it might be difficult to give as complete a description of the behavior of solutions near an isolated singularity. On the other hand, some studies only require certain partial information on the behavior of solutions near an isolated singularity. Part of this paper, e.g. Theorems 1.1 and 1.3, address some issues which arise naturally in applications of the method of moving planes.

Let $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$, where $\mathcal{S}^{n \times n}$ denotes the set of $n \times n$ real symmetric matrices and Ω is a domain (bounded connected open set) in the n -dimensional Euclidean space \mathbb{R}^n . Without loss of generality we assume that Ω contains the origin 0. Throughout the paper we use $B_r(x)$ to denote a ball of radius r and centered at x , and B_r to denote $B_r(0)$.

We assume that $F(x, v, \nabla v, \nabla^2 v)$ is an elliptic, but not necessarily uniformly elliptic, operator:

$$\left(-\frac{\partial F}{\partial M_{ij}}(x, s, p, M) \right) > 0 \quad \forall (x, s, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n}. \quad (1)$$

In our first theorem, the function $u \in C^2(\Omega \setminus \{0\})$ has the following property for some $\bar{r} > 0$:

$$\text{For any } V \in \mathbb{R}^n, w(x) := u(x) + V \cdot x \text{ satisfies } \inf_{B_r \setminus \{0\}} w = \min_{\partial B_r} w \quad \forall 0 < r < \bar{r}. \quad (2)$$

In dimension $n \geq 2$, a superharmonic function $u \in C^0(B_3 \setminus \{0\})$ satisfying $\inf_{B_3 \setminus \{0\}} u > -\infty$ has the above property. On the other hand, the function

$$u(x) := \begin{cases} 4, & -3 < x < 0, \\ 1, & 0 \leq x < 3, \end{cases} \quad (3)$$

a harmonic function in $(-3, 3) \setminus \{0\}$, does not satisfy condition (2) for any $\bar{r} > 0$. To see this, simply take $w(x) = u(x) + x$. We start with a result related to the strong maximum principle.

Theorem 1.1. *For $n \geq 1$, let Ω be a domain in \mathbb{R}^n containing the origin, and let $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n})$ satisfy (1). Assume that $u \in C^2(\Omega \setminus \{0\})$ satisfies (2) for some $\bar{r} > 0$, $v \in C^2(\Omega)$, and*

$$\begin{aligned} u &> v \quad \text{in } \Omega \setminus \{0\}, \\ F(x, u, \nabla u, \nabla^2 u) &\geq F(x, v, \nabla v, \nabla^2 v) \quad \text{in } \Omega \setminus \{0\}. \end{aligned}$$

Then

$$\liminf_{|x| \rightarrow 0} (u - v)(x) > 0. \quad (4)$$

Remark 1.1. In dimension $n \geq 2$, condition (2) in the above theorem can be replaced by $a_{ij}(x)\partial_{ij}u \leq C$ in $B_{\bar{r}} \setminus \{0\}$ for some $\bar{r} > 0$, where $(a_{ij}(x))$ is some positive definite matrix function which is continuous in $B_{\bar{r}}$ if $n \geq 3$ and Hölder continuous in $B_{\bar{r}}$ if $n = 2$, and C is some constant. Indeed, if $C = 0$, then u satisfies (2); if $C \neq 0$, we may reduce it to the $C = 0$ case by working with $u(x) - A|x|^2$ and $v(x) - A|x|^2$, for some large constant A , instead of u and v , and working with the new F . On the other hand, it is not the case for $n = 1$. This can be seen by taking $F(x, u, \nabla u, \nabla^2 u) = -u''$, $v(x) = 1 - x$ and u as in (3).

Remark 1.2. If the assumption “ $u > v$ in $\Omega \setminus \{0\}$ ” in the above theorem is replaced by “ $u \geq v$ in $\Omega \setminus \{0\}$ ”, then, in view of the strong maximum principle, either $u \equiv v$ in $\Omega \setminus \{0\}$ or (4) holds.

Theorem 1.1 is related to the following: Consider a viscosity supersolution u of a fully nonlinear elliptic equation outside a singular point 0 (or a singular set S). Under what condition is u also a supersolution across 0 (or across S)? We plan to address this issue in subsequent papers.

Condition (2) is related to the following

Definition 1.1. A function $u \in C^0(\Omega)$ is called *superaffine* in a domain Ω if in any subdomain D , the function $u + w$, where w is any linear function, achieves its minimum in \bar{D} on ∂D .

This notion was also introduced by Harvey and Lawson in [10] at the same time. For a $u \in C^2(\Omega)$, u is superaffine in Ω if and only if the lowest eigenvalue $\lambda_1(\nabla^2 u)$ of the Hessian of u is ≤ 0 in Ω —as is easily seen (see the proof of Proposition 1.1 below).

Throughout this paper we always order the eigenvalues of the Hessian of a C^2 function u as

$$\lambda_1(\nabla^2 u) \leq \dots \leq \lambda_n(\nabla^2 u).$$

Existence and uniqueness of continuous solutions to general degenerate equations, including

$$\lambda_p(\nabla^2 u) + \dots + \lambda_{p+q}(\nabla^2 u) = 0$$

for fixed p and q as examples, have been obtained in [10].

Condition (2) is slightly stronger than u being superaffine in $\Omega \setminus \{0\}$, for example it is not satisfied by $|x|^a$, $0 < a < 1$, which is superaffine in $\Omega \setminus \{0\}$. However, the following holds.

Lemma 1.1. For $n \geq 2$, let $u \in C^2(B_1 \setminus \{0\})$ satisfy

$$\lambda_1(\nabla^2 u) + \lambda_2(\nabla^2 u) \leq 0 \quad \text{in } B_1 \setminus \{0\}, \quad (5)$$

and

$$\inf_{B_1 \setminus \{0\}} u > -\infty. \quad (6)$$

Then condition (2), with $\bar{r} = 1$, holds.

Remark 1.3. In Lemma 1.1, condition (5) cannot be replaced by the following weaker condition: For some $0 < a < 1$,

$$\lambda_1(\nabla^2 u) + a\lambda_2(\nabla^2 u) \leq 0 \quad \text{in } B_1 \setminus \{0\}. \quad (7)$$

Indeed, the function $u(x) = |x|^{1-a}$ satisfies (7) (with equality).

Note that

$$\lambda_1(\nabla^2 u) + a\lambda_2(\nabla^2 u) \leq \frac{1+a}{2}(\lambda_1(\nabla^2 u) + \lambda_2(\nabla^2 u))$$

for $0 \leq a < 1$. So (7) is weaker than (5).

Remark 1.4. The assumption (6) in Lemma 1.1 cannot be removed, as can be seen by taking

$$u(x) = \begin{cases} \log|x|, & n = 2, \\ -|x|^{2-n}, & n \geq 3. \end{cases}$$

Theorem 1.1 would not hold if condition (2) were dropped. Here are some examples.

Example 1. For $n \geq 1$, $l = 2, 3, \dots$, let $\alpha = (2l - 2)/(2l - 1)$. Clearly $\alpha \in (0, 1)$ and $2l = (2 - \alpha)/(1 - \alpha)$. For $u(x) := |x|^\alpha$ and $v \equiv 0$, we have $u > v$ in $B_1 \setminus \{0\}$ and

$$F(\nabla u, \nabla^2 u) := -\Delta u + (n + \alpha - 2)\alpha^{1/(\alpha-1)}|\nabla u|^{2l} = 0 = F(\nabla v, \nabla^2 v) \quad \text{in } B_1 \setminus \{0\}.$$

But

$$\lim_{|x| \rightarrow 0} (u - v)(x) = 0. \quad (8)$$

Example 2. For $n = 2$, $0 < \alpha < 1$, and $\epsilon = 1 - \sqrt{\alpha} \in (0, 1)$, let

$$b_1(x) = -(1 - \epsilon)\frac{x_2}{|x|}, \quad b_2(x) = (1 - \epsilon)\frac{x_1}{|x|}, \quad a_{ij}(x) = \delta_{ij} - b_i(x)b_j(x).$$

Consider $u(x) := |x|^\alpha$, $v \equiv 0$. We have $u > v$ in $B_1 \setminus \{0\}$ and

$$\begin{aligned} F(x, \nabla^2 u) &:= -a_{ij}(x)u_{ij}(x) = -\alpha|x|^{\alpha-2}a_{ij}(x)\left(\delta_{ij} - (2 - \alpha)\frac{x_i x_j}{|x|^2}\right) \\ &= -(\alpha - (1 - \epsilon)^2)\alpha|x|^{\alpha-2} = 0 = F(x, \nabla^2 v) \quad \text{in } B_1 \setminus \{0\}. \end{aligned}$$

But (8) holds.

Example 3. For $n \geq 2$, $0 < \alpha < 1$, and $\epsilon = (1 - \alpha)/(n - 1) \in (0, 1)$, let $O(x)$ be an $n \times n$ orthogonal matrix function in $L^\infty(B_1 \setminus \{0\})$ such that

$$O(x)_{ij}\frac{x_j}{|x|} = \delta_{1i} \quad \forall x \in B_1 \setminus \{0\}.$$

Let $\hat{a}_{11} = 1$, $\hat{a}_{ii} = \epsilon$ for $2 \leq i \leq n$, and $\hat{a}_{ij} = 0$ for $i \neq j$. Define

$$a_{ij}(x) := O(x)_{li}\hat{a}_{lm}O(x)_{mj}.$$

Consider $u(x) := |x|^\alpha$, $v \equiv 0$. We have $u > v$ in $B_1 \setminus \{0\}$ and

$$\begin{aligned} F(x, \nabla^2 u) &:= -a_{ij}(x)u_{ij}(x) = -\alpha|x|^{\alpha-2}a_{ij}(x)\left(\delta_{ij} - (2 - \alpha)\frac{x_i x_j}{|x|^2}\right) \\ &= -\alpha|x|^{\alpha-2}(1 + (n - 1)\epsilon - (2 - \alpha)) = 0 = F(x, \nabla^2 v). \end{aligned}$$

But (8) holds.

In Theorem 1.1, the singular set of u is a point. Our next theorem allows the singular set to be a closed submanifold E of dimension $k \leq n - 2$. A submanifold is called *closed* if it is a compact manifold without boundary.

Theorem 1.2. For $n \geq 2$, let F satisfy (1), and let $E \subset \Omega$ be a smooth closed submanifold of dimension k , $0 \leq k \leq n - 2$. Assume that $u \in C^2(\Omega \setminus E)$ and $v \in C^2(\Omega)$ satisfy

$$\lambda_1(\nabla^2 u) + \cdots + \lambda_{k+2}(\nabla^2 u) \leq 0 \quad \text{in } \Omega \setminus E, \quad (9)$$

$$u > v \quad \text{in } \Omega \setminus E,$$

$$F(x, u, \nabla u, \nabla^2 u) \geq F(x, v, \nabla v, \nabla^2 v) \quad \text{in } \Omega \setminus E. \quad (10)$$

Then

$$\liminf_{\text{dist}(x, E) \rightarrow 0} (u - v)(x) > 0. \quad (11)$$

Remark 1.5. In the above theorem, condition (9) is only needed to be satisfied, for some $\bar{r} > 0$, in $E_{\bar{r}} \setminus E$, $E_{\bar{r}} = \{x \mid \text{dist}(x, E) < \bar{r}\}$, since we can apply the theorem with $\Omega = E_{\bar{r}}$.

Similar to Remark 1.2, we have

Remark 1.6. If the assumption “ $u > v$ in $\Omega \setminus E$ ” in the above theorem is replaced by “ $u \geq v$ in $\Omega \setminus E$ ”, then either $u \equiv v$ in $\Omega \setminus E$, or (11) holds.

Our proof of Theorem 1.2 makes use of the following maximum principle for functions satisfying (9).

Proposition 1.1. For $n \geq 2$, $-1 \leq k \leq n - 2$, let E be a smooth closed k -dimensional manifold in \mathbb{R}^n and $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in C^2(\Omega \setminus E) \cap C^0(\bar{\Omega} \setminus E)$ satisfies (9) and

$$\inf_{\Omega \setminus E} u > -\infty.$$

Then

$$u \geq \inf_{\partial\Omega \setminus E} u \quad \text{on } \Omega \setminus E.$$

Note that in the above, when $k = -1$, E is understood as \emptyset , the empty set; while for $k = 0$, E consists of finitely many points.

Lemma 1.1 is a special case of Proposition 1.1, by taking $k = 0$, $n \geq 2$, $\Omega = B_1$ and $E = \{0\}$.

We introduce, for $k = 1, \dots, n$, the notion of viscosity k -superaffine functions which extends that of superharmonic functions and superaffine functions.

We use $LSC(\Omega)$ to denote the set of lower semicontinuous functions in Ω , i.e. maps $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying

$$\liminf_{x \rightarrow \bar{x}} u(x) \geq u(\bar{x}), \quad \forall \bar{x} \in \Omega.$$

Similarly, we use $USC(\Omega)$ to denote the set of upper semicontinuous functions in Ω , i.e. maps $u : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying

$$\limsup_{x \rightarrow \bar{x}} u(x) \leq u(\bar{x}), \quad \forall \bar{x} \in \Omega.$$

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in LSC(\Omega)$ satisfy

$$\inf_{\Omega} u > -\infty. \quad (12)$$

For $1 \leq k \leq n$, we say that u is a *viscosity k -superaffine function* in Ω if the conditions

$$\varphi \in C^2(\Omega), \quad \varphi \leq u \quad \text{in } \Omega, \quad \varphi(\bar{x}) = u(\bar{x}) \quad \text{for some } \bar{x} \in \Omega$$

imply

$$\lambda_1(\nabla^2 \varphi) + \cdots + \lambda_k(\nabla^2 \varphi) \leq 0 \quad \text{at } \bar{x}.$$

We say that u is a *viscosity k -subaffine function* in Ω if $-u$ is a viscosity k -superaffine function in Ω .

We use $\mathcal{A}_k(\Omega)$ to denote the set of viscosity k -superaffine functions in Ω .

Clearly, $\mathcal{A}_n(\Omega)$ is the set of superharmonic functions in Ω , $\mathcal{A}_1(\Omega)$ is the set of superaffine functions in Ω , and

$$\mathcal{A}_n(\Omega) \subset \mathcal{A}_{n-1}(\Omega) \subset \cdots \subset \mathcal{A}_1(\Omega).$$

In Appendix A, we give some results on viscosity k -superaffine functions, including an extension of Proposition 1.1 to viscosity k -superaffine functions.

Theorem 1.2 in the case $k = n - 2$ can be extended as follows.

Theorem 1.3. For $n \geq 2$, let F satisfy (1), and let $E \subset \Omega$ be a closed subset of zero (Newtonian) capacity. Assume that $u \in LSC(\Omega \setminus E)$ and $v \in C^2(\Omega)$ satisfy, for some constant C ,

$$\Delta u \leq C \quad \text{in } \Omega \setminus E, \quad (13)$$

$$u > v \quad \text{in } \Omega \setminus E, \quad (14)$$

and

$$F(x, u, \nabla u, \nabla^2 u) \geq F(x, v, \nabla v, \nabla^2 v) \quad \text{in } \Omega \setminus E \text{ in the viscosity sense.} \quad (15)$$

Then (11) holds.

Remark 1.7. Theorem 1.3 gives complete answers to Questions 1.1–1.3 in [14].

Remark 1.8. In the above theorem, (14) can be replaced by

$$u \geq v \quad \text{in } \Omega \setminus E, \quad \text{and } u \text{ is not identically equal to } v \text{ in } \Omega \setminus E,$$

and (13) can be replaced by

$$a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u \leq C \quad \text{in } \Omega \setminus E \text{ in viscosity sense,} \quad (16)$$

where $(a_{ij}(x))$ is some Hölder continuous positive definite matrix function in Ω , $b_i(x)$ and $c(x)$ are continuous functions in Ω , and C is some constant; see [5].

Open Problem 1.1. Extend Proposition 1.1 and Theorem 1.2 for $1 \leq k < n - 2$ to more general E .

1.2.

As an application of Theorem 1.1, we establish some results on symmetry of positive solutions in a punctured ball using the method of moving planes.

For $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $M = (M_{ij}) \in \mathcal{S}^{n \times n}$, we use, in (18) below, the notation

$$\hat{p} := (-p_1, p_2, \dots, p_n), \quad \widehat{M} := (\widehat{M}_{ij})$$

with $\widehat{M}_{1j} = -M_{1j}$ for $2 \leq j \leq n$; and $M_{i1} = -M_{i1}$ for $2 \leq i \leq n$; $\widehat{M}_{11} = M_{11}$; $\widehat{M}_{ij} = M_{ij}$ for $2 \leq i, j \leq n$.

Theorem 1.4. *For $n \geq 1$, let $F \in C^1(\overline{B}_1 \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n}$ satisfy*

$$\left(\frac{\partial F}{\partial M_{ij}}(x, s, p, M) \right) > 0 \quad \forall (x, s, p, M) \in \overline{B}_1 \setminus \{0\} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n}, \quad (17)$$

(the reader is warned that $-F$ satisfies (1)). Suppose that for all $x \in B_1$, $0 < x_1 < 1$, $-1 < \hat{x}_1 \leq x_1$, $x_1 + \hat{x}_1 > 0$, $p \in \mathbb{R}^n$, $M \in \mathcal{S}^{n \times n}$,

$$F(\hat{x}_1, x_2, \dots, x_n, s, \hat{p}, \widehat{M}) \geq F(x_1, x_2, \dots, x_n, s, p, M). \quad (18)$$

Assume that $u \in C^2(\overline{B}_1 \setminus \{0\})$ satisfies (2) for some $\bar{r} > 0$,

$$F(x, u, \nabla u, \nabla^2 u) = 0, \quad u > 0 \quad \text{in } B_1 \setminus \{0\}, \quad (19)$$

and

$$u = 0 \quad \text{on } \partial B_1.$$

Then

$$\frac{\partial u}{\partial x_1}(x) < 0 \quad \forall x \in B_1, \quad 0 < x_1 < 1. \quad (20)$$

The theorem is proved in Section 4. An immediate consequence is

Corollary 1.1. *If in addition, F is symmetric under reflection in the plane $\{x_1 = 0\}$, then u is symmetric in x_1 .*

Another consequence, after applying the theorem in every direction, is

Corollary 1.2. *For $n \geq 1$, let $F(|x|, s, p, M)$ satisfy (1), and for some $\epsilon > 0$,*

$$\frac{\partial}{\partial r} F(r, s, p, M) \geq 0 \quad \forall 0 < r < 1 + \epsilon, \quad s > 0, \quad p \in \mathbb{R}^n, \quad M \in \mathcal{S}^{n \times n},$$

and

$$F(r, s, Op, O^t MO) \equiv F(r, s, p, M),$$

$$\forall 0 < r < 1 + \epsilon, \quad s > 0, \quad p \in \mathbb{R}^n, \quad M \in \mathcal{S}^{n \times n}, \quad O \in O(n),$$

and let $u \in C^2(\overline{B}_1 \setminus \{0\})$ satisfy (2) for some $\bar{r} > 0$,

$$F(|x|, u, \nabla u, \nabla^2 u) = 0, \quad u > 0 \quad \text{in } B_1 \setminus \{0\}$$

and

$$u = 0 \quad \text{on } \partial B_1.$$

Then u is radially symmetric about the origin and $u'(r) < 0$ for $0 < r < 1$.

Condition (2) cannot be dropped in Corollary 1.2, as shown by the following examples.

Example 1.1. Let $u \in C^2([-1, 1] \setminus \{0\})$ be defined by $u(x) = 1 + x$ if $-1 \leq x \leq 0$, $u(x) = 1/2 - x/2$ if $0 < x \leq 1$. Then

$$u'' = 0, \quad u > 0 \quad \text{in } (-1, 1) \setminus \{0\},$$

but u is not symmetric about the origin. This shows that condition (2) cannot be dropped in Corollary 1.2 in dimension $n = 1$.

Here is another example in dimension $n = 1$.

Example 1.2. Let

$$u(x) = \sin(\pi|x|), \quad 0 < |x| < 1.$$

Then

$$u'' + \pi^2 u = 0, \quad u > 0 \quad \text{in } (-1, 1) \setminus \{0\},$$

but u is not monotone in $(0, 1)$.

In a lecture about this paper in November 2008, the last author said that for $n > 1$ he did not know if the condition (2) could be dropped. In particular, he asked if one had spherical symmetry and monotonicity for a solution of

$$\begin{cases} \Delta u + f(u) = 0, & u > 0, & \text{in } B_1 \setminus \{0\}, \\ u = 0 & & \text{in } \partial B_1, \end{cases}$$

f locally Lipschitz—with no further assumption. Immediately after the talk, Susanna Terracini said she could prove it and she showed him the proof. It made use of an idea she had used in [18] which treats equations with singular potential in all of space. Her proof uses a nice variant of the argument of [1] and works also if f is permitted to depend also on x , $f = f(x, u)$, satisfying (18).

In a later paper [5], we will make use of her variant to prove symmetry and monotonicity in problems of the form (19), but under other conditions on F .

Recently we found an example for dimension $n \geq 2$ showing that condition (2) cannot be dropped. Consider

$$\mathcal{M}^+(M) := \sum_{i=1}^n h(\lambda_i(M)), \quad M \in \mathcal{S}^{n \times n},$$

where $\{\lambda_i(M)\}$ denotes eigenvalues of M and

$$h(s) := \begin{cases} 1, & s \geq 0, \\ \frac{n-1}{1-\alpha}, & s < 0. \end{cases}$$

$\mathcal{M}^+(\nabla^2 u)$, the Pucci extremal operator, is uniformly elliptic and is concave and Lipschitz in $\nabla^2 u$. See e.g. Section 2.2 of [2] for the Pucci extremal operators and their properties.

In the following we will consider radially symmetric functions $u(x)$, and we will slightly abuse notation by writing $u(x) = u(r)$, $r = |x|$.

For $0 < \alpha < 1$, let

$$u(r) = \begin{cases} r^\alpha - \frac{\alpha}{2}r^2, & 0 < r \leq 1, \\ -\frac{\alpha(2-\alpha)}{n} \left(\frac{1}{2}r^2 + \frac{1}{n-2}r^{2-n} \right) + \frac{(2-\alpha)(n-2+\alpha)}{2(n-2)}, & r > 1. \end{cases}$$

A calculation gives

$$\frac{u'(r)}{r} = \begin{cases} \alpha r^{\alpha-2} - \alpha, & 0 < r \leq 1, \\ -\frac{\alpha(2-\alpha)}{n}(1-r^{-n}), & r > 1, \end{cases}$$

and

$$u''(r) = \begin{cases} -\alpha(1-\alpha)r^{\alpha-2} - \alpha, & 0 < r \leq 1, \\ -\frac{\alpha(2-\alpha)}{n}(1+(n-1)r^{-n}), & r > 1. \end{cases}$$

The function u is in $C^{2,1}(0, \infty)$ and satisfies, for some $\bar{r} > 1$,

$$u > 0 \quad \text{in } (0, \bar{r}), \quad u(0) = u(\bar{r}) = 0.$$

The eigenvalues of the Hessian $\nabla^2 u$ are

$$\lambda_1(\nabla^2 u) = u''(r), \quad \lambda_2(\nabla^2 u) = \cdots = \lambda_n(\nabla^2 u) = u'(r)/r.$$

Since $u'' < 0$ in $(0, \infty)$, while $u' > 0$ in $(0, 1)$ and $u' < 0$ in $(1, \infty)$, we have

$$\begin{aligned} \mathcal{M}^+(\nabla^2 u)(x) &= \frac{n-1}{1-\alpha} \lambda_1(\nabla^2 u) + \sum_{i=2}^n \lambda_i(\nabla^2 u) = \frac{n-1}{1-\alpha} u''(r) + (n-1) \frac{u'(r)}{r} \\ &= -\frac{(n-1)\alpha(2-\alpha)}{1-\alpha}, \quad 0 < |x| \leq 1, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}^+(\nabla^2 u)(x) &= \frac{n-1}{1-\alpha} \sum_{i=1}^n \lambda_i(\nabla^2 u) = \frac{n-1}{1-\alpha} \left(u''(r) + \frac{n-1}{r} u'(r) \right) \\ &= -\frac{(n-1)\alpha(2-\alpha)}{1-\alpha}, \quad |x| > 1. \end{aligned}$$

Example 1.3. For $n \geq 2$, $0 < \alpha < 1$, let u and \mathcal{M}^+ be as above. Then $u \in C^{2,1}(\overline{B_{\bar{r}}(0)} \setminus \{0\})$ satisfies

$$\mathcal{M}^+(\nabla^2 u) = -\frac{(n-1)\alpha(2-\alpha)}{1-\alpha} \quad \text{in } B_{\bar{r}}(0) \setminus \{0\},$$

and

$$u > 0 \quad \text{in } B_{\bar{r}}(0) \setminus \{0\}, \quad u = 0 \quad \text{on } \partial B_{\bar{r}}(0).$$

But $u'(r)$ changes sign in $(0, \bar{r})$.

1.3. More general equations

We extend Theorem 1.2 to more general operators which are motivated by conformally invariant operators. An operator $F(\cdot, u, \nabla u, \nabla^2 u)$ is *conformally invariant* if

$$F(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv F(\cdot, u, \nabla u, \nabla^2 u) \circ \psi \quad \text{on } \mathbb{R}^n$$

for all Möbius transformations ψ and all positive functions $u \in C^2(\mathbb{R}^n)$, where $u_\psi := |J_\psi|^{(n-2)/2n}(u \circ \psi)$ and J_ψ denotes the Jacobian of ψ . It was proved in [12, Theorem 1.2] that conformally invariant operators are all of the form $f(\lambda(A^u))$, where $f(\lambda)$ is a symmetric function of $\lambda = (\lambda_1, \dots, \lambda_n)$, and $\lambda(A^u)$ denotes the eigenvalues of the $n \times n$ symmetric matrix function

$$\begin{aligned} A^u := & -\frac{2}{n-2}u^{-(n+2)/(n-2)}\nabla^2 u \\ & + \frac{2n}{(n-2)^2}u^{-2n/(n-2)}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-2n/(n-2)}|\nabla u|^2 I, \end{aligned} \quad (21)$$

and I is the $n \times n$ identity matrix. After setting

$$w = u^{-2/(n-2)},$$

we have

$$A^u = A_w := w\nabla^2 w - \frac{|\nabla w|^2}{2}I,$$

which has been called in some literature the *conformal Hessian* of w . The ellipticity of $f(\lambda(A^u))$ amounts to $\partial f/\partial \lambda_i > 0, 1 \leq i \leq n$, in the relevant region.

If $F(x, u, \nabla u, \nabla^2 u)$ is a conformally invariant second order elliptic operator, (4) was proved in [14, Theorem 16] if u satisfies, instead of (2), a stronger assumption $\Delta u \leq 0$ in $B_3 \setminus \{0\}$. As explained in the introduction of [13], the general Liouville theorem for conformally invariant elliptic operators of second order established there (Theorem 1.3 in that paper) would have followed from the proof of Theorem 1.4 in [12] if the above mentioned theorem in [14] had been known at the time. The proof of this theorem in [14] makes use of ideas in the proof of the Liouville theorem in [13]. Given (4) for conformally invariant elliptic operators, one can also adapt the moving planes procedure of [8] and [3] to show that all solutions are radially symmetric about some point, and then classify radially symmetric entire solutions.

In [14], (4) was also established for some special classes of $F(x, u, \nabla u, \nabla^2 u)$ which include $F(\nabla u, \nabla^2 u)$, again for superharmonic u ; see Theorem 1.7 and Corollary 1.6 in [14]. Our proof of Theorem 1.1, very different from the arguments in [14], is also extended to the situations where the singular set $\{0\}$ is replaced by a closed submanifold of dimension $k \leq n-2$ (see Theorem 1.2). Our proof of Theorem 1.3, different from that of Theorem 1.1 and the arguments in [14], makes use of the Aleksandrov–Bakelman–Pucci estimate; see e.g. Section 3 of [2] and Section 9.1 of [9] for the ABP estimate.

Here we consider

$$\tilde{A}^u := -\nabla^2 u + \tilde{L}(\cdot, u, \nabla u) \quad (22)$$

where \tilde{L} satisfies

$$\tilde{L} \in C_{\text{loc}}^{0,1}(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n). \quad (23)$$

One example of such \tilde{L} is

$$\tilde{L}(\cdot, u, \nabla u) = \frac{n}{(n-2)u} \nabla u \otimes \nabla u - \frac{1}{(n-2)u} |\nabla u|^2 I.$$

With this \tilde{L} , \tilde{A}^u takes the form

$$\tilde{A}^u = \frac{n-2}{2} u^{(n+2)/(n-2)} A^u, \quad (24)$$

where A^u is the conformally invariant one given in (21).

Fully nonlinear elliptic equations of second order with the Hessian $\nabla^2 u$ in appropriate regions of $\mathcal{S}^{n \times n}$ have been investigated in the classical paper [6] and many subsequent ones.

Let $U \subset \mathcal{S}^{n \times n}$ be an open set. We assume that

$$M + N \in U, \quad \forall M \in U, N \in \mathcal{S}_+^{n \times n}. \quad (25)$$

For example, for $1 \leq k \leq n$, let

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the k -th elementary symmetric function and let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone $\Gamma_n \equiv \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \forall i\}$. Then

$$U_k := \{M \in \mathcal{S}^{n \times n} \mid \lambda(M) \in \Gamma_k\} \quad (26)$$

satisfies (25)—see e.g. [6].

Theorem 1.5. *For $n \geq 2$, let Ω be a domain in \mathbb{R}^n containing the origin $\{0\}$, U be an open subset of $\mathcal{S}^{n \times n}$ satisfying (25), and let \tilde{A}^u satisfy (22) and (23). Assume that $u \in C^2(\Omega \setminus \{0\})$ satisfies (2) for some $\bar{r} > 0$, $v \in C^2(\Omega)$,*

$$\begin{aligned} \tilde{A}^u &\in \bar{U} && \text{in } \Omega \setminus \{0\}, \\ \tilde{A}^v &\in \mathcal{S}^{n \times n} \setminus U && \text{in } \Omega, \\ u &> v && \text{in } \Omega \setminus \{0\}. \end{aligned} \quad (27)$$

Then

$$\liminf_{|x| \rightarrow 0} (u - v)(x) > 0. \quad (28)$$

Question 1.1. *Does (28) still hold if (27) in Theorem 1.5 is replaced by a weaker assumption*

$$u \geq v \quad \text{in } \Omega \setminus \{0\}, \quad \text{and} \quad \liminf_{\text{dist}(x, \partial\Omega) \rightarrow 0} (u - v)(x) > 0? \quad (29)$$

For an open subset U of $\mathcal{S}^{n \times n}$, we introduce, for $1 \leq k \leq n-1$,

Property P_k : Given any constant $C > 0$, there exist some constants $\bar{\delta} = \bar{\delta}(C) > 0$ and $\bar{\epsilon} = \bar{\epsilon}(C) > 0$ such that for any $0 < \epsilon < \bar{\epsilon}$ and $N \in \mathcal{S}^{n \times n}$, $M \in \bar{U}$ with $M \leq CI$, $\lambda_i(N) = -\bar{\delta}$ for $1 \leq i \leq k$ and $\lambda_j(N) = 1$ for $k+1 \leq j \leq n$ we have $M + \epsilon N \in U$.

Remark 1.9. Property P_k is satisfied by an open convex subset U of $\mathcal{S}^{n \times n}$ if U satisfies one of the following:

- (i) $\partial U \cap \{M \mid M \leq CI\}$ is compact for any constant C , and for any $M \in \partial U$, there exists a constant $a > 0$ such that $M + N \in U$ if the eigenvalues of N consist of k zeros and $n-k$ a 's.
- (ii) $M \in U$ implies $aM \in U$ for any positive constant a , and N with k zeros and $n-k$ ones as eigenvalues belongs to U .
- (iii) $U = U_l$, as in (26), for some $1 \leq l \leq n-k$.

See the end of Section 6 for an explanation of the above remark.

In Theorem 1.5, the singular set is a point, $\{0\}$. The following theorem allows singular sets to be submanifolds of dimension k , $1 \leq k \leq n-2$.

Theorem 1.6. For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a domain, $E \subset \Omega$ be a smooth closed submanifold of dimension k , $1 \leq k \leq n-2$, U be an open subset of $\mathcal{S}^{n \times n}$ satisfying (25) and Property P_k , and let \tilde{A}^u satisfy (22) and (23). Assume that $u \in C^2(\Omega \setminus E)$ satisfies (9), $v \in C^2(\Omega)$, and for some $\bar{r} > 0$,

$$\tilde{A}^u \in \bar{U} \quad \text{in } \Omega \setminus E, \quad (30)$$

$$\tilde{A}^v \in \mathcal{S}^{n \times n} \setminus U \quad \text{in } \Omega, \quad (31)$$

$$u > v \quad \text{in } \Omega \setminus E. \quad (32)$$

Then

$$\liminf_{\text{dist}(x,E) \rightarrow 0} (u-v)(x) > 0. \quad (33)$$

Similar to Question 1.1, we have

Question 1.2. Does (33) still hold if (32) in Theorem 1.6 is replaced by a weaker assumption

$$u \geq v \quad \text{in } \Omega \setminus E, \quad \text{and} \quad \liminf_{\text{dist}(x,\partial\Omega) \rightarrow 0} (u-v)(x) > 0?$$

We suspect that the answers to Question 1.1 and 1.2 are “No”, and some structure assumptions on \tilde{L} are needed in order to have an affirmative answer. These issues will be addressed in a subsequent paper [4].

If \tilde{A}^u is the conformally invariant Hessian given by (24), and if condition (2) is replaced by a stronger assumption “ $\Delta u \leq 0$ in $B_{\bar{r}} \setminus \{0\}$ for some $\bar{r} > 0$ ”, then the answer to Question 1.1 is “Yes”, as proved in [15]. The assumption (29) is indeed weaker than (27) since the strong maximum principle might not hold—indeed, as shown in [17], both the strong maximum principle and the Hopf Lemma fail for

$U = U_k$, $2 \leq k \leq n$: there exist $u_1 \in C^\infty(B_1)$ such that u_1 and $u_2 \equiv 1$ satisfy $0 < u_1 < u_2$ in $B_1 \setminus \{x_1 = 0\}$, $u_1 = u_2$ on $\{x_1 = 0\}$, $A^{u_i} \in \partial U_k$ for $i = 1, 2$, but $\partial_{x_1}(u_1 - u_2)(0) = 0$.

The above mentioned result in [15] was established under a weaker $C^{1,1}$ regularity assumption on u and v . The regularity assumption was further weakened in [16] to locally Lipschitz viscosity solutions of (30) and (31), which is crucial in the proof there of the local gradient estimates to general conformally invariant second order elliptic equations without concavity assumptions on the equations.

The proofs of the above mentioned results in [15] and [16], as well as those of the previously mentioned results in [14] for conformally invariant elliptic operators, make use of the following

Lemma A ([13]). *For $n \geq 2$, $B_1 \subset \mathbb{R}^n$, let $u \in L^1_{\text{loc}}(B_1 \setminus \{0\})$ satisfy*

$$\liminf_{x \rightarrow 0} u(x) = 0, \quad \Delta u \leq 0 \quad \text{in } B_1 \setminus \{0\}$$

in the distribution sense. Let P denote the set of vectors p in \mathbb{R}^n satisfying

$$u(x) \geq p \cdot x + o(|x|) \quad \text{in } B_1 \setminus \{0\}. \quad (34)$$

Then P contains at most one element.

In the above, $\Delta u \leq 0$ can be replaced by $\Delta u \leq C$ for some constant C —since we can work with $u(x) - \frac{C}{2n}|x|^2$ instead of u .

In Appendix F we give an improvement of the above lemma (see Lemma 12.3 there), which is used in the proof of Theorem 1.3 (see the proof of Lemma 3.1). We also mention an extension of the lemma for functions satisfying (2) instead of superharmonic functions in the punctured ball. It is not used in this paper.

Lemma 1.2. *Let u be defined in $B_1 \setminus \{0\}$ in \mathbb{R}^n , $n \geq 2$, with*

$$\liminf_{x \rightarrow 0} u(x) = 0.$$

Suppose that u satisfies (2) with $\bar{r} = 1$. Let P denote the set of vectors p in \mathbb{R}^n satisfying (34). Then there are at most n vectors p_0, \dots, p_{n-1} in P such that

$$p_1 - p_0, \dots, p_{n-1} - p_0 \text{ are linearly independent}, \quad (35)$$

i.e. the convex hull of all the vectors in P is at most $(n - 1)$ -dimensional.

Remark 1.10. Condition (35) is equivalent to: for any $i \leq n - 1$, $\{p_j - p_i \mid 0 \leq j \leq n - 1, j \neq i\}$ are linearly independent.

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in Section 2, Theorem 1.3 in Section 3, Theorem 1.4 in Section 4, Theorem 1.5 in Section 5, and Theorem 1.6 in Section 6; this last proof is technical and we suggest omitting that section on a first reading. Appendix A contains some properties of distance functions, Appendix B contains some linear algebra lemmas. Proposition 1.1 and an extension of it, Theorem 10.1, are proved in Appendices C and D, Lemma 1.2 is proved in Appendix E. Appendix F contains some improvement of Lemma A.

2. Proof of Theorems 1.1 and 1.2

2.1. Proof of Theorem 1.1

We argue by contradiction. Suppose

$$\liminf_{x \rightarrow 0} (u - v)(x) = 0. \quad (36)$$

Let

$$v_\epsilon(x) := v(x) + \epsilon|x|.$$

Lemma 2.1. *For any $\epsilon > 0$, there exists $0 < \bar{r}_\epsilon < 1$ such that $\inf_{\partial B_r} (u - v_\epsilon) < 0$ for all $0 < r < \bar{r}_\epsilon$.*

Proof. By condition (2) and (36),

$$\begin{aligned} \inf_{x \in \partial B_r} [u(x) - (v(0) + \nabla v(0) \cdot x)] &= \inf_{B_r \setminus \{0\}} [u(x) - (v(0) + \nabla v(0) \cdot x)] \\ &\leq \liminf_{x \rightarrow 0} [u(x) - (v(0) + \nabla v(0) \cdot x)] = 0. \end{aligned}$$

Thus

$$\begin{aligned} \inf_{\partial B_r} (u - v_\epsilon) &\leq \inf_{x \in \partial B_r} [u(x) - (v(0) + \nabla v(0) \cdot x)] \\ &\quad + \sup_{x \in \partial B_r} [(v(0) + \nabla v(0) \cdot x) - v(x) - \epsilon|x|] \\ &\leq \frac{1}{2} (\sup_{B_1} \|\nabla^2 v\|) r^2 - \epsilon r. \end{aligned}$$

Lemma 2.1 follows. \square

By Lemma 2.1,

$$\lambda(\epsilon) := - \inf_{B_3 \setminus \{0\}} (u - v_\epsilon) > 0.$$

Since

$$\liminf_{x \rightarrow 0} (u - v_\epsilon)(x) = \liminf_{x \rightarrow 0} (u - v)(x) = 0,$$

there exists $x_\epsilon \in \bar{B}_3 \setminus \{0\}$ such that

$$u - \tilde{v}_\epsilon \geq 0 \quad \text{in } B_3 \setminus \{0\}, \quad (u - \tilde{v}_\epsilon)(x_\epsilon) = 0, \quad (37)$$

where

$$\tilde{v}_\epsilon(x) := v_\epsilon(x) - \lambda(\epsilon) = v(x) + \epsilon|x| - \lambda(\epsilon).$$

Using the positivity of $u - v$ in $\bar{B}_3 \setminus \{0\}$, we deduce from the above that

$$\lambda(\epsilon) = -(u - v_\epsilon)(x_\epsilon) = v(x_\epsilon) - u(x_\epsilon) + \epsilon|x_\epsilon| \leq \epsilon|x_\epsilon|, \quad (38)$$

and

$$\lim_{\epsilon \rightarrow 0} |x_\epsilon| = 0.$$

We will show, for small $\epsilon > 0$, that

$$F(x_\epsilon, v(x_\epsilon), \nabla v(x_\epsilon), \nabla^2 v(x_\epsilon)) > F(x_\epsilon, \tilde{v}_\epsilon(x_\epsilon), \nabla \tilde{v}_\epsilon(x_\epsilon), \nabla^2 \tilde{v}_\epsilon(x_\epsilon)). \quad (39)$$

This would lead to a contradiction. Indeed this would imply, together with (10), that

$$F(\cdot, u, \nabla u, \nabla^2 u)(x_\epsilon) > F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 \tilde{v}_\epsilon)(x_\epsilon).$$

We deduce from (37) that $\nabla u(x_\epsilon) = \nabla \tilde{v}_\epsilon(x_\epsilon)$, $\nabla^2 u(x_\epsilon) \geq \nabla^2 \tilde{v}_\epsilon(x_\epsilon)$, and, using the ellipticity assumption (1),

$$F(\cdot, u, \nabla u, \nabla^2 u)(x_\epsilon) \leq F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 \tilde{v}_\epsilon)(x_\epsilon),$$

a contradiction.

Now we prove (39) for small ϵ . By (38),

$$|\tilde{v}_\epsilon - v| + |\nabla(\tilde{v}_\epsilon - v)| = O(\epsilon), \quad (40)$$

and therefore

$$\begin{aligned} F(\cdot, v, \nabla v, \nabla^2 v)(x_\epsilon) &= F(\cdot, \tilde{v}_\epsilon + O(\epsilon), \nabla \tilde{v}_\epsilon + O(\epsilon), \nabla^2 v)(x_\epsilon) \\ &\geq F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 v)(x_\epsilon) - C\epsilon, \end{aligned} \quad (41)$$

where C is independent of ϵ .

By the mean value theorem,

$$\begin{aligned} F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 v)(x_\epsilon) - F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 \tilde{v}_\epsilon)(x_\epsilon) &= \\ \int_0^1 F_{ij}(x_\epsilon, \tilde{v}_\epsilon(x_\epsilon), \nabla \tilde{v}_\epsilon(x_\epsilon), \nabla^2 v(x_\epsilon) + t[\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)]) dt (\partial_{ij} v(x_\epsilon) - \partial_{ij} \tilde{v}_\epsilon(x_\epsilon)). \end{aligned}$$

Consider the compact set

$$S := \{(x, s, p, M) \mid |x| \leq 1, \|(s, p, M) - (v(x), \nabla v(x), \nabla^2 v(x))\| \leq 1\}.$$

By the ellipticity assumption (1), there exists some constant $b > 0$ such that

$$\left(-\frac{\partial F}{\partial M_{ij}} \right) \geq bI \quad \text{on } S,$$

where I is the $n \times n$ identity matrix.

A calculation gives, for some $O(x) \in O(n)$,

$$\nabla^2 \tilde{v}_\epsilon(x) - \nabla^2 v(x) = \epsilon \nabla^2(|x|) = \epsilon O(x)^t \left(\text{diag} \left(0, \frac{1}{|x|}, \dots, \frac{1}{|x|} \right) \right) O(x). \quad (42)$$

Define

$$\bar{t}_\epsilon := \begin{cases} 1 & \text{if } \|\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)\| \leq 1/\sqrt{3}, \\ 1/(\sqrt{3}\|\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)\|) & \text{if } \|\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)\| > 1/\sqrt{3}. \end{cases}$$

In view of (40) and the definition of \bar{t}_ϵ ,

$$(x_\epsilon, \tilde{v}_\epsilon(x_\epsilon), \nabla \tilde{v}_\epsilon(x_\epsilon), \nabla^2 v(x_\epsilon) + t[\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)]) \in S, \quad \forall 0 \leq t \leq \bar{t}_\epsilon.$$

It follows that

$$\begin{aligned}
& F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 v)(x_\epsilon) - F(\cdot, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon, \nabla^2 \tilde{v}_\epsilon)(x_\epsilon) \\
& \geq \int_0^{\bar{t}_\epsilon} F_{ij}(x_\epsilon, \tilde{v}_\epsilon(x_\epsilon), \nabla \tilde{v}_\epsilon(x_\epsilon), \nabla^2 v(x_\epsilon) + t[\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)]) dt \\
& \quad \cdot (\partial_{ij} v(x_\epsilon) - \partial_{ij} \tilde{v}_\epsilon(x_\epsilon)) \\
& \geq b \bar{t}_\epsilon \sum_{i=1}^n (\partial_{ii} \tilde{v}_\epsilon(x_\epsilon) - \partial_{ii} v(x_\epsilon)) \geq b \bar{t}_\epsilon \|\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)\| \\
& \geq b \min\{\|\nabla^2 \tilde{v}_\epsilon(x_\epsilon) - \nabla^2 v(x_\epsilon)\|, 1/\sqrt{3}\} \geq b \min\{\epsilon/|x_\epsilon|, 1/\sqrt{3}\}.
\end{aligned}$$

Since $|x_\epsilon| \rightarrow 0$, estimate (39) follows from (41) and the above. Theorem 1.1 is established. \square

2.2. Proof of Theorem 1.2

Before the proof, we first take up a simpler case.

Case $k = n - 2$. Shrinking Ω slightly, we may assume that $u > v$ on $\bar{\Omega} \setminus E$, $u \in C^2(\bar{\Omega} \setminus E)$, and $v \in C^2(\bar{\Omega})$. We argue by contradiction. Suppose that for some $\bar{x} \in E$,

$$\liminf_{x \in \bar{\Omega} \setminus E, x \rightarrow \bar{x}} (u - v)(x) = 0. \quad (43)$$

Without loss of generality, $\bar{x} = 0$.

$$\text{For } C_1 := \frac{1}{2n} \sup_{\Omega} (-\Delta v),$$

$$\Delta(u - v - C_1|x|^2) \leq 0, \quad x \in \Omega \setminus E.$$

Since $\text{Cap}(E)$, the capacity of E , is equal to 0, and $u - v - C_1|x|^2$ is bounded from below on $\Omega \setminus E$,

$$\Delta(u - v - C_1|x|^2) \leq 0 \quad \text{in } \Omega \text{ in the distribution sense.}$$

It follows, in view of (43), that

$$\int_{\partial B_r(0)} (u - v - C_1|x|^2) \leq 0, \quad 0 < r < 1. \quad (44)$$

Consider, for $0 < \epsilon < 1$,

$$v_\epsilon(x) := v(x) + \epsilon d(x), \quad (45)$$

where $d(x) := \text{dist}(x, E)$ denotes the distance function from x to E . Then, by (44), there exist some constants $\bar{r}, c_1, c_2 > 0$, depending only on n and E , such that

$$\begin{aligned}
\int_{\partial B_r(0)} (u - v_\epsilon) &= \int_{\partial B_r(0)} (u - v - C_1|x|^2) - \int_{\partial B_r(0)} (\epsilon d(x) - C_1|x|^2) \\
&\leq - \int_{\partial B_r(0)} (\epsilon d(x) - C_1|x|^2) \leq -c_1 \epsilon r^n + c_2 C_1 r^{n+1}, \quad \forall 0 < r < \bar{r}.
\end{aligned}$$

It follows, for small $\epsilon > 0$, that

$$\lambda(\epsilon) := - \inf_{\Omega \setminus E} (u - v_\epsilon) > 0. \quad (46)$$

Since $v_\epsilon = v$ on E , and $u - v > 0$ on $\bar{\Omega} \setminus E$, we have, for small $\epsilon > 0$,

$$u - v_\epsilon > 0 \quad \text{on } \partial\Omega, \quad \liminf_{x \in \Omega \setminus E, d(x) \rightarrow 0} (u - v_\epsilon)(x) \geq 0.$$

Therefore, there exists $x_\epsilon \in \Omega \setminus E$ such that

$$u - \tilde{v}_\epsilon \geq 0 \quad \text{in } \Omega \setminus E, \quad (u - \tilde{v}_\epsilon)(x_\epsilon) = 0, \quad (47)$$

where

$$\tilde{v}_\epsilon(x) := v_\epsilon(x) - \lambda(\epsilon) = v(x) + \epsilon d(x) - \lambda(\epsilon). \quad (48)$$

Using the positivity of $u - v$ in $\bar{\Omega} \setminus E$, we deduce from the above that

$$\lambda(\epsilon) = -(u - v_\epsilon)(x_\epsilon) = v(x_\epsilon) - u(x_\epsilon) + \epsilon d(x_\epsilon) \leq \epsilon d(x_\epsilon), \quad (49)$$

and

$$\lim_{\epsilon \rightarrow 0} d(x_\epsilon) = 0.$$

We will establish (39), with our new \tilde{v}_ϵ and x_ϵ , for small $\epsilon > 0$. This leads to a contradiction as shown in the proof of Theorem 1.1.

Now we prove (39) for small $\epsilon > 0$. By (49), we still have (40) and (41).

By the result in Appendix A,

$$\nabla^2 \tilde{v}_\epsilon(x) - \nabla^2 v(x) = \epsilon O(x)^t \left(\text{diag} \left(0, \dots, 0, \frac{1}{d(x)}, \dots, \frac{1}{d(x)} \right) + O(1) \right) O(x), \quad (50)$$

where $O(x)$ are orthogonal matrices, and there are $k + 1$ zeros in the diagonal matrix.

Using (50) instead of (42), and using $d(x_\epsilon) \rightarrow 0$ instead of $|x_\epsilon| \rightarrow 0$, we establish (39) the same way as in the proof of Theorem 1.1. Theorem 1.2 for $k = n - 2$ is established.

Proof of Theorem 1.2: general case. Shrinking Ω slightly, we may assume that $u > v$ on $\bar{\Omega} \setminus E$, $u \in C^2(\bar{\Omega} \setminus E)$, and $v \in C^2(\bar{\Omega})$. We prove the theorem by contradiction. Suppose that (43) holds for some $\bar{x} \in E$. We may assume $\bar{x} = 0 \in E$, and the tangent space of E at 0 is spanned by e_{n-k+1}, \dots, e_n , where $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ are the standard basis of \mathbb{R}^n . We write $x = (x', x'')$, where $x' = (x_1, \dots, x_{n-k})$ and $x'' = (x_{n-k+1}, \dots, x_n)$.

For x close to 0, we have, for some constant C ,

$$\frac{3}{4}|x'| - C|x''|^2 \leq |x'| - C|x|^2 \leq d(x) \equiv \text{dist}(x, E) \leq |x'| + C|x| \leq \frac{5}{4}|x'| + C|x''|^2. \quad (51)$$

For example, this follows easily from (70).

Let v_ϵ be defined as in (45). We have

Lemma 2.2. *For small $\epsilon > 0$, (46) holds.*

Proof. Suppose not; then, for some $\bar{r}, \bar{\epsilon}, \bar{C} > 0$,

$$u(x) \geq v_{\bar{\epsilon}}(x) = \bar{\epsilon}d(x) + v(x) \geq v(0) + \nabla v(0) \cdot x + \frac{\bar{\epsilon}}{2}|x'| - \bar{C}|x''|^2, \quad x \in B_{\bar{r}} \setminus E.$$

For $0 < a < \min\{\bar{r}, \bar{\epsilon}/4\}$ which will be chosen later, let

$$h(x) := v(0) + \nabla v(0) \cdot x + \frac{\bar{\epsilon}}{4a}|x'|^2 - (\bar{C} + 1)|x''|^2 + \frac{a^2}{2}.$$

On $\partial B_a(0) \setminus E$,

$$u(x) \geq v(0) + \nabla v(0) \cdot x + \frac{\bar{\epsilon}}{2a}|x'|^2 - \bar{C}|x''|^2 = h(x) + \frac{\bar{\epsilon}}{4a}|x'|^2 + |x''|^2 - \frac{a^2}{2} \geq h(x).$$

By Lemma 8.2 in Appendix B and the assumption (9), there exists some positive constant $\bar{a} > 0$ such that if we further require $0 < a < \bar{a}$, we have

$$\sum_{i=1}^{k+2} \lambda_i(\nabla^2(u-h)) \leq \sum_{i=1}^{k+2} \lambda_i(\nabla^2 u) \leq 0.$$

Indeed, the first inequality follows from Lemma 8.2 by taking $l = k + 2$, $M = (2a/\bar{\epsilon})\nabla^2 u$, $D = \text{diag}(1, \dots, 1, -\delta_1, \dots, -\delta_k)$, and $\delta_1 = \dots = \delta_k = 4a(\bar{C} + 1)/\bar{\epsilon}$; while the second inequality is (9). Thus, in view of Proposition 1.1,

$$u - h \geq \inf_{\partial B_a(0) \setminus E} (u - h) \geq 0,$$

and therefore

$$\liminf_{x \rightarrow 0, x \in \Omega \setminus E} u(x) \geq h(0) = v(0) + \frac{a^2}{2} > v(0),$$

violating (43). Lemma 2.2 is established. \square

Given Lemma 2.2, we have, as in the proof of Theorem 1.2 for $k = n - 2$, (47) for some $x_\epsilon \in \Omega \setminus E$, where \tilde{v}_ϵ is given in (48). The rest of the proof is the same as in the proof of Theorem 1.2 for $k = n - 2$. Theorem 1.2 is established.

3. Proof of Theorem 1.3

By (13) and Lemma 12.1,

$$\Delta(u - v) \leq C - \Delta v \leq C_3 := C - \min_{\bar{\Omega}} \Delta v < \infty \quad \text{in } \Omega. \quad (52)$$

We take u to be the canonical representative in Ω , i.e.

$$u(x) = \lim_{r \rightarrow 0^+} \int_{B_r(x)} u, \quad x \in \Omega$$

(see Appendix F).

Supposing the conclusion of the theorem is false, we have, after a possible translation, $0 \in E$, and

$$\liminf_{x \in \Omega \setminus E, x \rightarrow 0} (u - v)(x) = 0.$$

Shrinking Ω slightly, we may assume without loss of generality that $v \in C^2(\bar{\Omega})$ and, for some constants $\delta_1, \delta_2 > 0$,

$$(u - v)(x) \geq \delta_1, \quad x \in \Omega, \text{ dist}(x, \partial\Omega) \leq \delta_2. \quad (53)$$

For $\epsilon \in (0, \delta_1)$, let, for $d = \text{diam}(\Omega)$,

$$w_\epsilon(x) := \begin{cases} \min\{(u - v)(x) - \epsilon, 0\}, & x \in \Omega, \\ 0, & x \in B_{2d}(0) \setminus \bar{\Omega}, \end{cases}$$

and

$$\Gamma_{w_\epsilon}(x) := \sup\{a + b \cdot x \mid a \in \mathbb{R}, b \in \mathbb{R}^n, a + b \cdot z \leq w_\epsilon(z) \forall z \in B_{2d}(0)\}$$

be the convex envelope of w_ϵ on $B_{2d}(0)$.

Since $\Gamma_{w_\epsilon} = 0$ outside $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta_2\}$, and $\min_{B_{2d}(0)} \Gamma_{w_\epsilon} \leq w_\epsilon(0) = -\epsilon < 0$, the contact set of w_ϵ and Γ_{w_ϵ} satisfies

$$\{x \in B_{2d}(0) \mid w_\epsilon(x) = \Gamma_{w_\epsilon}(x)\} \subset \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta_2\} \subset B_d(0). \quad (54)$$

Lemma 3.1. *There exist some positive constants K and δ such that for any point $\bar{x} \in \{x \in B_{2d}(0) \mid w_\epsilon(x) = \Gamma_{w_\epsilon}(x)\}$, there exists $\bar{p} \in \mathbb{R}^n$ so that*

$$\Gamma_{w_\epsilon}(x) \leq \Gamma_{w_\epsilon}(\bar{x}) + \bar{p} \cdot (x - \bar{x}) + K|x - \bar{x}|^2, \quad \forall |x - \bar{x}| < \delta. \quad (55)$$

Proof. By (53) and (54), $\bar{x} \in \Omega$, $\text{dist}(\bar{x}, \partial\Omega) \geq \delta_2$, and

$$u(\bar{x}) - v(\bar{x}) - \epsilon = w_\epsilon(\bar{x}) = \Gamma_{w_\epsilon}(\bar{x}) < 0, \quad \text{and} \quad u - v - \epsilon \geq w_\epsilon \geq \Gamma_{w_\epsilon} \text{ in } \Omega.$$

By (52) and Lemma A in the introduction, Γ_{w_ϵ} has a unique supporting plane at $(\bar{x}, \Gamma_{w_\epsilon}(\bar{x}))$. So $\nabla \Gamma_{w_\epsilon}(\bar{x})$, as the slope of the supporting plane, is well defined. Let \bar{y} be a point near \bar{x} and let $q \in \mathbb{R}^n$ be the slope of a supporting plane of Γ_{w_ϵ} at $(\bar{y}, \Gamma_{w_\epsilon}(\bar{y}))$. Then, for all z in Ω ,

$$(u - v)(z) \geq (u - v)(\bar{x}) + \nabla \Gamma_{w_\epsilon}(\bar{x}) \cdot (z - \bar{x}),$$

and

$$\begin{aligned} (u - v)(z) &\geq \epsilon + \Gamma_{w_\epsilon}(\bar{y}) + q \cdot (z - \bar{y}) \\ &\geq \epsilon + \Gamma_{w_\epsilon}(\bar{x}) + \nabla \Gamma_{w_\epsilon}(\bar{x}) \cdot (\bar{y} - \bar{x}) + q \cdot (z - \bar{y}) \\ &= (u - v)(\bar{x}) + \nabla \Gamma_{w_\epsilon}(\bar{x}) \cdot (\bar{y} - \bar{x}) + q \cdot (z - \bar{y}). \end{aligned}$$

Thus, by Lemma 12.3 and in view of (52),

$$|q - \nabla \Gamma_{w_\epsilon}(\bar{x})| \leq C_4 C_3 |\bar{y} - \bar{x}|$$

for some dimensional constant C_4 . This implies (55). Indeed, since $\Gamma_{w_\epsilon}(\bar{x}) \geq \Gamma_{w_\epsilon}(\bar{y}) + q \cdot (\bar{x} - \bar{y})$, we have

$$\Gamma_{w_\epsilon}(\bar{y}) \leq \Gamma_{w_\epsilon}(\bar{x}) + \nabla \Gamma_{w_\epsilon}(\bar{x}) \cdot (\bar{y} - \bar{x}) + [q - \nabla \Gamma_{w_\epsilon}(\bar{x})] \cdot (\bar{y} - \bar{x}). \quad \square$$

With Lemma 3.1, we can apply Lemma 3.5 of [2] to infer that $\Gamma_{w_\epsilon} \in C_{\text{loc}}^{1,1}(B_{2d}(0))$. Note that in the statement of Lemma 3.5 of [2], the function is assumed to be continuous, but the proof there applies to LSC functions, as in our

case. By the Aleksandrov–Bakelman–Pucci estimate,

$$\epsilon^n = |\inf_{\Omega} w_{\epsilon}|^n \leq \int_{\{w_{\epsilon}=\Gamma_{w_{\epsilon}}\}} \det(\nabla^2 \Gamma_{w_{\epsilon}}) = \int_{\{w_{\epsilon}=\Gamma_{w_{\epsilon}}\} \cap (\Omega \setminus E)} \det(\nabla^2 \Gamma_{w_{\epsilon}}). \quad (56)$$

For the last equality, we have used the fact that E has zero Lebesgue measure.

At a point x in $\{w_{\epsilon} = \Gamma_{w_{\epsilon}}\} \cap (\Omega \setminus E)$, there exist $a \in \mathbb{R}$ and b in \mathbb{R}^n such that

$$0 > (u - v)(x) - \epsilon = w_{\epsilon}(x) = \Gamma_{w_{\epsilon}}(x) \geq -\epsilon,$$

$$(u - v)(z) - \epsilon = w_{\epsilon}(z) \geq \Gamma_{w_{\epsilon}}(z) \geq \Gamma_{w_{\epsilon}}(x) + \nabla \Gamma_{w_{\epsilon}}(x) \cdot (z - x).$$

If $\nabla \Gamma_{w_{\epsilon}}$ is differentiable at x , then, for any $\delta > 0$,

$$\begin{aligned} u(z) &\geq (\Gamma_{w_{\epsilon}} + v)(x) + \epsilon + \nabla(\Gamma_{w_{\epsilon}} + v)(x) \cdot (z - x) \\ &\quad + \frac{1}{2}(z - x)^t \nabla^2(\Gamma_{w_{\epsilon}} + v)(x) \cdot (z - x) + o(|z - x|^2) \\ &\geq (\Gamma_{w_{\epsilon}} + v)(x) + \epsilon + \nabla(\Gamma_{w_{\epsilon}} + v)(x) \cdot (z - x) \\ &\quad + \frac{1}{2}(z - x)^t \nabla^2(\Gamma_{w_{\epsilon}} + v)(x) \cdot (z - x) - \frac{\delta}{2}|z - x|^2, \quad \text{near } x. \end{aligned}$$

By (13), (15) and the above,

$$\Delta(\Gamma_{w_{\epsilon}} + v)(x) - \delta n \leq C,$$

and

$$\begin{aligned} F(x, (\Gamma_{w_{\epsilon}} + v)(x) + \epsilon, \nabla(\Gamma_{w_{\epsilon}} + v)(x), \nabla^2(\Gamma_{w_{\epsilon}} + v)(x) - \delta I) \\ \geq F(x, v(x), \nabla v(x), \nabla^2 v(x)). \end{aligned}$$

Sending δ to 0 leads to

$$\Delta(\Gamma_{w_{\epsilon}} + v)(x) \leq C,$$

and

$$F(x, (\Gamma_{w_{\epsilon}} + v)(x) + \epsilon, \nabla(\Gamma_{w_{\epsilon}} + v)(x), \nabla^2(\Gamma_{w_{\epsilon}} + v)(x)) \geq F(x, v(x), \nabla v(x), \nabla^2 v(x)).$$

Clearly,

$$|\Gamma_{w_{\epsilon}}(x)| \leq \epsilon, \quad |\nabla \Gamma_{w_{\epsilon}}(x)| \leq \epsilon/d.$$

Since $v \in C^2(\overline{\Omega})$, we have $\Delta \Gamma_{w_{\epsilon}} \leq C - \Delta v(x) \leq \widehat{C}$. By the convexity of $\Gamma_{w_{\epsilon}}$, $\nabla^2 \Gamma_{w_{\epsilon}}(x) \geq 0$, therefore

$$|\nabla^2 \Gamma_{w_{\epsilon}}(x)| \leq \widehat{C}$$

for some positive constant \widehat{C} independent of ϵ and x . In the following \widehat{C} will denote various such constants.

Thus, by (1) and the regularity of F ,

$$\begin{aligned} 0 &\leq F(x, v(x), \nabla v(x), \nabla^2(\Gamma_{w_{\epsilon}} + v)(x)) - F(x, v(x), \nabla v(x), \nabla^2 v(x)) + \widehat{C}\epsilon \\ &= \int_0^1 \sum_{ij} \frac{\partial F}{\partial M_{ij}}(x, v(x), \nabla v(x), t \nabla^2 \Gamma_{w_{\epsilon}}(x)) \partial_{ij} \Gamma_{w_{\epsilon}}(x) dt + \widehat{C}\epsilon \\ &\leq -\frac{1}{\widehat{C}} \det(\nabla^2 \Gamma_{w_{\epsilon}}(x))^{1/n} + \widehat{C}\epsilon. \end{aligned}$$

By a classical theorem of Rademacher, $\nabla\Gamma_{w_\epsilon}$ is differentiable almost everywhere. Therefore, in view of (56),

$$\epsilon^n \leq \int_{\{w_\epsilon = \Gamma_{w_\epsilon}\} \cap (\Omega \setminus E)} \det(\nabla^2 \Gamma_{w_\epsilon}) \leq \widehat{C} \epsilon^n |\{w_\epsilon = \Gamma_{w_\epsilon}\} \cap (\Omega \setminus E)|.$$

Since $u > v$ in $\Omega \setminus E$, E has zero Lebesgue measure, and

$$\{w_\epsilon = \Gamma_{w_\epsilon}\} \cap (\Omega \setminus E) \subset \{x \in \Omega \setminus E \mid (u - v)(x) < \epsilon\},$$

we have, as $\epsilon \rightarrow 0$,

$$\begin{aligned} & |\{w_\epsilon = \Gamma_{w_\epsilon}\} \cap (\Omega \setminus E)| \\ & \leq |\{x \in \Omega \setminus E \mid (u - v)(x) < \epsilon\}| \rightarrow |\{x \in \Omega \setminus E \mid (u - v)(x) \leq 0\}| = 0. \end{aligned}$$

This leads to a contradiction if we choose ϵ very small from the beginning. Theorem 1.3 is established.

4. Symmetry of a singular solution

In this section we prove Theorem 1.4. The proof follows closely arguments in [1], with Theorem 1.1 as an additional ingredient to handle the possible isolated singularity of u at the origin.

For $0 < \lambda < 1$, let, with $x_\lambda := (2\lambda - x_1, x_2, \dots, x_n)$,

$$\begin{aligned} \Sigma_\lambda &:= \{x \mid |x| < 1, \lambda < x_1 < 1\}, & T_\lambda &:= \{x \mid |x| < 1, x_1 = \lambda\}, \\ u_\lambda(x) &:= u(x_\lambda) & \text{and} & & w_\lambda &:= u_\lambda - u. \end{aligned}$$

We will prove that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \setminus \{0_\lambda\} \quad \forall 0 < \lambda < 1.$$

Step 1. There exists $1/2 < \lambda_0 < 1$ such that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \quad \forall \lambda_0 < \lambda < 1.$$

Using (19) and (18), we have in Σ_λ , for all $3/4 < \lambda < 1$,

$$\begin{aligned} 0 &= F(x, u(x), \nabla u(x), \nabla^2 u(x)) - F(x_\lambda, u(x_\lambda), \nabla u(x_\lambda), \nabla^2 u(x_\lambda)) \\ &\leq F(x, u(x), \nabla u(x), \nabla^2 u(x)) - F(x, u_\lambda(x), \nabla u_\lambda(x), \nabla^2 u_\lambda(x)). \end{aligned} \quad (57)$$

By (17) and the fact that u is in $C^2(\overline{B}_1 \setminus \{0\})$, there exists some positive constant C_1 independent of λ , and some functions $\{a_{ij}(x)\}$, $\{b_i(x)\}$, $c(x)$ satisfying, with I denoting the $n \times n$ identity matrix,

$$\frac{1}{C_1} I \leq (a_{ij}(x)) \leq C_1 I, \quad |b_i(x)| + |c(x)| \leq C_1, \quad \text{in } \Sigma_\lambda,$$

such that

$$-a_{ij}(x)\partial_{ij}w_\lambda + b_i(x)\partial_i w_\lambda + c(x)w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda. \quad (58)$$

It is clear that $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$. Using the maximum principle for domains of small measure as in [1], we have $w_\lambda \geq 0$ in Σ_λ if $1 - \lambda > 0$ is smaller than some positive constant which depends only on C_1 and n . Step 1 is established.

Define

$$\bar{\lambda} := \inf\{\mu \mid 0 < \mu < 1, w_\lambda \geq 0 \text{ in } \Sigma_\lambda \setminus \{0\} \forall \mu < \lambda < 1\}.$$

Because of Step 1, $0 \leq \bar{\lambda} < 1$.

Step 2. $\bar{\lambda} = 0$.

There are several cases to consider.

Case 1: $1/2 < \bar{\lambda} < 1$. In this case we argue exactly as in [1]. For $\bar{\epsilon} := \frac{1}{4}(\frac{1}{2} + \bar{\lambda})$ and $\frac{1}{2} < \frac{1}{2}\bar{\lambda} - \bar{\epsilon} \leq \lambda \leq \bar{\lambda}$, w_λ satisfies a uniformly elliptic inequality (58) in Σ_λ as before, with ellipticity constants independent of λ —though they may depend on $\bar{\epsilon}$ due to the possible singularity of u at 0. By continuity, $w_{\bar{\lambda}} \geq 0$ in $\Sigma_{\bar{\lambda}}$. Since $w_{\bar{\lambda}} > 0$ on $\partial\Sigma_{\bar{\lambda}} \cap B_1$, it follows from the strong maximum principle that $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$. For $\delta > 0$ small, let D_δ be the set of points in $\Sigma_{\bar{\lambda}}$ whose distance to its boundary is $\geq \delta$. Then, in D_δ , $w_{\bar{\lambda}} \geq \alpha$ for some positive constant α .

For $0 < \epsilon < \bar{\epsilon}$ small, $w_{\bar{\lambda}-\epsilon} \geq \alpha/2$ in D_δ . Thus

$$w_{\bar{\lambda}-\epsilon} \geq 0 \quad \text{on } \partial(\Sigma_{\bar{\lambda}-\epsilon} \setminus D_\delta).$$

For δ and ϵ small we conclude again, by using the maximum principle in domains of small measure as in [1], that

$$w_{\bar{\lambda}-\epsilon} \geq 0 \quad \text{in } \Sigma_{\bar{\lambda}-\epsilon} \setminus D_\delta \text{ and hence in } \Sigma_{\bar{\lambda}-\epsilon},$$

contradicting the definition of $\bar{\lambda}$.

Case 2: $\bar{\lambda} = 1/2$. Observe first that by property (2),

$$\liminf_{x \rightarrow 0} u(x) \geq a > 0. \quad (59)$$

By continuity, $w_{\bar{\lambda}} \geq 0$ in $\Sigma_{\bar{\lambda}}$. Since u is small near $(1, 0, \dots, 0)$, we deduce from (59) that for $\rho > 0$ small,

$$w_\lambda \geq a/2 \quad \text{in } (B_\rho(0_\lambda) \cap \Sigma_\lambda) \setminus \{0_\lambda\}, \quad \forall \bar{\lambda} - \rho \leq \lambda \leq \bar{\lambda}. \quad (60)$$

For $\bar{\lambda} - \rho \leq \lambda \leq \bar{\lambda}$, w_λ satisfies a uniformly elliptic inequality (58) in $\Sigma_\lambda \setminus B_\rho(0_\lambda)$ as before, with ellipticity constants independent of λ —though they may depend on ρ . By the strong maximum principle, $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}} \setminus B_\rho(0_{\bar{\lambda}})$. From now on ρ is fixed.

As in Case 1, for δ small, consider $D_\delta =$ set of points in $\Sigma_{\bar{\lambda}}$ whose distance to its boundary is $\geq \delta$. As before,

$$w_{\bar{\lambda}} > 0 \quad \text{in } D_\delta.$$

By continuity, since D_δ is closed, we have, for small $0 < \epsilon < \rho$,

$$w_{\bar{\lambda}-\epsilon} > 0 \quad \text{in } D_\delta.$$

This and (60) with $\lambda = \bar{\lambda} - \epsilon$ imply that $w_{\bar{\lambda}-\epsilon} \geq 0$ on $\partial(\Sigma_{\bar{\lambda}-\epsilon} \setminus (D_\delta \cup B_\rho(0_{\bar{\lambda}-\epsilon})))$.

For ϵ, δ small, we have, since the region has small measure,

$$\Sigma_{\bar{\lambda}-\epsilon} \setminus (D_\delta \cup B_\rho(0_{\bar{\lambda}-\epsilon})),$$

which has small volume, and conclude that $w_{\bar{\lambda}-\epsilon} \geq 0$ there, and hence in $\Sigma_{\bar{\lambda}-\epsilon} \setminus \{0_{\bar{\lambda}-\epsilon}\}$. This again contradicts the definition of $\bar{\lambda}$.

Case 3: $0 < \bar{\lambda} < 1/2$. Once more, $w_{\bar{\lambda}} \geq 0$ in $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$ and $w_{\bar{\lambda}} > 0$ on $\partial\Sigma_{\bar{\lambda}} \cap B_1$. Since $w_{\bar{\lambda}}$ is C^2 in $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$, and $w_{\bar{\lambda}}$ satisfies a uniformly elliptic inequality of the form (58) on any compact subset of $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$, by the strong maximum principle, $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$.

Since inequality (57) holds in $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$, an application of Theorem 1.1 yields

$$\liminf_{x \rightarrow 0} w_{\bar{\lambda}}(x) > 0,$$

and so, in some ball $B_{2\rho}(0_{\bar{\lambda}})$ with $\rho > 0$,

$$w_{\bar{\lambda}} > \alpha \quad \text{in } B_{2\rho}(0_{\bar{\lambda}}) \setminus \{0_{\bar{\lambda}}\} \quad \text{for some } \alpha > 0. \quad (61)$$

For $0 < \lambda \leq \bar{\lambda}$, w_{λ} satisfies a uniformly elliptic inequality of the form (58) in $\Sigma_{\lambda} \setminus B_{\rho}(0_{\lambda})$ with ellipticity constants independent of λ , though they may depend on ρ . From now on ρ is fixed.

As before, for $\delta > 0$ small, let D_{δ} be the set of points in $\Sigma_{\bar{\lambda}}$ whose distance to its boundary is $\geq \delta$. We can take δ so small that the origin $0_{\bar{\lambda}}$ is in the interior of D_{δ} . Using (61) and the positivity of $w_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}} \setminus \{0_{\bar{\lambda}}\}$, we have

$$w_{\bar{\lambda}} \geq \gamma \quad \text{in } D_{\delta} \setminus \{0_{\bar{\lambda}}\} \quad \text{for some } \gamma > 0. \quad (62)$$

Now consider $\Sigma_{\bar{\lambda}-\epsilon}$ for $0 < \bar{\lambda} - \epsilon$.

By continuity and by (62), we have, for ϵ small,

$$w_{\bar{\lambda}-\epsilon}(x) > \gamma/2 \quad \forall x \in D_{\delta} \setminus \{0_{\bar{\lambda}-\epsilon}\}.$$

The above for $x \in (D_{\delta} \cap B_{\rho}(0_{\bar{\lambda}})) \setminus \{0_{\bar{\lambda}}\}$ requires some explanation as follows. We write

$$w_{\bar{\lambda}-\epsilon}(x) = w_{\bar{\lambda}}(\tilde{x}) + u(\tilde{x}) - u(x),$$

where $\tilde{x} = (x_1 + 2\epsilon, x_2, \dots, x_n)$. For ϵ small, $\tilde{x} \in B_{2\rho}(0_{\bar{\lambda}})$. So, by (62), and by the continuity of u in \bar{D}_{δ} , we have, for small ϵ ,

$$w_{\bar{\lambda}-\epsilon}(x) = w_{\bar{\lambda}}(\tilde{x}) + u(\tilde{x}) - u(x) \geq \alpha + u(\tilde{x}) - u(x) \geq \alpha/2.$$

Clearly, $w_{\bar{\lambda}-\epsilon} \geq 0$ on $\partial(\Sigma_{\bar{\lambda}-\epsilon} \setminus D_{\delta})$. As pointed out earlier, $w_{\bar{\lambda}-\epsilon}$ satisfies a uniformly elliptic inequality like (58) in $\Sigma_{\bar{\lambda}-\epsilon} \setminus D_{\delta}$ with ellipticity constants independent of ϵ and δ . Applying the maximum principle in the region, which has small measure, we conclude that $w_{\bar{\lambda}-\epsilon} \geq 0$ there, and hence in $\Sigma_{\bar{\lambda}-\epsilon} \setminus \{0_{\bar{\lambda}-\epsilon}\}$, a contradiction.

Step 2 is established.

Since $w_{\lambda} = 0$ on T_{λ} , it follows from Step 2 that

$$2 \frac{\partial u}{\partial x_1} = \frac{\partial w_{\lambda}}{\partial x_1} \leq 0 \quad \text{on } T_{\lambda}, \quad \forall 0 < \lambda < 1,$$

that is,

$$\frac{\partial u}{\partial x_1}(x) \leq 0 \quad \forall x \in B_1, \quad 0 < x_1 < 1.$$

For $0 < \lambda < 1$, we already know that $w_\lambda \geq 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$. We also know that w_λ is not identically zero since it is positive on $\partial\Sigma_\lambda \cap B_1$. Because of (57), we can apply the Hopf Lemma to w_λ to obtain

$$\frac{\partial w_\lambda}{\partial x_1} < 0 \quad \text{on } T_\lambda, \quad \forall 0 < \lambda < 1,$$

that is, (20) holds. Theorem 1.4 is established.

5. Proof of Theorem 1.5

The proof is similar to that of Theorem 1.2. Shrinking Ω slightly, we may assume that $u > v$ on $\bar{\Omega} \setminus \{0\}$, $u \in C^2(\bar{\Omega} \setminus \{0\})$, and $v \in C^2(\bar{\Omega})$. We prove the theorem by contradiction. Suppose that (36) holds.

Fix some $0 < \alpha < 1$. Let

$$v_\epsilon(x) = v(x) + \epsilon|x|^{1+\alpha}.$$

Similar to Lemma 2.2 we have

Lemma 5.1. *For $\epsilon > 0$,*

$$\lambda(\epsilon) := - \inf_{\Omega \setminus \{0\}} (u - v_\epsilon) > 0.$$

Proof. Suppose not; then, for some $\epsilon > 0$ and $C > 0$,

$$u(x) \geq v_\epsilon(x) \geq \epsilon|x|^{1+\alpha} + v(0) + \nabla v(0) \cdot x - C|x|^2, \quad x \in \Omega \setminus \{0\}.$$

For small $r > 0$,

$$\min_{x \in \partial B_r} [u(x) - \nabla v(0) \cdot x] \geq v(0) + \epsilon r^{1+\alpha} - Cr^2 \geq v(0) + \frac{\epsilon}{2} r^{1+\alpha}.$$

On the other hand, in view of (36),

$$\inf_{x \in B_r \setminus \{0\}} [u(x) - \nabla v(0) \cdot x] \leq \liminf_{x \rightarrow 0} [u(x) - \nabla v(0) \cdot x] = v(0) < v(0) + \frac{\epsilon}{2} r^{1+\alpha}.$$

The above contradicts condition (2). \square

By Lemma 5.1, there exists $x_\epsilon \in \Omega \setminus \{0\}$ such that

$$u - \tilde{v}_\epsilon \geq 0 \quad \text{in } \Omega \setminus \{0\}, \quad (u - \tilde{v}_\epsilon)(x_\epsilon) = 0,$$

where

$$\tilde{v}_\epsilon(x) := v_\epsilon(x) - \lambda(\epsilon) = v(x) + \epsilon|x|^{1+\alpha} - \lambda(\epsilon).$$

Using the positivity of $u - v$ in $\bar{\Omega} \setminus \{0\}$, we obtain

$$0 < \lambda(\epsilon) = -(u - v_\epsilon)(x_\epsilon) = v(x_\epsilon) - u(x_\epsilon) + \epsilon|x_\epsilon|^{1+\alpha} \leq \epsilon|x_\epsilon|^{1+\alpha}, \quad (63)$$

and

$$\lim_{\epsilon \rightarrow 0} |x_\epsilon| = 0.$$

We will show that

$$\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \in \mathcal{S}^{n \times n} \setminus U. \quad (64)$$

A calculation gives

$$\nabla^2 \tilde{v}_\epsilon(x) - \nabla^2 v(x) = (1 + \alpha)\epsilon O(x)^t (\text{diag}(\alpha|x|^{\alpha-1}, |x|^{\alpha-1}, \dots, |x|^{\alpha-1}) + O(1)) O(x),$$

where $O(x)$ are orthogonal matrices.

At x_ϵ , using (63), we have

$$|L(x_\epsilon, v, \nabla v) - L(x_\epsilon, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon)| \leq C(|\nabla v - \nabla \tilde{v}_\epsilon| + |v - \tilde{v}_\epsilon|) \leq C\epsilon|x_\epsilon|^\alpha.$$

It follows, for $\epsilon > 0$ small, that

$$\tilde{A}^v(x_\epsilon) \geq \tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) + \alpha(1 + \alpha)\epsilon|x_\epsilon|^{\alpha-1}I - C\epsilon|x_\epsilon|^\alpha I > \tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon).$$

Since $\tilde{A}^v(x_\epsilon)$ is in $\mathcal{S}^{n \times n} \setminus U$, so is $\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon)$ —due to (25).

Since x_ϵ is an interior local minimum point of $u - \tilde{v}_\epsilon$, we have

$$u(x_\epsilon) = \tilde{v}_\epsilon(x_\epsilon), \quad \nabla u(x_\epsilon) = \tilde{v}_\epsilon(x_\epsilon), \quad \nabla^2 u(x_\epsilon) \geq \nabla^2 v_\epsilon(x_\epsilon).$$

Thus

$$\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \geq \tilde{A}^u(x_\epsilon).$$

Since $\tilde{A}^u(x_\epsilon) \in \bar{U}$, we know from the above, in view of (25), that $\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \in \bar{U}$. This violates (64). Theorem 1.5 is established.

6. Proof of Theorem 1.6

The proof uses arguments of the proofs of Theorems 1.2 and 1.5.

We start with the first paragraph of the proof of Theorem 1.2. Fix some constant $0 < \alpha < 1$. For x close to 0, we have, for some constant C ,

$$\frac{3}{4}|x'|^{1+\alpha} - C|x''|^2 \leq d(x)^{1+\alpha} \equiv \text{dist}(x, E)^{1+\alpha} \leq \frac{5}{4}|x'|^{1+\alpha} + C|x''|^2.$$

We only prove the left inequality. From (51) we have

$$|x'| \leq d(x) + C|x''|^2,$$

so

$$\begin{aligned} |x'|^{1+\alpha} &\leq d(x)^{1+\alpha} \left(1 + C \left(\frac{|x''|^2}{d(x)} \right) \right)^{1+\alpha} \\ &\leq d(x)^{1+\alpha} \left(1 + K \frac{|x''|^2}{d(x)} + K \left(\frac{|x''|^2}{d(x)} \right)^{1+\alpha} \right) \\ &\leq d(x)^{1+\alpha} + Kd(x)^\alpha |x''|^2 + K|x''|^{2(1+\alpha)}, \end{aligned}$$

which yields the first inequality.

Let

$$v_\epsilon(x) = v(x) + \epsilon d(x)^{1+\alpha}.$$

Claim. *The conclusion of Lemma 2.2 holds with the above v_ϵ .*

To prove the claim, we only need to follow the proof of Lemma 2.2 with the following modifications:

1. In the first displayed formulas, change “ $\bar{\epsilon}d(x)$ ” to “ $\bar{\epsilon}d(x)^{1+\alpha}$ ”, and change “ $\frac{\bar{\epsilon}}{2}|x'|$ ” to “ $\frac{\bar{\epsilon}}{2}|x'|^{1+\alpha}$ ”.
2. Change “ $0 < a < \min\{\bar{r}, \bar{\epsilon}/4\}$ ” to “ $0 < a < \min\{\bar{r}, (\bar{\epsilon}/4)^{1-\alpha}\}$ ”.
3. In the next two displayed formulas, change the two “ $\frac{\bar{\epsilon}}{4a}$ ” to “ $\frac{\bar{\epsilon}}{4a^{1-\alpha}}$ ”, and “ $\frac{\bar{\epsilon}}{2a}$ ” to “ $\frac{\bar{\epsilon}}{2a^{1-\alpha}}$ ”.
4. Later on, change “ $M = (2a/\bar{\epsilon})\nabla^2 u$ ” to “ $M = (2a^{1-\alpha}/\bar{\epsilon})\nabla^2 u$ ”, and “ $\delta_1 = \dots = \delta_k = 4a(\bar{C} + 1)/\bar{\epsilon}$ ” to “ $\delta_1 = \dots = \delta_k = 4a^{1-\alpha}(\bar{C} + 1)/\bar{\epsilon}$ ”.

Given the claim above, we have, as in the proof of Theorem 1.2 for $k = n - 2$, (47) for some $x_\epsilon \in \Omega \setminus E$, where

$$\tilde{v}_\epsilon(x) := v_\epsilon(x) - \lambda(\epsilon) = v(x) + \epsilon d(x)^{1+\alpha} - \lambda(\epsilon).$$

Using the positivity of $u - v$ in $\bar{\Omega} \setminus E$, we obtain

$$0 < \lambda(\epsilon) = -(u - v_\epsilon)(x_\epsilon) = v(x_\epsilon) - u(x_\epsilon) + \epsilon d(x_\epsilon)^{1+\alpha} \leq \epsilon d(x_\epsilon)^{1+\alpha}, \quad (65)$$

and

$$\lim_{\epsilon \rightarrow 0} d(x_\epsilon) = 0.$$

We will show that

$$\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \in \mathcal{S}^{n \times n} \setminus U.$$

Since x_ϵ is an interior local minimum point of $u - \tilde{v}_\epsilon$, we have

$$u(x_\epsilon) = \tilde{v}_\epsilon(x_\epsilon), \quad \nabla u(x_\epsilon) = \nabla \tilde{v}_\epsilon(x_\epsilon), \quad \nabla^2 u(x_\epsilon) \geq \nabla^2 \tilde{v}_\epsilon(x_\epsilon).$$

Thus

$$\tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \geq \tilde{A}^u(x_\epsilon).$$

By the result in Appendix A,

$$\begin{aligned} & \nabla^2 \tilde{v}_\epsilon(x) - \nabla^2 v(x) \\ &= (1 + \alpha)\epsilon O(x)^t (\text{diag}(0, \dots, 0, \alpha d(x)^{\alpha-1}, d(x)^{\alpha-1}, \dots, d(x)^{\alpha-1}) + O(1)) O(x), \end{aligned} \quad (66)$$

where $O(x)$ are orthogonal matrices, and there are k zeros in the diagonal matrix.

At x_ϵ , using (65), we have

$$|L(x_\epsilon, v, \nabla v) - L(x_\epsilon, \tilde{v}_\epsilon, \nabla \tilde{v}_\epsilon)| \leq C_1(|\nabla v - \nabla \tilde{v}_\epsilon| + |v - \tilde{v}_\epsilon|) \leq C_1 \epsilon d(x_\epsilon)^\alpha,$$

where $C_1 > 0$ is some constant independent of ϵ . In the following we use $\{C_i\}$ to denote various positive constants which are independent of ϵ .

Since $d(x_\epsilon) \rightarrow 0$ and $\alpha < 1$, we have

$$\tilde{A}^v(x_\epsilon) - \tilde{A}^u(x_\epsilon) \geq \tilde{A}^v(x_\epsilon) - \tilde{A}^{\tilde{v}_\epsilon}(x_\epsilon) \geq \hat{N}, \quad (67)$$

where

$$\hat{N} := \alpha \epsilon O(x_\epsilon)^t (\text{diag}(-C_2, \dots, -C_2, d(x_\epsilon)^{\alpha-1}, d(x_\epsilon)^{\alpha-1}, \dots, d(x_\epsilon)^{\alpha-1})) O(x_\epsilon).$$

Since v is C^2 in Ω ,

$$M := \tilde{A}^u(x_\epsilon) \leq \tilde{A}^v(x_\epsilon) - \hat{N} \leq \tilde{A}^v(x_\epsilon) + \alpha C_2 \epsilon I \leq C_3 I.$$

Let $\bar{\delta} = \bar{\delta}(C_3)$ and $\bar{\epsilon} = \bar{\epsilon}(C_3)$ be the positive numbers in Property P_k , and let

$$N := \frac{\alpha\bar{\delta}}{C_2} O(x_\epsilon)^t (\text{diag}(-\bar{\delta}, \dots, -\bar{\delta}, 1, \dots, 1)) O(x_\epsilon).$$

Since $M = \tilde{A}^u(x_\epsilon) \in \bar{U}$, we know from Property P_k that $M + N \in U$ for small $\epsilon > 0$.

For small $\epsilon > 0$, $d(x_\epsilon)^{\alpha-1} > C_2/\bar{\delta}$, so $\hat{N} \geq N$.

By (67) and the above, $\tilde{A}^v(x_\epsilon) \geq M + \hat{N} \geq M + N$. Since $M + N \in U$ for small ϵ , we have, and in view of (25), $\tilde{A}^v(x_\epsilon) \in U$ for small ϵ . This violates (31). Theorem 1.6 is established.

Concerning Remark 1.9. For (ii): Let $\mathcal{N} \subset \mathcal{S}^{n \times n}$ denote the set of matrices whose eigenvalues consist of k zeros and $n - k$ 1's. We know that $\mathcal{N} \subset U$. Since \mathcal{N} is compact, there exists some positive number $\bar{\delta}$ such that the $n\bar{\delta}$ -neighborhood of \mathcal{N} is contained in U . Consequently, $N \in U$ if its eigenvalues consist of k $-\bar{\delta}$'s and $n - k$ 1's. For $M \in \bar{U}$, we know that $2M \in \bar{U}$, since U is invariant under multiplication by positive constants. Similarly, $2\epsilon N \in U$ for any $0 < \epsilon < 1$. By the convexity of U , $M + \epsilon N = \frac{1}{2}(2M + 2\epsilon N) \in U$.

For (iii): It is easy to see that U_l satisfies (ii).

For (i): For any $M \in \partial U$ with $M \leq CI$, there exist some $a = a(M) > 0$ and $\beta = \beta(M) > 0$ such that if $\hat{M} \in \partial U$ and $|\hat{M} - M| < \beta$, then

$$\hat{M} + aO^t \text{diag}(-\delta, \dots, -\delta, 1, \dots, 1)O \in U, \quad \forall 0 < \delta \leq \beta,$$

where there are k $-\delta$'s and $n - k$ 1's, and $O \in O(n + 1)$ is an orthogonal matrix. By the convexity of U , we have

$$\hat{M} + \epsilon O^t \text{diag}(-\delta, \dots, -\delta, 1, \dots, 1)O \in U, \quad \forall 0 < \delta \leq \beta, 0 < \epsilon \leq a.$$

By the compactness of $\partial U \cap \{M \leq CI\}$, there exist $\bar{\delta} > 0$ and $\bar{\epsilon} > 0$ such that

$$M + \epsilon O^t \text{diag}(-\delta, \dots, -\delta, 1, \dots, 1)O \in U, \quad \forall 0 < \delta \leq \bar{\delta}, 0 < \epsilon \leq \bar{\epsilon}.$$

7. Appendix A

For $n \geq 1$ and $0 \leq k \leq n - 1$, let E be a smooth compact k -dimensional submanifold in \mathbb{R}^n , and let

$$d(x) := \text{dist}(x, E) := \inf\{|x - y| \mid y \in E\}, \quad x \in \mathbb{R}^n,$$

denote the distance function from x to E . For $x \in \mathbb{R}^n \setminus E$ close to E , we use $\nabla^2 d(x)$ to denote the Hessian of $d(x)$.

Lemma 7.1. *Let $E \subset \mathbb{R}^n$ be as above, and let G be a smooth one-variable function, $\hat{G}(x) = G(d(x))$. Then there exists some matrix-valued function $O(x) \in O(n)$ such*

that

$$\begin{aligned} & O(x)^t (\nabla^2 \widehat{G}(x)) O(x) \\ &= \text{diag}(0, \dots, 0, G''(d(x)), d(x)^{-1} G'(d(x)), \dots, d(x)^{-1} G'(d(x))) \\ &\quad + O(d|G''(d)| + |G'(d)|), \quad \text{as } d(x) \rightarrow 0, \end{aligned} \quad (68)$$

with k zeros in the diagonal matrix.

Proof. For x close to E , there exists a unique $\bar{x} = \bar{x}(x) \in E$ such that $d(x) = |x - \bar{x}|$. Without loss of generality, we may assume that $\bar{x} = 0$. Then $d(x) = |x|$. We may also assume that $T_0 E$, the tangent space of E at 0, is spanned by e_{n-k+1}, \dots, e_n , where $(e_i)_{i=1}^n$ the standard basis of \mathbb{R}^n . If we fix a smooth local orthonormal frame $\{\epsilon_1(p), \dots, \epsilon_{n-k}(p)\}$ of the orthogonal complement of $T_p E$, the tangent space of E at $p \in E$, there is a unique representation of $x \in \mathbb{R}^n$ near 0 given by

$$x = p(x) + \varphi_1(x)\epsilon_1(p(x)) + \dots + \varphi_{n-k}(x)\epsilon_{n-k}(p(x))$$

where φ_α and p are smooth functions near 0.

We may also assume that $\epsilon_\alpha(0) = e_\alpha$ for $1 \leq \alpha \leq n-k$. Clearly

$$\varphi_\alpha(0) = 0, \quad \nabla \varphi_\alpha(0) = e_\alpha, \quad 1 \leq \alpha \leq n-k. \quad (69)$$

Near 0,

$$d(x) = \sqrt{\sum_{\alpha=1}^{n-k} \varphi_\alpha(x)^2}. \quad (70)$$

By the chain rule,

$$\partial_{ij} \widehat{G}(x) = G''(d(x)) \partial_i d(x) \partial_j d(x) + G'(d(x)) \partial_{ij} d(x).$$

A calculation gives

$$\partial_i d = d^{-1} \sum_{\alpha} \varphi_\alpha (\partial_i \varphi_\alpha),$$

$$\partial_{ij} d = -d^{-3} \sum_{\alpha, \beta} \varphi_\alpha \varphi_\beta (\partial_i \varphi_\alpha) (\partial_j \varphi_\beta) + d^{-1} \sum_{\alpha} (\partial_i \varphi_\alpha) (\partial_j \varphi_\alpha) + d^{-1} \sum_{\alpha} \varphi_\alpha (\partial_{ij} \varphi_\alpha).$$

Using (69) and the fact $|\varphi_\alpha(x)| \leq d(x)$, we have, as $|x| \rightarrow 0$,

$$\begin{aligned} \partial_i d(x) \partial_j d(x) &= O(d(x)) && \text{if } \max\{i, j\} > n-k, \\ \partial_i d(x) \partial_j d(x) &= d^{-2} \varphi_i \varphi_j + O(d(x)) && \text{if } 1 \leq i, j \leq n-k, \\ d(x) \partial_{ij} d(x) &= O(d(x)) && \text{if } \max\{i, j\} > n-k, \\ d(x) \partial_{ij} d(x) &= \sum_{\alpha} (\partial_i \varphi_\alpha) (\partial_j \varphi_\alpha) - d^{-2} \sum_{\alpha, \beta} (\varphi_\beta \partial_j \varphi_\beta) (\varphi_\alpha \partial_i \varphi_\alpha) + \sum_{\alpha} \varphi_\alpha (\partial_{ij} \varphi_\alpha) \\ &= \delta_{ij} - d^{-2} \varphi_i \varphi_j + O(d(x)) && \text{if } 1 \leq i, j \leq n-k. \end{aligned}$$

Since the $(n-k) \times (n-k)$ matrix $(d^{-2} \varphi_i \varphi_j)$ has eigenvalues 0 and 1, with 0 having the multiplicity $n-k-1$, (68) follows immediately.

8. Appendix B

Lemma 8.1. *Let $M \in \mathcal{S}^{n \times n}$ and $1 \leq k \leq n$. Then*

$$\sum_{i=1}^k \lambda_i(M) = \min \left\{ \sum_{i=1}^k \epsilon_i^t M \epsilon_i \mid \epsilon_i \in \mathbb{R}^n, \epsilon_i^t \epsilon_j = \delta_{ij} \right\}.$$

Proof. There exists an orthonormal basis $\{E_i\}_{i=1}^n$ such that

$$ME_i = \lambda_i(M)E_i.$$

Clearly

$$\sum_{i=1}^k E_i^t M E_i = \sum_{i=1}^k \lambda_i(M) E_i^t E_i = \sum_{i=1}^k \lambda_i(M). \quad (71)$$

Let $\{\epsilon_i\}_{i=1}^k$ satisfy $\epsilon_i^t \epsilon_j = \delta_{ij}$. There exists $O \in O(n)$ such that

$$\epsilon_i = \sum_{j=1}^n O_{ij} E_j, \quad 1 \leq i \leq k.$$

Thus

$$\begin{aligned} & \sum_{i=1}^k \epsilon_i^t M \epsilon_i \\ &= \sum_{i=1}^k \sum_{j=1}^n \sum_{l=1}^n O_{ij} E_j^t O_{il} M E_l = \sum_{i=1}^k \sum_{j=1}^n \sum_{l=1}^n O_{ij} O_{il} \lambda_l(M) E_j^t E_l = \sum_{i=1}^k \sum_{j=1}^n O_{ij}^2 \lambda_j(M) \\ &\geq \sum_{j=1}^k \lambda_j(M) \left(\sum_{i=1}^k O_{ij}^2 \right) + \lambda_k(M) \left(\sum_{j=k+1}^n \sum_{i=1}^k O_{ij}^2 \right). \end{aligned}$$

Since $O \in O(n)$,

$$\sum_{j=1}^n \sum_{i=1}^k O_{ij}^2 = k, \quad \sum_{i=1}^k O_{ij}^2 \leq 1.$$

It follows that

$$\begin{aligned} & \sum_{i=1}^k \epsilon_i^t M \epsilon_i \\ &\geq \sum_{j=1}^k \lambda_j(M) + \sum_{j=1}^k \lambda_j(M) \left(\sum_{i=1}^k O_{ij}^2 - 1 \right) + \lambda_k(M) \left(k - \sum_{j=1}^k \sum_{i=1}^k O_{ij}^2 \right) \\ &\geq \sum_{j=1}^k \lambda_j(M) + \sum_{j=1}^k \lambda_k(M) \left(\sum_{i=1}^k O_{ij}^2 - 1 \right) + \lambda_k(M) \left(k - \sum_{j=1}^k \sum_{i=1}^k O_{ij}^2 \right) = \sum_{j=1}^k \lambda_j(M). \end{aligned}$$

Lemma 8.1 follows from this and (71). \square

Lemma 8.2. *Let $n \geq 2$, $1 \leq k < l \leq n$, $0 < \delta_i \leq (l-k)/k$, $1 \leq i \leq k$, and $M \in \mathcal{S}^{n \times n}$. Then*

$$\lambda_1(M-D) + \cdots + \lambda_l(M-D) \leq \lambda_1(M) + \cdots + \lambda_l(M),$$

where $D = \text{diag}(1, \dots, 1, -\delta_1, \dots, -\delta_k)$.

Proof. There exists an orthonormal basis $\{E_i\}$ of \mathbb{R}^n such that

$$ME_i = \lambda_i(M)E_i, \quad 1 \leq i \leq n.$$

By Lemma 8.1,

$$\sum_{i=1}^l \lambda_i(M-D) \leq \sum_{i=1}^l E_i^t (M-D) E_i = \sum_{i=1}^l \lambda_i(M) - \sum_{i=1}^l E_i^t D E_i. \quad (72)$$

Let $(e_i)_{i=1}^n$ be the standard basis. We know that

$$De_m = \begin{cases} e_m, & 1 \leq m \leq n-k, \\ -\delta_{m-(n-k)} e_m, & n-k+1 \leq m \leq n. \end{cases}$$

There exists $O \in O(n)$ such that

$$E_i = \sum_{j=1}^n O_{ij} e_j, \quad 1 \leq i \leq n.$$

It follows that

$$\begin{aligned} \sum_{i=1}^l E_i^t D E_i &= \sum_{i=1}^l \sum_{j=1}^n \sum_{m=1}^n O_{ij} e_j^t O_{im} D e_m \\ &= \sum_{i=1}^l \sum_{j=1}^n \sum_{m=1}^{n-k} O_{ij} O_{im} \delta_{jm} - \sum_{i=1}^l \sum_{j=1}^n \sum_{m=n-k+1}^n O_{ij} O_{im} \delta_{m-(n-k)} \delta_{jm} \\ &= \sum_{i=1}^l \sum_{m=1}^{n-k} O_{im}^2 - \sum_{i=1}^l \sum_{m=n-k+1}^n O_{im}^2 \delta_{m-(n-k)}. \end{aligned}$$

Since $O \in O(n)$, we have

$$\sum_{m=1}^n O_{im}^2 = 1, \quad \sum_{i=1}^l O_{im}^2 \leq 1.$$

Thus

$$\begin{aligned} \sum_{i=1}^l E_i^t D E_i &= \sum_{i=1}^l \sum_{m=1}^n O_{im}^2 - \sum_{i=1}^l \sum_{m=n-k+1}^n O_{im}^2 - \sum_{i=1}^l \sum_{m=n-k+1}^n O_{im}^2 \delta_{m-(n-k)} \\ &\geq l - k - \frac{l-k}{k} k = 0. \end{aligned}$$

Lemma 8.2 follows from this and (72). \square

9. Appendix C. Proof of Proposition 1.1

For the sake of exposition, we first prove the result for $k = -1$, then for $k = 0$, and finally for $k \geq 1$.

The proof for $k = -1$. For any $\epsilon > 0$, let $u_\epsilon(x) := u(x) - \epsilon|x|^2$. Since $\lambda_1(\nabla^2 u_\epsilon) = \lambda_1(\nabla^2 u) - 2\epsilon \leq -2\epsilon < 0$, we easily see that the minimum of u_ϵ can only be attained on $\partial\Omega$. Indeed, if x_ϵ is a minimum point of u_ϵ in Ω , then $\nabla^2 u_\epsilon(x_\epsilon)$ is a semi-positive definite matrix, violating $\lambda_1(\nabla^2 u_\epsilon) < 0$. Sending ϵ to 0 yields $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

The proof for $k = 0$. For simplicity we assume that E contains only one point, say $\{0\}$, in $\overline{\Omega}$, since the proof requires only some modification when E consists of finitely many points; see the proof for $k \geq 1$. Let

$$\hat{u} := u - \inf_{\Omega \setminus \{0\}} u.$$

Then

$$\inf_{\Omega \setminus \{0\}} \hat{u} = 0. \quad (73)$$

We need to prove that

$$\inf_{\partial\Omega \setminus \{0\}} \hat{u} = 0.$$

Suppose that

$$\inf_{\partial\Omega \setminus \{0\}} \hat{u} = \delta > 0.$$

Let $\bar{r} = 6 \operatorname{diam}(\Omega)$. For any $\epsilon > 0$ small, let

$$h_\epsilon(r) := 1 - \frac{\log(r/\bar{r})}{\log(\epsilon/\bar{r})}, \quad \epsilon \leq r \leq \bar{r}; \quad \hat{h}_\epsilon(x) := h_\epsilon(|x|).$$

Clearly,

$$\begin{aligned} h_\epsilon(\epsilon) &= 0, & h_\epsilon(r) &\leq 1, & \epsilon < r < \bar{r}, \\ h'_\epsilon(r) &= -\frac{1}{r \log(\epsilon/\bar{r})} > 0, & \epsilon < r < \bar{r}, \\ h''_\epsilon(r) &= \frac{1}{r^2 \log(\epsilon/\bar{r})} < 0, & \epsilon < r < \bar{r}, \\ \frac{h'_\epsilon(r)}{r} &= -\frac{1}{r^2 \log(\epsilon/\bar{r})} > 0, & \epsilon < r < \bar{r}. \end{aligned}$$

For $x \in \Omega \setminus B_\epsilon$, the n eigenvalues of $\nabla^2 \hat{h}_\epsilon$ are, with $r = |x|$,

$$h''_\epsilon(r) < \frac{h'_\epsilon(r)}{r} = \dots = \frac{h'_\epsilon(r)}{r},$$

i.e.

$$\begin{aligned} \lambda_1(\nabla^2 \hat{h}_\epsilon(x)) &= h''_\epsilon(r) < 0, \\ \lambda_2(\nabla^2 \hat{h}_\epsilon(x)) &= \dots = \lambda_n(\nabla^2 \hat{h}_\epsilon(x)) = \frac{h'_\epsilon(r)}{r} = -h''_\epsilon(r) > 0. \end{aligned}$$

Since

$$\lambda_1(\nabla^2(\delta\hat{h}_\epsilon)) + \lambda_2(\nabla^2(\delta\hat{h}_\epsilon)) = 0 \quad \text{in } \Omega \setminus \overline{B}_\epsilon,$$

and

$$\lambda_1(\nabla^2\hat{u}) + \lambda_2(\nabla^2\hat{u}) \leq 0 \quad \text{in } \Omega \setminus \overline{B}_\epsilon,$$

we have

$$\lambda_1(\nabla^2(\hat{u} - \delta\hat{h}_\epsilon)) = \lambda_1(\nabla^2\hat{u} - \nabla^2(\delta\hat{h}_\epsilon)) \leq 0 \quad \text{in } \Omega \setminus \overline{B}_\epsilon.$$

Indeed, if the above does not hold, then $\nabla^2\hat{u} - \nabla^2(\delta\hat{h}_\epsilon)$ is a positive definite matrix, and therefore $\lambda_i(\nabla^2\hat{u}) > \lambda_i(\nabla^2(\delta\hat{h}_\epsilon))$ for every i . In particular, $\lambda_1(\nabla^2\hat{u}) + \lambda_2(\nabla^2\hat{u}) > \lambda_1(\nabla^2(\delta\hat{h}_\epsilon)) + \lambda_2(\nabla^2(\delta\hat{h}_\epsilon))$, which leads to a contradiction.

Thus, by Proposition 1.1 for $k = -1$,

$$\hat{u} - \delta\hat{h}_\epsilon \geq \inf_{\partial(\Omega \setminus B_\epsilon)} (\hat{u} - \delta\hat{h}_\epsilon) \geq 0 \quad \text{in } \Omega \setminus \overline{B}_\epsilon.$$

Now, for any $x \in \Omega \setminus \{0\}$, we have, from the above,

$$\hat{u}(x) \geq \delta\hat{h}_\epsilon(x), \quad \forall 0 < \epsilon < |x|.$$

Sending ϵ to 0 leads to $\hat{u}(x) \geq \delta$, i.e. $u \geq \delta$ on $\Omega \setminus \{0\}$. This violates (73). Proposition 1.1 for $k = 0$ is established.

The proof for $k \geq 1$. Let

$$g(s) := -\ln s + \ln(-\ln s), \quad 0 < s < 1,$$

$$\hat{g}(x) := g(d(x)/\bar{r}), \quad x \in \Omega \setminus E,$$

where $d(x) := \text{dist}(x, E)$, and $\bar{r} > 9 \text{ diam}(\Omega)$ is some large positive number whose value will be determined below. Clearly

$$g'(s) = -s^{-1} + s^{-1}(\ln s)^{-1}, \quad g''(s) = s^{-2} - s^{-2}(\ln s)^{-1} - s^{-2}(\ln s)^{-2},$$

and, in view of Lemma 7.1, for some matrix-valued function $O(x) \in O(n)$,

$$\begin{aligned} (\bar{r}^2)O(x)^t(\nabla^2\hat{g}(x))O(x) &= \text{diag}(0, \dots, 0, g''(\tilde{d}), \tilde{d}^{-1}g'(\tilde{d}), \dots, \tilde{d}^{-1}g'(\tilde{d})) \\ &\quad + O(1)(\bar{r}|g'(\tilde{d})| + dg''(\tilde{d})), \end{aligned}$$

where $\tilde{d} = d(x)/\bar{r}$, $|O(1)| \leq C$ for some constant C depending only on E —it is in particular independent of \bar{r} . Note that there are k zeros and $n - k - 1$ of $\tilde{d}^{-1}g'(\tilde{d})$ in the above diagonal matrix. Thus, for x close to E ,

$$\begin{aligned} &(\bar{r}^2)\lambda_i(-\nabla^2\hat{g}(x)) \\ &= \begin{cases} -\tilde{d}^{-2} + \tilde{d}^{-2}(\ln \tilde{d})^{-1} + \tilde{d}^{-2}(\ln \tilde{d})^{-2} + O(1)\bar{r}\tilde{d}^{-1}, & i = 1, \\ O(1)\bar{r}\tilde{d}^{-1}, & 2 \leq i \leq k+1, \\ \tilde{d}^{-2} - \tilde{d}^{-2}(\ln \tilde{d})^{-1} + O(1)\bar{r}\tilde{d}^{-1}, & k+2 \leq i \leq n, \end{cases} \end{aligned}$$

and therefore, for some small $\tilde{r} \in (0, 1/9)$ which depends only on E ,

$$\sum_{i=1}^{k+2} \lambda_i(-\nabla^2\hat{g}(x)) = d(x)^{-2}[(\ln \tilde{d})^{-2} + O(1) \text{diam}(\Omega)] > 0, \quad \forall \tilde{d} < \tilde{r}. \quad (74)$$

To guarantee $\tilde{d} < \tilde{r}$ for all x in $\Omega \setminus E$, we only need to choose from the beginning $\tilde{r} > \text{diam}(\Omega)/\tilde{r}$.

Let

$$\hat{u} := u - \inf_{\Omega \setminus E} u.$$

Then

$$\inf_{\Omega \setminus E} \hat{u} = 0. \quad (75)$$

We need to prove that

$$\inf_{\partial\Omega \setminus E} \hat{u} = 0.$$

Suppose that

$$\inf_{\partial\Omega \setminus E} \hat{u} = \delta > 0. \quad (76)$$

For $\bar{r} = \tilde{r}^{-1} \text{diam}(\Omega)$, let

$$\hat{h}_\epsilon(x) := 1 - \frac{g(d(x)/\bar{r})}{g(\epsilon/\bar{r})}.$$

Then, for $E_\epsilon := \{x \in \mathbb{R}^n \mid d(x) < \epsilon\}$,

$$\hat{h}_\epsilon = 0 \quad \text{on } \partial E_\epsilon, \quad \hat{h}_\epsilon < 1 \quad \text{on } \Omega \setminus E_\epsilon. \quad (77)$$

By (9) and (74),

$$\sum_{i=1}^{k+2} \lambda_i(\nabla^2 \hat{u}) \leq 0 < \sum_{i=1}^{k+2} \lambda_i(\nabla^2 \hat{h}_\epsilon) \quad \text{in } \Omega \setminus E_\epsilon.$$

It follows that

$$\lambda_1(\nabla^2(\hat{u} - \delta \hat{h}_\epsilon)) < 0 \quad \text{in } \Omega \setminus E_\epsilon.$$

Therefore, in view of (75)–(77),

$$\hat{u} - \delta \hat{h}_\epsilon \geq \inf_{\partial(\Omega \setminus E_\epsilon)} (\hat{u} - \delta \hat{h}_\epsilon) \geq 0.$$

Now, for any $\Omega \setminus E$, we have, from the above,

$$\hat{u}(x) \geq \delta \hat{h}_\epsilon(x), \quad \forall 0 < \epsilon < d(x).$$

Sending ϵ to 0 leads to $\hat{u}(x) \geq \delta$, i.e. $u \geq \delta$ on $\Omega \setminus E$. This violates (75). Proposition 1.1 is established.

10. Appendix D. Some properties of viscosity k -superaffine functions

We give an extension of Proposition 1.1. As always, we order the eigenvalues of a C^2 function φ as

$$\lambda_1(\nabla^2 \varphi) \leq \dots \leq \lambda_n(\nabla^2 \varphi).$$

Recall that we use $\mathcal{A}_k(\Omega)$ to denote the set of viscosity k -superaffine functions on a bounded open set Ω in \mathbb{R}^n , and use $LSC(\Omega)$ to denote the set of lower semicontinuous functions in Ω .

Theorem 10.1. *For $n \geq 2$, $0 \leq k \leq n - 2$, let E be a smooth closed k -dimensional manifold in \mathbb{R}^n and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $u \in \mathcal{A}_{k+2}(\Omega \setminus E)$ and set $u(y) := \liminf_{x \rightarrow y} u(x)$ for $y \in E$. Then $u \in \mathcal{A}_1(\Omega)$.*

Theorem 10.1 is somewhat sharp, as explained below.

Definition 10.1. For $n \geq 1$, $0 \leq k \leq n - 2$, and $0 \leq a \leq 1$, let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in LSC(\Omega)$ satisfy (12). We say that $u \in \mathcal{A}_{k+1,a}(\Omega)$ if the conditions

$$\varphi \in C^2(\Omega), \quad \varphi \leq u \quad \text{in } \Omega, \quad \varphi(\bar{x}) = u(\bar{x}) \quad \text{for some } \bar{x} \in \Omega$$

imply

$$\lambda_1(\nabla^2 \varphi) + \cdots + \lambda_{k+1}(\nabla^2 \varphi) + a\lambda_{k+2}(\nabla^2 \varphi) \leq 0 \quad \text{at } \bar{x}.$$

Clearly

$$\mathcal{A}_{k+1,0}(\Omega) = \mathcal{A}_{k+1}(\Omega), \quad \mathcal{A}_{k+1,1}(\Omega) = \mathcal{A}_{k+2}(\Omega),$$

and

$$\mathcal{A}_{k+2}(\Omega) \subset \mathcal{A}_{k+1,b}(\Omega) \subset \mathcal{A}_{k+1,a}(\Omega) \quad \forall 0 \leq a \leq b \leq 1.$$

For $u \in C^2(\Omega)$, $u \in \mathcal{A}_{k+1,a}(\Omega)$ means

$$\lambda_1(\nabla^2 u) + \cdots + \lambda_{k+1}(\nabla^2 u) + a\lambda_{k+2}(\nabla^2 u) \leq 0 \quad \text{in } \Omega.$$

The following example shows that in Theorem 10.1 the assumption $u \in \mathcal{A}_{k+2}(\Omega \setminus E)$ cannot be replaced by $u \in \mathcal{A}_{k+1,a}(\Omega \setminus \{0\})$ for any $0 < a < 1$.

Example 10.1. (i) Let $n \geq 2$, $k = 0$ and $E = \{0\}$. For any $0 < a < 1$, let $u(x) = |x|^{1-a}$. Then $u'(r) = (1-a)r^{-a} > 0$, $u''(r) = -a(1-a)r^{-1-a} < 0$ and $u'(r)/r = (1-a)r^{-1-a}$. So, by Lemma 7.1, the eigenvalues of $\nabla^2 u$ are

$$-a(1-a)r^{-1-a} < (1-a)r^{-1-a} = \cdots = (1-a)r^{-1-a}.$$

So

$$\lambda_1(\nabla^2 u) + a\lambda_2(\nabla^2 u) = 0 \quad \text{in } B_1 \setminus \{0\}.$$

But u does not belong to $\mathcal{A}_1(B)$ since $u(0) = 0 < u(1) = 1$.

(ii) For $n \geq 3$, $1 \leq k \leq n - 2$, we write $x = (x', x'') \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-k}$ and $x'' \in \mathbb{R}^k$. Let $\Omega = B_1$ be the unit ball in \mathbb{R}^n , and let E be a k -dimensional smooth closed manifold satisfying $E \cap B_2 = \{x = (x', x'') \mid x' = 0, |x''| < 2\}$. For any $0 < a < 1$, let $u(x) = |x''|^{1-a-\epsilon} + \epsilon^2|x'|^2$ where $\epsilon > 0$ is some small number to be specified below. Then the eigenvalues of $\nabla^2 u$ in $B_1 \setminus E$ are

$$\lambda_i(\nabla^2 u) = \begin{cases} -(a+\epsilon)(1-a-\epsilon)|x''|^{-1-a-\epsilon}, & i = 1, \\ 2\epsilon^2, & 2 \leq i \leq k+1, \\ (1-a-\epsilon)|x''|^{-1-a-\epsilon}, & k+2 \leq i \leq n. \end{cases}$$

So $u \in C^2(B_1 \setminus E)$ and, for small $\epsilon > 0$,

$$\begin{aligned} & \lambda_1(\nabla^2 u) + \cdots + \lambda_{k+1}(\nabla^2 u) + a\lambda_{k+2}(\nabla^2 u) \\ &= -\epsilon(1-a-\epsilon)|x''|^{-1-a-\epsilon} + 2k\epsilon^2 \leq -\epsilon(1-a-\epsilon) + 2k\epsilon^2 < 0 \quad \text{in } B_1 \setminus E. \end{aligned}$$

But $u(0) = 0 < \epsilon^2 = \min_{\partial B_1} u$.

The following two simple lemmas give an equivalent definition of superaffine functions.

Lemma 10.1. *For $n \geq 1$, let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathcal{A}_1(\Omega)$. Then for every open subset D satisfying $\bar{D} \subset \Omega$, and for every $V \in \mathbb{R}^n$, the function $w(x) := u(x) + V \cdot x$ satisfies*

$$\min_{\bar{D}} w = \min_{\partial D} w. \quad (78)$$

Proof. Since $u \in LSC(\Omega)$, it is clear that both $\inf_{\bar{D}} u$ and $\inf_{\partial D} u$ are achieved. We prove (78) by contradiction. Suppose that for some D and some V ,

$$-\infty < \min_{x \in \bar{D}} (u(x) + V \cdot x) < \min_{x \in \partial D} (u(x) + V \cdot x).$$

Let

$$\tilde{u}(x) := u(x) + V \cdot x.$$

Then $\tilde{u} \in \mathcal{A}_1(\Omega)$, and

$$\delta := \min_{\partial D} \tilde{u} > \min_{\bar{D}} \tilde{u} = 0, \quad \text{say.} \quad (79)$$

Since D is bounded, there exists some constant $\bar{\epsilon} > 0$ such that

$$\bar{\epsilon}|x|^2 + \frac{\delta}{4} < \frac{\delta}{2}, \quad \forall x \in \bar{D}. \quad (80)$$

Let

$$\varphi_a(x) := \bar{\epsilon}|x|^2 + a.$$

For very negative a , $\varphi_a \leq \tilde{u}$ on \bar{D} . Let

$$\bar{a} := \sup\{a \mid \varphi_a \leq \tilde{u} \text{ on } \bar{D}\}.$$

By (79) and (80),

$$\min_{\bar{D}} (\tilde{u} - \varphi_{\bar{a}}) = 0, \quad \bar{a} \leq 0, \quad \tilde{u} - \varphi_{\bar{a}} > \delta/4 > 0 \quad \text{on } \partial D.$$

It follows that for some $\bar{x} \in D$,

$$(\tilde{u} - \varphi_{\bar{a}})(\bar{x}) = \min_{\bar{D}} (\tilde{u} - \varphi_{\bar{a}}) = 0.$$

With the above, we can easily find a smooth $\tilde{\varphi}$ satisfying

$$\tilde{u} - \tilde{\varphi} \geq 0 \quad \text{in } \Omega, \quad (\tilde{u} - \tilde{\varphi})(x) = 0, \quad \nabla^2 \tilde{\varphi}(\bar{x}) = \nabla^2 \varphi_{\bar{a}}(\bar{x}) = 2\bar{\epsilon}I > 0.$$

Since u is in $\mathcal{A}_1(\Omega)$, we have

$$\lambda_1(\nabla^2 \tilde{\varphi})(\bar{x}) \leq 0.$$

This is a contradiction. □

Lemma 10.2. *For $n \geq 1$, let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in LSC(\Omega)$ satisfy (12); u is bounded below. If for every $\bar{x} \in \Omega$, there exists some $0 < \bar{\epsilon} < \text{dist}(\bar{x}, \partial\Omega)$ such that for every $0 < \epsilon < \bar{\epsilon}$, and for every $V \in \mathbb{R}^n$, the function $w(x) := u(x) + V \cdot x$ satisfies*

$$\min_{\bar{B}_\epsilon(\bar{x})} w = \min_{\partial B_\epsilon(\bar{x})} w,$$

then $u \in \mathcal{A}_1(\Omega)$.

Proof. Suppose that u is not in $\mathcal{A}_1(\Omega)$. Then for some $\varphi \in C^2(\Omega)$ and some $\bar{x} \in \Omega$, we have

$$\varphi \leq u \quad \text{in } \Omega, \quad \varphi(\bar{x}) = u(\bar{x}), \quad \lambda_1(\nabla^2\varphi(\bar{x})) > 0.$$

For x close to \bar{x} ,

$$\varphi(x) \geq \varphi(\bar{x}) + \nabla\varphi(\bar{x})(x - \bar{x}) + \frac{1}{2}\lambda_1(\nabla^2\varphi(\bar{x}))|x - \bar{x}|^2.$$

Thus, for small $\epsilon > 0$, and for $V := \nabla\varphi(\bar{x})$,

$$u(x) - V \cdot x \geq \varphi(x) - V \cdot x > \varphi(\bar{x}) - V \cdot \bar{x}, \quad \forall x \in \partial B_\epsilon(\bar{x}),$$

while

$$u(\bar{x}) - V \cdot \bar{x} = \varphi(\bar{x}) - V \cdot \bar{x}.$$

This implies

$$\min_{x \in \bar{B}_\epsilon(\bar{x})} (u(x) - V \cdot x) < \min_{x \in \partial B_\epsilon(\bar{x})} (u(x) - V \cdot x),$$

a contradiction. □

Lemma 10.3. *For $1 \leq k \leq n$, if $u \in \mathcal{A}_k(\Omega)$ and $v \in C^2(\Omega)$ satisfies*

$$\lambda_1(\nabla^2 v) + \cdots + \lambda_k(\nabla^2 v) \geq 0 \quad \text{in } \Omega,$$

then $u - v \in \mathcal{A}_1(\Omega)$.

Proof. Suppose not. Then for some $u \in \mathcal{A}_k(\Omega)$ and some $v \in C^2(\Omega)$ satisfying

$$\lambda_1(\nabla^2 v) + \cdots + \lambda_k(\nabla^2 v) \geq 0 \quad \text{in } \Omega,$$

$u - v$ is not in $\mathcal{A}_1(\Omega)$. Then for some $\varphi \in C^2(\Omega)$ and some $\bar{x} \in \Omega$, we have

$$\varphi \leq u - v \quad \text{in } \Omega, \quad \varphi(\bar{x}) = (u - v)(\bar{x}), \quad \lambda_1(\nabla^2\varphi(\bar{x})) > 0.$$

Since $u \in \mathcal{A}_k(\Omega)$, it follows from the above that

$$\lambda_1(\nabla^2(\varphi + v)(\bar{x})) + \cdots + \lambda_k(\nabla^2(\varphi + v)(\bar{x})) \leq 0.$$

Since $\nabla^2\varphi(\bar{x})$ is positive definite, we arrive at

$$\lambda_1(\nabla^2 v(\bar{x})) + \cdots + \lambda_k(\nabla^2 v(\bar{x})) < 0,$$

a contradiction. □

Proof of Theorem 10.1. We argue by contradiction. Suppose that there exists $u \in \mathcal{A}_{k+2}(\Omega \setminus E)$ such that, after setting $u(y) := \liminf_{x \rightarrow y} u(x)$, u does not belong to $\mathcal{A}_1(\Omega)$. By Lemma 10.2, there exist $\bar{x} \in \Omega$, $\epsilon_i > 0$ with $\epsilon_i \rightarrow 0$, and $V_i \in \mathbb{R}^n$ such that $w_i := u(x) + V_i \cdot x$ satisfies

$$\min_{\overline{B_{\epsilon_i}(\bar{x})}} w_i < \min_{\partial B_{\epsilon_i}(\bar{x})} w_i. \quad (81)$$

Since $u \in \mathcal{A}_{k+2}(\Omega \setminus \{0\}) \subset \mathcal{A}_1(\Omega \setminus \{0\})$, we must have $\bar{x} = 0$.

Let

$$g(s) = \begin{cases} -\ln s & \text{if } k = 0, \\ -\ln s + \ln(-\ln s) & \text{if } 1 \leq k \leq n-2. \end{cases}$$

It was proved in Appendix C (see (74)) that for some small positive constant β ,

$$\hat{g}(x) := g(d(x)/\beta), \quad x \in \Omega \setminus E,$$

satisfies

$$\hat{g}(x) > 0, \quad \sum_{i=1}^{k+2} \lambda_i (-\nabla^2 \hat{g}(x)) > 0, \quad \forall x \in \Omega \setminus E.$$

Fix some $i = \bar{i}$ for which (81) holds, and set $\bar{r} := \epsilon_{\bar{i}}$ and

$$\hat{u} := w_{\bar{i}} - \inf_{B_{\bar{r}}} w_{\bar{i}}.$$

Clearly, $\hat{u} \in \mathcal{A}_{k+2}(B_{\bar{r}} \setminus E)$, and

$$\inf_{B_{\bar{r}}} \hat{u} = 0, \quad (82)$$

$$\hat{u} \geq \delta := \min_{\partial B_{\bar{r}}} \hat{u} = \min_{\partial B_{\bar{r}}} w_{\bar{r}} - \inf_{B_{\bar{r}}} w_{\bar{r}} > 0 \quad \text{on } \partial B_{\bar{r}}.$$

For any $\epsilon > 0$ small, let

$$\hat{h}_\epsilon(x) := 1 - \frac{g(d(x)/\bar{r})}{g(\epsilon/\bar{r})}.$$

Then (77) holds for $E_\epsilon := \{x \in \mathbb{R}^n \mid d(x) < \epsilon\}$.

By the property of \hat{g} ,

$$\lambda_1(\nabla^2 \hat{h}_\epsilon) + \cdots + \lambda_k(\nabla^2 \hat{h}_\epsilon) \geq 0 \quad \text{in } B_{\bar{r}} \setminus \overline{E}_\epsilon.$$

Thus, by Lemma 10.3, $\hat{u} - \hat{h}_\epsilon \in \mathcal{A}_1(B_{\bar{r}} \setminus \overline{E}_\epsilon)$. It follows that

$$\hat{u} - \delta \hat{h}_\epsilon \geq \inf_{\partial(B_{\bar{r}} \setminus E_\epsilon)} (\hat{u} - \delta \hat{h}_\epsilon) \geq 0.$$

Sending ϵ to 0 leads to $\hat{u} \geq \delta$ on $B_{\bar{r}} \setminus E$. This violates (82). Theorem 10.1 is established. \square

11. Appendix E. Proof of Lemma 1.2

Suppose that the conclusion of the lemma does not hold. Then there are $n + 1$ vectors p_0, p_1, \dots, p_n satisfying (34) with $p_1 - p_0, \dots, p_n - p_0$ linearly independent.

After a linear transformation of variables, and adding a linear function to u —this preserves (2)—we may assume that

$$p_0 = 0, \quad p_1 = e_1, \quad \dots, \quad p_n = e_n,$$

where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc. So

$$u(x) \geq o(|x|) \quad \text{and} \quad u(x) \geq x_i + o(|x|), \quad i = 1, \dots, n. \quad (83)$$

Consider

$$w = u(x) - \frac{1}{2n} \sum_{i=1}^n x_i.$$

Claim.

$$w(x) \geq \frac{1}{2n}|x| + o(|x|).$$

This would then contradict (2).

Proof of Claim. Consider any x near the origin. Suppose, say,

$$x_1, \dots, x_k > 0, \quad x_{k+1}, \dots, x_n \leq 0.$$

Then

$$\begin{aligned} w(x) &= \frac{1}{n} \sum_i (u(x) - \frac{1}{2} x_i) \\ &\geq \frac{1}{2n} \sum_{i=1}^k x_i - \frac{1}{2n} \sum_{i=k+1}^n x_i + o(|x|) \quad \text{by (83)} \\ &\geq \frac{1}{2n}|x| + o(|x|). \quad \square \end{aligned}$$

Under the assumption of Lemma A (see (34)), either $P = \emptyset$ or P contains only one element. However, this would not be true under the assumption of Lemma 1.2, as shown by the following example.

Example 11.1. In \mathbb{R}^n , $n \geq 2$, take

$$u(x) = \max\{0, x_1\} = \begin{cases} 0, & x_1 \leq 0, \\ x_1, & x_1 > 0. \end{cases}$$

Then $P = \{c e_1 \mid 0 \leq c \leq 1\}$, where P is defined as in Lemma 1.2 and $e_1 = (1, 0, \dots, 0)$. Clearly, $u \in \mathcal{A}_{n-1}(\mathbb{R}^n) \subset \mathcal{A}_1(\mathbb{R}^n)$.

Here is another example.

Example 11.2. For $1 \leq k < n$, let

$$u(x) = \max\{0, x_1, \dots, x_k\}.$$

Then $P = \{c_1 e_1 + \dots + c_k e_k \mid 0 \leq c_i \leq 1\}$. Since u is a function of k variables only, we have $u \in \mathcal{A}_{n-k}(\mathbb{R}^n) \subset \mathcal{A}_1(\mathbb{R}^n)$.

12. Appendix F

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $u \in L^1_{\text{loc}}(\Omega)$ satisfy, for some constant C ,

$$\Delta u \leq C \quad \text{in } \Omega \quad (84)$$

in the distribution sense. Then $u(y) - (C/2)|y|^2$ is superharmonic in Ω , and $\int_{\partial B_r(x)} [u(y) - (C/2)|y|^2] dy$ and $\int_{B_r(x)} [u(y) - (C/2)|y|^2] dy$ are nonincreasing functions in r in $(0, \text{dist}(x, \partial\Omega))$. It follows that

$$u^*(x) := \lim_{r \rightarrow 0^+} \int_{B_r(x)} u = \lim_{r \rightarrow 0^+} \int_{\partial B_r(x)} u \in (-\infty, \infty], \quad x \in \Omega,$$

satisfies

$$u^*(x) = \liminf_{y \rightarrow x} u^*(y), \quad x \in \Omega,$$

and, by a classical theorem of Lebesgue, $u^* = u$ a.e. in Ω . We call u^* the *canonical representative* of u in Ω .

Lemma 12.1. *Let $E \subset \Omega$ be a closed subset of zero capacity. Assume that $u \in L^1_{\text{loc}}(\Omega \setminus E)$ satisfies, for some constant C ,*

$$\Delta u \leq C \quad \text{in } \Omega \setminus E \text{ in the distribution sense,} \quad (85)$$

and

$$\inf_{\Omega \setminus E} u > -\infty.$$

Then $u \in L^1_{\text{loc}}(\Omega)$, and

$$\Delta u \leq C \quad \text{in } \Omega \text{ in the distribution sense.}$$

Proof. We may assume that u is the canonical representative in $\Omega \setminus E$. Subtracting a quadratic polynomial from u , we may assume that $C = 0$ and $u \geq 0$ in $\Omega \setminus E$. Since E has zero capacity, there exists a positive harmonic function h in $\Omega \setminus E$ satisfying

$$\lim_{\text{dist}(x, E) \rightarrow 0} h(x) = \infty;$$

see [7]. For constants $\epsilon > 0$ and $M > 1$, consider

$$u_{M, \epsilon} := \min\{u + \epsilon h, M\}.$$

Since $u \geq 0$ and $h(x)$ tends to infinity as x approaches E , $u_{M, \epsilon} = M$ in a neighborhood of E . Thus

$$\Delta u_{M, \epsilon} \leq 0 \quad \text{in } \Omega \text{ in the distribution sense,} \quad (86)$$

and, for almost all \bar{x} in $\Omega \setminus E$, and every $0 < r < \text{dist}(\bar{x}, \partial\Omega)$,

$$u_{M, \epsilon}(\bar{x}) \geq \int_{B_r(\bar{x})} u_{M, \epsilon}.$$

Sending M to infinity, and then ϵ to zero, we obtain, using the Fatou theorem,

$$u(\bar{x}) \geq \int_{B_r(\bar{x})} u \quad \text{a.e. } \bar{x} \in \Omega \setminus E, \quad 0 < r < \text{dist}(\bar{x}, \partial\Omega).$$

Since $u \in L^1_{\text{loc}}(\Omega \setminus E)$, u is finite a.e. in $\Omega \setminus E$. Therefore $u \in L^1_{\text{loc}}(\Omega)$. By (86),

$$\int_{\Omega} u_{M,\epsilon} \Delta \varphi \leq 0, \quad \forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0.$$

Sending ϵ to zero, then M to infinity, we have, using the Lebesgue dominated convergence theorem,

$$\int_{\Omega} u \Delta \varphi \leq 0, \quad \forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0,$$

i.e. (85) for $C = 0$. □

Definition 12.1. u is called *quasi-superharmonic* in Ω if $u \in L^1_{\text{loc}}(\Omega)$ satisfies (85) for some constant C .

In the following, if u is quasi-superharmonic in Ω , we work with its canonical representative in Ω , and still denote it by u .

We establish the following improvement of Lemma A.

Lemma 12.2. *Let u be quasi-superharmonic in Ω , and let ω be a nonnegative non-decreasing continuous function on $(0, 2d)$, $d := \text{diam}(\Omega)$. Assume that u satisfies, for some $\bar{x}, \bar{y} \in \Omega$ and $p, q \in \mathbb{R}^n$,*

$$u(y) \geq u(\bar{x}) + p \cdot (y - \bar{x}) - |y - \bar{x}| \omega(|y - \bar{x}|), \quad y \in \Omega, \quad (87)$$

and

$$u(y) \geq u(\bar{y}) + q \cdot (y - \bar{y}) - |y - \bar{y}| \omega(|y - \bar{y}|), \quad y \in \Omega. \quad (88)$$

Then, for some constant C_1 depending only on n ,

$$|p - q| \leq C_1 \omega(2|\bar{x} - \bar{y}|) + C_1 |\bar{x} - \bar{y}|. \quad (89)$$

Lemma 12.2 follows from the following more general lemma, which is used in the proof of Theorem 1.3 (see the proof of Lemma 3.1).

Lemma 12.3. *Let u be quasi-superharmonic in Ω , and let ω be a nonnegative non-decreasing continuous function on $(0, 2d)$, $d := \text{diam}(\Omega)$. Assume that u satisfies, for some $\bar{x}, \bar{y} \in \Omega$, $\bar{x} \neq \bar{y}$, and $p, q \in \mathbb{R}^n$,*

$$u(y) \geq u(\bar{x}) + p \cdot (y - \bar{x}) - |y - \bar{x}| \omega(|y - \bar{x}|), \quad y \in \Omega,$$

and

$$u(z) \geq u(\bar{x}) + p \cdot (\bar{y} - \bar{x}) - |\bar{y} - \bar{x}| \omega(|\bar{y} - \bar{x}|) + q \cdot (z - \bar{y}) - |z - \bar{y}| \omega(|z - \bar{y}|), \quad z \in \Omega. \quad (90)$$

Then, for some positive constants C_1 and C_2 depending only on n ,

$$|p - q| \leq C_1 \omega(2|\bar{x} - \bar{y}|) + C_2 C |\bar{x} - \bar{y}|,$$

where C is the constant in (84).

Proof. Working with $\tilde{u}(z) = u(z + \bar{x}) - [u(\bar{x}) + p \cdot z]$ instead of $u(z)$, we may assume that $\bar{x} = 0$, $u(0) = 0$, $p = 0$, $\bar{y} \neq 0$, $q \neq 0$:

$$u(z) \geq -|z|\omega(|z|), \quad z \in \Omega, \quad (91)$$

$$u(z) \geq -|\bar{y}|\omega(|\bar{y}|) + q \cdot (z - \bar{y}) - |z - \bar{y}|\omega(|z - \bar{y}|), \quad z \in \Omega. \quad (92)$$

By (91),

$$u(z) \geq -2|\bar{y}|\omega(2|\bar{y}|), \quad 0 < |z| \leq 2|\bar{y}|. \quad (93)$$

For $|z - \bar{y}| \leq \frac{1}{2}|\bar{y}|$, we deduce from (92) that

$$u(z) \geq -|\bar{y}|\omega(|\bar{y}|) + q \cdot (z - \bar{y}) - \frac{1}{2}|\bar{y}|\omega\left(\frac{1}{2}|\bar{y}|\right) \geq -2|\bar{y}|\omega(2|\bar{y}|) + q \cdot (z - \bar{y}).$$

It follows that

$$u(z) \geq -2|\bar{y}|\omega(2|\bar{y}|) + \frac{1}{4}|q||\bar{y}|, \quad \forall z \in (B_{2|\bar{y}|}(0) \setminus B_{|\bar{y}|/2}(\bar{y})) \cap U, \quad (94)$$

where

$$U := \{z \in \mathbb{R}^n \mid q \cdot (z - \bar{y}) \geq \frac{1}{2}|q||z - \bar{y}|\}.$$

Since $(B_{2|\bar{y}|}(0) \setminus B_{|\bar{y}|/2}(\bar{y})) \cap U$ is a subset of $B_{2|\bar{y}|}(0)$ of measure $\geq |\bar{y}|^n/C_1$ for some positive dimensional constant C_1 , and since $u(x) - \frac{C}{2n}|x|^2$ is superharmonic by (84), we deduce from (93) and (94), in view of the mean value theorem, that

$$0 = u(0) \geq \int_{B_{2|\bar{y}|}(0)} \left(u(x) - \frac{C}{2n}|x|^2 \right) \geq -2|\bar{y}|\omega(2|\bar{y}|) + \frac{1}{C_1}|\bar{y}||q| - C_2C|\bar{y}|^2.$$

It follows that

$$|q| \leq C_1\omega(2|\bar{y}|) + C_2C|\bar{y}|.$$

Lemma 12.3 is established. \square

Proof of Lemma 12.2. Taking $y = \bar{y}$ in (87), and using this to substitute for $u(\bar{y})$ in (88), we have (90). Thus (89) in the case $\bar{x} \neq \bar{y}$ follows from Lemma 12.3.

Now we discuss the case when $\bar{x} = \bar{y}$. Replacing $u(y)$ by $u(y) - [u(\bar{x}) + p \cdot (y - \bar{x})]$, we may assume that $\bar{x} = \bar{y} = 0$, $p = 0$, and $q \neq 0$. So we have

$$u(y) \geq -|y|\omega(|y|), \quad y \in \Omega, \quad (95)$$

$$u(y) \geq q \cdot y - |y|\omega(|y|), \quad y \in \Omega. \quad (96)$$

It is easy to see that for some large constant $\alpha > 1$ independent of ϵ ,

$$w_\epsilon(y) := \alpha\omega(\alpha|y| + \alpha\epsilon)$$

satisfies

$$|y|\omega(|y|) \leq \epsilon|q| + \epsilon\omega(\epsilon) + |y - \epsilon q/|q||\omega_\epsilon(|y - \epsilon q/|q||).$$

It follows from (96) and the above that

$$u(y) \geq -|y_\epsilon|\omega_\epsilon(|y_\epsilon|) + q \cdot (y - y_\epsilon) - |y - y_\epsilon|\omega_\epsilon(|y - y_\epsilon|), \quad y \in \Omega,$$

where $y_\epsilon = \epsilon q/|q| \neq 0$. We can apply Lemma 12.3, with $\bar{y} = y_\epsilon$, to obtain, for some constants C_1, C_2 depending only on n ,

$$|q| \leq C_1\omega_\epsilon(2|y_\epsilon|) + C_2C|y_\epsilon|,$$

where C is the constant in (84). Sending ϵ to zero in the above leads to $|q| \leq C_1\omega(0)$. Lemma 12.2 is established. \square

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Luis Caffarelli
Department of Mathematics
The University of Texas
Austin, TX 78712, USA
e-mail: caffarel@math.utexas.edu

YanYan Li
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854, USA
e-mail: yli@math.rutgers.edu

Louis Nirenberg
Courant Institute
251 Mercer Street
New York, NY 10012, USA
e-mail: nirenberg@cims.nyu.edu

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