A Harnack type inequality for some conformally invariant equations on half Euclidean space

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Dedicated to Professor Wu Wenjun on the occasion of his $90'$s birthday

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We establish a Harnack type inequality for general conformally invariant fully nonlinear elliptic equations of second order. Let u be a positive function in $\mathbb{R}^n$, and let $\psi : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ be a Möbius transformation, i.e. a transformation generated by translations, multiplications by nonzero constants and the inversion $x \to x/|x|^2$. Set

$$u_\psi := |J_\psi|^{\frac{n-2}{2n}}(u \circ \psi),$$

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where J_ψ is the Jacobian of ψ.

It is proved in [3] that an operator $H(u, \nabla u, \nabla^2 u)$ is conformally invariant, i.e.

$$H(u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv H(u, \nabla u, \nabla^2 u) \circ \psi \text{ holds for all positive } u \text{ and all Möbius } \psi,$$

if and only if H is of the form

$$H(u, \nabla u, \nabla^2 u) \equiv F(A^u)$$

where

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I, \quad (1)$$

I is the $n \times n$ identity matrix, and $F(M)$ is a function depending only on the eigenvalues of the $n \times n$ real symmetric matrix M.

For a Möbius transformation φ, there exists some $n \times n$ orthogonal matrix functions $O(x)$ (i.e. $O(x)O(x)^t = I$), depending on φ, such that

$$A^{u_\varphi}(x) \equiv O(x) A^u(\varphi(x)) O^t(x). \quad (2)$$

So $A^{u_\varphi}(x)$ and $A^u(\varphi(x))$ have the same eigenvalues.

Also, we have

$$\sum_{i=1}^n \lambda_i(A^u) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u, \quad (3)$$

where $\{\lambda_i(A^u)\}$ are the eigenvalues of A^u,

Let $\mathcal{S}^{n\times n}$ be the set of $n \times n$ real symmetric matrices, $\mathcal{S}_+^{n\times n} \subset \mathcal{S}^{n\times n}$ be the set of positive definite matrices, $O(n)$ be the set of $n \times n$ real orthogonal matrices, $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying

$$O^{-1} U O = U, \qquad \forall\, O \in O(n), \quad (4)$$

and

$$U \cap \{M + tN \mid 0 < t < \infty\} \text{ is convex} \qquad \forall\, M \in \mathcal{S}^{n\times n}, N \in \mathcal{S}_+^{n\times n}. \quad (5)$$

Let $F \in C^1(U)$ satisfy

$$F(O^{-1} M O) = F(M), \qquad \forall\, M \in U, \quad (6)$$

$$\sum_i M_{ii} \geq 0, \quad \forall\, M = (M_{ij}) \in U \text{ with } F(M) = 1, \quad (7)$$

$$(F_{ij}(M)) > 0, \qquad \forall\, M \in U, \quad (8)$$

where $F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M)$, and, for some $\delta > 0$,

$$F(M) \neq 1, \qquad \forall M = (M_{ij}) \in U \text{ with } |M| < \delta. \tag{9}$$

Examples of such (F, U) include those given by elementary symmetric functions. For $1 \leq k \leq n$, let

$$\sigma_k(\lambda) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone Γ_n, which is equal to $\{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}$, and let

$$U_k = \{M \in \mathcal{S}^{n \times n} \mid \lambda(M) \in \Gamma_k\}, \qquad F_k(M) = \sigma_k(\lambda(M))^{1/k}.$$

Then (F_k, U_k) satisfies (4)-(9); see e.g. [1].

We will use notations

$$x = (x', x_n), \quad \text{and} \quad x' = (x_1, \cdots, x_{n-1}).$$

$$B_R(x) = \{x \in \mathbb{R}^n \mid |x| < R\}, \quad B_R^T(x) = B_R(x) \cap \{x_n > T\}, \quad B_R^T = B_R^T(0),$$
$$B_R^+(x) = B_R(x) \cap \{x_n > 0\}, \quad B_R^+ = B_R^+(0),$$
$$\partial'' B_R^T(x) = \partial B_R^T(x) \cap \{x_n > T\}, \quad \partial' B_R^T(x) = \partial B_R^T(x) \cap \{x_n = T\},$$
$$\partial' B_R^+(x) = \partial B_R^+(x) \cap \{x_n = 0\}, \quad \partial'' B_R^+(x) = \partial B_R^+(x) \cap \{x_n > 0\}.$$

We establish a Harnack type inequality for solutions of

$$\begin{cases} F(A^u) = 1, \quad \text{in } B_{3R}^+ \\ \frac{\partial u}{\partial x_n} = c u^{\frac{n}{n-2}} \quad \text{on } \partial' B_{3R}^+, \\ u > 0, \quad A^u \in U, \quad \text{in } \overline{B_{3R}^+} \end{cases} \tag{10}$$

where $R > 0$ and c are constants, and A^u is given in (1).

If U satisfies (7), then, in view of (3), $A^u \in U$ implies $\Delta u \leq 0$.

Theorem 1.1 *Assume that (F, U) satisfies (4)-(9). For $n \geq 3$, $R > 0$ and $c \in \mathbb{R}$, let $u \in C^2(\overline{B_{3R}^+})$ be a solution of (10). Then, for some constant C depending only on n, c and δ, we have*

$$(\max_{B_R^+} u)(\min_{\partial B_{2R}^+} u) \leq C R^{2-n}. \tag{11}$$

Remark 1.1 *The corresponding results in Euclidean balls were proved in [6], which is an extension of the result for* $(F,U)=(F_1,U_1)$ *in [10]. Theorem 1.1 for* $(F,U)=(F_1,U_1)$ *was proved in [9], and for* $(F,U)=(F_k,U_k)$*,* $2\le k\le n$*, in [4] (announced in [5] and [7]). Theorem 1.1 extends the result in [4] to general* (F,U) *by making use of the more recent result on local gradient estimates in [8] (theorem 1.3); while the proof in [4] uses the local gradient estimates for* (F_k,U_k) *in [2].*

Remark 1.2 *It suffices to prove Theorem 1.1 for* $R=1$ *— working with* $\tilde u(x)=R^{\frac{n-2}{2}}u(Rx)$ *reduces the general case to this case.*

Large parts of the proofs of Theorem 1.1 follow closely arguments in [9], [6] and [7]. The new ingredients are Lemma 1.3 and the above mentioned local gradient estimates in [8].

Proof of Theorem 1.1. We prove it by contradiction. Suppose the contrary, then, for some n,c,δ, there exist $\{R_j\}$ and $\{u_j\}$, $R_j>0$ and $u_j\in C^2(\overline{B^+_{3R_j}})$ satisfying (10) in $B^+_{3R_j}$, such that

$$u_j(x_j)\min_{\partial B^+_{2R_j}}u_j>jR_j^{2-n},\quad j=1,2,3,\cdots,\tag{12}$$

where $x_j\in\overline{B^+_{R_j}}$ and $u_j(x_j)=\max_{\overline{B^+_{R_j}}}u_j$. Since $A^{u_j}\in U$ and U satisfies (7), we have $\Delta u_j\le 0$, and therefore

$$\min_{\partial B^+_{2R_j}}u_j=\min_{\overline{B^+_{2R_j}}}u_j\le u_j(x_j).$$

It follows that

$$u_j(x_j)R_j^{\frac{n-2}{2}}\to\infty,\quad\text{as } j\to\infty.\tag{13}$$

Applying an elementary lemma in [9] (lemma 7.1) to $u_j(x_j+\frac{R_j}{4}\cdot)$ with $a=\frac{n-2}{2}$ and $T=\frac{4x_{jn}}{R_j}$ (x_{jn} denotes the n-th component of x_j), we find $z_j\in B_{\frac{R_j}{4}}(x_j)\cap\overline{\mathbb{R}^n_+}$ such that

$$u_j(z_j)\ge 2^{\frac{2-n}{2}}u_j\quad\text{on } B_{\sigma_j}(z_j)\cap\overline{\mathbb{R}^n_+},\tag{14}$$

and

$$(2\sigma_j)^{\frac{n-2}{2}}u_j(z_j)\ge(\frac{R_j}{4})^{\frac{n-2}{2}}u_j(x_j)\to\infty,\quad \textit{as } j\to\infty,\tag{15}$$

where

$$\sigma_j:=\frac{1}{2}(\frac{R_j}{4}-|z_j-x_j|)\le\frac{R_j}{8}.\tag{16}$$

Set

$$\gamma_j := u_j(z_j)^{\frac{2}{n-2}}\sigma_j, \qquad \Gamma_j := 2u_j(z_j)^{\frac{2}{n-2}}R_j.$$

By (15) and (16),

$$u_j(z_j) \geq u_j(x_j), \quad \Gamma_j \geq 16\gamma_j \to \infty. \tag{17}$$

Therefore, in view of (12),

$$u_j(z_j) \inf_{\partial'' B^+_{2R_j}} u_j > jR_j^{2-n}. \tag{18}$$

Let

$$T_j := u_j(z_j)^{\frac{2}{n-2}} z_{jn},$$

$$\Omega_j := \{y|\ z_j + \frac{y}{u_j(z_j)^{\frac{2}{n-2}}} \in B^+_{2R_j}\},$$

and

$$v_j(y) := \frac{1}{u_j(z_j)} u_j(z_j + \frac{y}{u_j(z_j)^{\frac{2}{n-2}}}), \qquad y \in \Omega_j.$$

Since $z_j \in B_{\frac{R_j}{4}}(x_j) \cap \overline{\mathbb{R}^n_+}$ and $x_j \in \overline{B^+_{R_j}}$, we have $0 \leq z_{jn} \leq \frac{5}{4}R_j$ and $T_j \leq \frac{5}{8}\Gamma_j$.

In view of the equation satisfied by u_j and the conformal invariance of the equation, v_j satisfies

$$\begin{cases} F(A^{v_j}) = 1, \quad \text{in } \Omega_j \\ \frac{\partial v_j}{\partial y_n} = c v_j^{\frac{n}{n-2}} \quad \text{on } \partial'\Omega_j := \bar{\Omega}_j \cap \{y_n = -T_j\} \\ v_j(0) = 1, \quad v_j \leq 2^{\frac{n-2}{2}} \quad \text{on } \Omega_j \cap \bar{B}_{\gamma_j}, \quad A^{v_j} \in U, \quad v_j > 0 \text{ on } \overline{\Omega_j}. \end{cases} \tag{19}$$

Let

$$\partial''\Omega_j := \partial\Omega_j \cap \{y|\ y_n > -T_j\}.$$

We have

$$\frac{3}{8}\Gamma_j \leq dist(0, \partial''\Omega_j) \leq \frac{13}{8}\Gamma_j.$$

Thus, recalling $\Gamma_j := 2u_j(z_j)^{\frac{2}{n-2}}R_j$, (18) implies

$$\inf_{y \in \partial''\Omega_j} (|y|^{n-2} v_j(y)) \geq \frac{u_j(z_j) \inf_{\partial'' B^+_{2R_j}} u_j}{u_j(z_j)^2} \inf_{y \in \partial''\Omega_j} |y|^{n-2} \to \infty. \tag{20}$$

Passing to a subsequence, we have

$$\lim_{j\to\infty} T_j = T \in [0,\infty].$$

We divide the rest of the proof into two cases.
Case 1. $T = \infty$.
Case 2. $T < \infty$.
Reaching a contradiction in Case 1. We know that $\min\{\gamma_j, T_j\} \to \infty$, and $\{v_j\}_{j=1,2..}$ is uniformly bounded on compact subsets of $\mathbb{R}^n$. It follows from theorem 1.3 in [8] that for every $R > 0$, there exists constant $C = C(R)$ and $\bar{j} = \bar{j}(R)$ such that

$$|\nabla v_j| \le C|v_j| \le C, \quad \text{on } B_R, \qquad \forall\ j \ge \bar{j}. \tag{21}$$

It follows, using $v_j(0) = 1$,

$$\frac{1}{C} \le v_j \le C \quad \text{on } B_R, \qquad \forall\ j \ge \bar{j}, \tag{22}$$

for some constant C, which possibly differs from the previous one, depending only on R.

For $x \in \mathbb{R}^n$ and $\lambda < T_j/2$, let $(v_j)_{x,\lambda}$ denote the Kelvin transformation of v_j with respect to $B_\lambda(x)$, i.e.

$$(v_j)_{x,\lambda}(y) = (\frac{\lambda}{|y-x|})^{n-2} v_j(x + \frac{\lambda^2(y-x)}{|y-x|^2}), \qquad y \in \Sigma^\lambda_{j,x} := \Omega_j \backslash \overline{B_\lambda(x)}.$$

Clearly $(v_j)_{x,\lambda}$ satisfies the same equation of v_j in $\Sigma^\lambda_{j,x}$.

By essentially the same arguments in the proof of lemma 2.1 in [9], we can find $\lambda_{j,x} > 0$ such that

$$(v_j)_{x,\lambda}(y) < v_j(y) \ \text{ for } \ y \in \Sigma^\lambda_{j,x} \ \text{ and } \ 0 < \lambda \le \lambda_{j,x}.$$

Define

$$\bar{\lambda}_j(x) := \sup\{\mu > 0 : (v_j)_{x,\lambda}(y) \le v_j(y) \ \text{ for } \ y \in \overline{\Sigma^\lambda_{j,x}} \ \text{ and } \ 0 < \lambda \le \mu\}.$$

Lemma 1.1 *For any $R > 1$, $\inf_{|x|\le R} \bar{\lambda}_j(x) \to \infty$ as $j \to \infty$.*

Proof. Suppose the contrary, for some $|x_j| \le R$ and along a subsequence, $\bar{\lambda}_j(x_j) \le C$ for some constant C independent of j. Without loss of generality, we assume that

$x_j = 0$. Denote $w_\lambda = v_j - (v_j)_{0,\lambda}$. To reach a contradiction we only need to show that

$$\frac{\partial w_{\bar\lambda_j}}{\partial \nu}(y) > 0 \qquad \text{for } \ y \in \partial B_{\bar\lambda_j}, \tag{23}$$

and

$$w_{\bar\lambda_j}(y) > 0 \qquad \text{for } \ y \in \overline{\Sigma_{\bar\lambda_j}} \setminus \partial B_{\bar\lambda_j}, \tag{24}$$

where ν denotes the unit outer normal of $\partial B_{\bar\lambda_j}$.

Indeed we easily deduce from (23) and (24) that $w_\lambda \geq 0$ on $\overline{\Sigma_\lambda}$ for λ close to $\bar\lambda_j$, violating the definition of $\bar\lambda_j$.

It is clear that

$$w_{\bar\lambda_j} \geq 0 \ \text{ in } \ \Sigma_{\bar\lambda_j}.$$

By (20) and the boundedness of $\{\bar\lambda_j\}$, $w_{\bar\lambda_j} > 0$ on $\overline{\partial''\Omega_j}$ for large j. Arguing as in the proof of lemma 3 in [7], using the strong maximum principle and the Hopf Lemma, we have (23) and

$$w_{\bar\lambda_j}(y) > 0 \qquad \text{for } \ y \in \Sigma_{\bar\lambda_j}.$$

To show (24), we only need to establish

$$w_{\bar\lambda_j}(y) > 0 \quad \text{on } \ \{y_n = -T_j\} \cap \partial\Omega_j.$$

This will follow from

Lemma 1.2 *Suppose $T_j \to \infty$ and $\{\bar\lambda_j\}$ are bounded. Then for any $N > 0$, there exists $j_0 > 1$ such that for $j > j_0$,*

$$\frac{\partial v_j^{\bar\lambda_j}(z)}{\partial z_n} > N v_j^{\bar\lambda_j}(z)^{\frac{n}{n-2}}, \qquad \forall z \in \partial\Omega_j \cap \{z_n = -T_j\}.$$

Indeed, if for some z with $z_n = -T_j$,

$$w_{\bar\lambda_j}(z) = 0.$$

Then z is a minimum point and, by Lemma 1.2 and for large j,

$$0 \leq \frac{\partial w_{\bar\lambda_j}}{\partial z_n}(z) = c v_j(z)^{\frac{n}{n-2}} - \frac{\partial v_j^{\bar\lambda_j}}{\partial z_n}(z) = c(v_j^{\bar\lambda_j}(z))^{\frac{n}{n-2}} - \frac{\partial v_j^{\bar\lambda_j}}{\partial z_n}(z) < 0.$$

A contradiction.

Given (21), the proof of Lemma 1.2 is the same as that of lemma 7.3 in [9]. For reader′s convenience we include the details.

Proof of Lemma 1.2. Since $T_j \to \infty$ and $\{\bar\lambda_j\}$ is bounded, we have, in view of (21) and (22), for some positive constant C independent of j,

$$\frac{1}{C} < v_j(\frac{\bar\lambda_j^2 z}{|z|^2}) < C \quad \text{and} \quad |\nabla v_j(\frac{\bar\lambda_j^2 z}{|z|^2})| \le C, \quad \forall z \in \partial\Omega_j \cap \{z_n = -T_j\}.$$

By a direct computation, we have, for large j,

$$\begin{aligned} \frac{\partial v_j^{\bar\lambda_j}}{\partial z_n}(z) &\ge (n-2)\bar\lambda_j^{n-2} T_j |z|^{-n} v_j(\frac{\bar\lambda_j^2 z}{|z|^2}) - \bar\lambda_j^n |z|^{-n} |\nabla v_j(\frac{\bar\lambda_j^2 z}{|z|^2})| \\ &\ge m\bar\lambda_j^{n-2} T_j |z|^{-n} > N v_j^{\bar\lambda_j}(z)^{\frac{n}{n-2}}, \end{aligned}$$

where m is a positive constant independent of j. Lemma 1.2 is established. So is Lemma 1.1.

□

The rest of the proof in Case 1 is the same as some arguments used in the proof of theorem 1.2 in [6], which we include for reader′s convenience.

Let $R > 1$ be a large constant to be chosen later, by Lemma 1.1, there exists $\bar j(R)$ such that

$$\bar\lambda_j(x) \ge 8R. \qquad \forall\ |x| \le 8R.$$

It follows that

$$v_{j,x}^{\lambda}(y) < v_j(y) \quad \forall\ |x| \le 8R,\ |y - x| \le 8R,\ 0 < \lambda < 8R.$$

By lemma A.2 in [6],

$$|\nabla \log v_j| \le \frac{C}{R}, \qquad \text{on } B_R$$

for some constant C depending only on n. It follows, using $v_j(0) = 1$ and $v_j \le 2^{(n-2)/2}$ in $\Omega_j \cap B_{\gamma_j}$ which contains B_R for large j, that

$$|v_j(y) - 1| \le \frac{C}{R}|y| \le \frac{C}{\sqrt{R}}, \qquad \forall\ |y| \le \sqrt{R}, \quad \text{for large } j.$$

Let ϵ (depending on j) be the number such that

$$\xi(y) := \frac{1-\epsilon}{R}(R - |y|^2), \qquad |y| < \sqrt{R}$$

satisfies

$$v_j \geq \xi, \quad \text{on } B_{\sqrt{R}},$$

and, for some $|\bar{y}| < \sqrt{R}$ ($\bar{y}$ depends on j),

$$v_j(\bar{y}) = \xi(\bar{y}).$$

Since $1 = v_j(0) \geq \xi(0) = 1 - \epsilon$ and $v_j(\bar{y}) > 0$, we have $0 \leq \epsilon < 1$.

By the above estimates on $|v_j(\bar{y}) - 1|$, we have

$$1 - CR^{-1/2} \leq v_j(\bar{y}) = \xi(\bar{y}) \leq 1 - \epsilon.$$

Clearly,

$$\nabla v_j(\bar{y}) = \nabla \xi(\bar{y}), \quad |\nabla \xi(\bar{y})| \leq \frac{2}{\sqrt{R}}, \quad D^2 v_j(\bar{y}) \geq D^2 \xi(\bar{y}) = -2(1-\epsilon)R^{-1} I.$$

It follows that

$$A^{v_j}(\bar{y}) \leq A^{\xi}(\bar{y}) \leq CR^{-1} I.$$

Since $F(A^{v_j}(\bar{y})) = 1$, we have, by (9),

$$CR^{-1} \geq \delta,$$

which leads to contradiction if we choose from the beginning the value of R satisfying $CR^{-1} < \delta$. We have reached a contradiction in Case 1.

Reaching a contradiction in Case 2.

Lemma 1.3 *Assume $T < \infty$. Then there exists $C = C(n) > 0$, depending only on n, such that for all $R > 1$, there exists $j_0 = j_0(R) > 0$, we have*

$$v_j(x', T_j) \geq 2^{\frac{2-n}{2}}, \quad \forall \ |x'| \leq R, \ j \geq j_0.$$

Proof. Since $T < \infty$ and $\gamma_j \to \infty$, there exists $\tilde{j} = \tilde{j}(R)$ such that $\gamma_j > 10(R^2 + T_j)$ for $\forall \ j \geq \tilde{j}$. For $x = (x', T_j)$ with $1 < |x'| < R$, the line through x and 0 intersects the hyperplane $\{y \mid y_n = -T_j\}$ at the point $\tilde{z}_j = -x$ and $\tilde{z}_j \in \Omega_j$ for $j \geq \tilde{j}$. Consider

$$f_j(s) := v_j(\tilde{z}_j + s\frac{x - \tilde{z}_j}{|x - \tilde{z}_j|^2}) = v_j(-x + \frac{sx}{2|x|^2}).$$

For $s_2 = 2s_1 = 4|x|^2 \le 4(T_j + R^2)$,

$$\tilde{z}_j + s_1 \frac{x - \tilde{z}_j}{|x - \tilde{z}_j|^2} = 0, \quad \text{and} \quad \tilde{z}_j + s_2 \frac{x - \tilde{z}_j}{|x - \tilde{z}_j|^2} = x.$$

By essentially the same arguments in the proof of lemma 2.1 in [9], there exists some $\lambda_0(j) > 0$ such that

$$(v_j)_{\tilde{z}_j,\lambda} \le v_j \quad \text{on } \overline{\Omega_j} \setminus B_\lambda(\tilde{z}_j), \quad \forall\, 0 < \lambda \le \lambda_0(j),$$

where

$$(v_j)_{\tilde{z}_j,\lambda}(y) := \left(\frac{\lambda}{|y - \tilde{z}_j|}\right)^{n-2} v_j\left(\tilde{z}_j + \frac{\lambda^2 (y - \tilde{z}_j)}{|y - \tilde{z}_j|^2}\right).$$

Let

$$\bar{\lambda}_j := \sup\{\mu \mid (v_j)_{\tilde{z}_j,\lambda} \le v_j \quad \text{on } \overline{\Omega_j} \setminus B_\lambda(\tilde{z}_j),\ \forall\, 0 < \lambda \le \mu\}.$$

Claim. $\bar{\lambda}_j \ge 10(T_j + R^2)$ for large j.

Indeed, if $\lambda_j < 10(T_j + R^2)$. We have

$$(v_j)_{\tilde{z}_j,\bar{\lambda}_j} \le v_j \quad \text{on } \overline{\Omega_j} \setminus B_{\bar{\lambda}_j}(\tilde{z}_j).$$

Since $\{\bar{\lambda}_j\}$ and $\{T_j\}$ stay bounded, we have, by the third line in (19),

$$v_j \le 2^{\frac{n-2}{2}} \quad \text{in } \Omega_j \cap B_{\bar{\lambda}_j}(\tilde{z}_j) \ \text{ for large } j.$$

It follows that

$$(v_j)_{\tilde{z}_j,\bar{\lambda}_j}(y) \le C|y|^{2-n}, \qquad \forall\, y \in \partial''\Omega_j,$$

where C is some constant independent of j. Thus, in view of (20),

$$(v_j)_{\tilde{z}_j,\bar{\lambda}_j} < v_j, \qquad \text{on } \partial''\Omega_j, \qquad \text{for large } j.$$

Using the strong maximum principle and the Hopf Lemma as in Case 1, we have

$$(v_j)_{\tilde{z}_j,\bar{\lambda}_j}(y) < v_j(y), \qquad \forall\, y \in \overline{\Omega}_j \cap \{|y - \tilde{z}_j| > \bar{\lambda}_j\},$$

$$\frac{\partial w_j}{\partial \nu}(y) > 0 \qquad \text{for } \ |y - \tilde{z}_j| = \bar{\lambda}_j,\ y_n > -T_j,$$

where $w_j := v_j - (v_j)_{\tilde{z}_j,\bar{\lambda}_j}$, and ν denotes the unit outer normal of $\partial B_{\bar{\lambda}_j}(\tilde{z}_j)$.

We also need to establish

$$\frac{\partial w_j}{\partial \nu}(y) > 0 \qquad \text{for } \ |y - \tilde{z}_j| = \bar{\lambda}_j, \ y_n = -T_j,$$

which follows from the same arguments in the proof of lemma 3 in [7].

With the above estimates we can prove, for some $\epsilon_j > 0$, that

$$(v_j)_{\tilde{z}_j,\lambda} \le v_j, \qquad \text{on } \Omega_j, \qquad \forall\ 0 < \lambda < \bar{\lambda}_j + \epsilon_j,$$

violating the definition of $\bar{\lambda}_j$. We have proved the claim.

A consequence of the claim is that for large j

$$(v_j)_{\tilde{z}_j,\lambda} \le v_j \quad \text{on } \overline{\Omega_j} \setminus B_\lambda(\tilde{z}_j), \quad \forall\ 0 < \lambda \le 10(T_j + R^2), .$$

This implies $s^{\frac{n-2}{2}} f_j(s)$ is an increasing function on $s \in (0, 2s_1)$. In particular,

$$v_j(x) = f_j(s_2) \ge (\frac{s_1}{s_2})^{\frac{n-2}{2}} f_j(s_2) = (\frac{s_1}{s_2})^{\frac{n-2}{2}} v_j(0) = 2^{\frac{2-n}{2}}.$$

Lemma 1.3 is proved.

Let ξ be the continuous solution of

$$\begin{cases} -\Delta \xi = 0 \qquad \text{in } B_1^+ \\ \xi = 0 \quad \text{on } \partial'' B_1^+ \\ \xi = 1 \ \text{ on } \partial' B_{\frac{1}{2}}, \quad 0 \le \xi \le 1 \ \text{ on } \partial' B_1 \ \text{ and } \xi = 0 \ \text{ on } \partial B_1 \cap \partial \mathbb{R}^n_+ \end{cases} \tag{25}$$

Let $\xi_j(y) = 2^{\frac{2-n}{2}} \xi(\frac{1}{R}(y - (0', T_j)))$ for $y \in B^j := \{y |\ y - (0', T_j) \in \overline{B_1^+}\}$. By (7) and Lemma 1.3,

$$\begin{cases} -\Delta(v_j - \xi_j) \ge 0 \qquad \text{in } B^j \\ v_j - \xi_j \ge 0 \qquad \text{on } \partial B^j \end{cases} \tag{26}$$

Applying the maximum principle, we have, for some dimensional positive constant $c_1(n) < 1$,

$$v_j \ge \xi_j \ge c_1(n) 2^{\frac{2-n}{2}} \ \text{ on } \ \{y |\ |y'| \le \frac{R}{2},\ T_j \le y_n \le T_j + \frac{R}{2}\}. \tag{27}$$

By (19), we have, for large j,

$$v_j \le 2^{\frac{n-2}{2}}, \qquad \text{on } \ \{y |\ |y'| \le \frac{R}{2},\ T_j \le y_n \le T_j + \frac{R}{2}\}.$$

With the above, the rest of the proof is essentially the same as that in Case 1. For reader′s convenience, we include the details.

For $z^* = (0', T_j + \frac{R}{4})$ and $\beta \geq 0$, let $\xi^\beta(y) := \frac{\beta}{\bar{R}^2}(\bar{R}^2 - |y - z^*|^2)$ for $|y - z^*| \leq \bar{R} := \frac{R}{8}$. We know when $\beta = 0$, $\xi^0 < v_j$ on $\overline{B_{\bar{R}}(z^*)}$, let $\bar{\beta}_j$ be the smallest positive number such that

$$v_j \geq \xi^{\bar{\beta}_j} \quad \text{on } \overline{B_{\bar{R}}(z^*)}, \quad v_j(\bar{y}_j) = \xi^{\bar{\beta}_j}(\bar{y}_j) \quad \text{for some } \bar{y}_j \in B_{\bar{R}}(z^*).$$

W,l,g, assume $\overline{B_{\bar{R}}(z^*)} \subset \Omega_j \cap \bar{B}_{\gamma_j}$, so $2^{\frac{n-2}{2}} \geq v_j(z^*) \geq \xi^{\bar{\beta}_j}(z^*) = \bar{\beta}_j$, we know $\bar{\beta}_j \leq 2^{\frac{n-2}{2}}$. Recall (27),

$$2^{\frac{2-n}{2}} c_1(n) \leq v_j(\bar{y}_j) = \xi^{\bar{\beta}_j}(\bar{y}_j) = \frac{\bar{\beta}_j}{\bar{R}^2}(\bar{R}^2 - |\bar{y}_j - z^*|^2) \leq 2^{\frac{n-2}{2}}(1 - (\frac{|\bar{y}_j - z^*|}{\bar{R}})^2)$$

from which, we have

$$|\bar{y}_j - z^*| \leq (1 - 4^{\frac{2-n}{2}} c_1(n))\bar{R}. \tag{28}$$

On the other hand, at $\bar{y}_j$, $A^{v_j}(\bar{y}_j) \leq A^{\xi^{\bar{\beta}_j}}(\bar{y}_j)$, so

$$A^{v_j}(\bar{y}_j) \leq A^{\xi^{\bar{\beta}_j}}(\bar{y}_j) \leq -\frac{2}{n-2}\xi^{\bar{\beta}_j}(\bar{y}_j)^{-\frac{n+2}{n-2}} \nabla^2 \xi^{\bar{\beta}_j}(\bar{y}_j) + \frac{2n}{(n-2)^2}\xi^{\bar{\beta}_j}(\bar{y}_j)^{-\frac{2n}{n-2}} I_{n\times n} \tag{29}$$

By $\xi^{\bar{\beta}_j}(\bar{y}_j) \geq c_1(n)2^{\frac{2-n}{2}}$, $|\nabla \xi^{\bar{\beta}_j}(\bar{y}_j)| = |-\frac{2\bar{\beta}_j}{\bar{R}^2}(\bar{y}_j - z^*)| \leq (1 - 4^{\frac{2-n}{2}} c_1(n))\frac{2^{\frac{n}{2}}}{\bar{R}}$ and $|\nabla^2 \xi^{\bar{\beta}_j}(\bar{y}_j)| = -\frac{2\bar{\beta}_j}{\bar{R}^2} I_{n\times n}$, (29) implies

$$A^{v_j}(\bar{y}_j) \leq c(n)\frac{\bar{\beta}_j}{\bar{R}^2} I_{n\times n} + c(n)\frac{\bar{\beta}_j^2}{\bar{R}} \leq c(n)\frac{1}{\bar{R}} I_{n\times n}.$$

Therefore by $A^{v_j}(\bar{y}_j) \in \Gamma_1$, $|A^{v_j}(\bar{y}_j)| \leq c(n)\frac{1}{\bar{R}}$. But (9) and $F(A^{v_j}(\bar{y}_j)) = 1$ together forces $\delta \leq c(n)\frac{1}{\bar{R}}$, which is impossible if $\bar{R}$ is large enough.

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