Gradient Estimates for the Perfect and Insulated Conductivity Problems with Multiple Inclusions

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Abstract

In this paper, we study the perfect and the insulated conductivity problems with multiple inclusions imbedded in a bounded domain in $\mathbb{R}^n, n \geq 2$. For these two extreme cases of the conductivity problems, the gradients of their solutions may blow up as two inclusions approach each other. We establish the gradient estimates for the perfect conductivity problems and an upper bound of the gradients for the insulated conductivity problems in terms of the distances between any two closely spaced inclusions.

0 Introduction

In this paper, a continuation of [5], we establish gradient estimates for the perfect conductivity problems in the presence of multiple closely spaced inclusions in a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. We also establish an upper bound of the gradients for the insulated conductivity problems. For these two extreme cases of the conductivity problems, the electric field, which is represented by the gradient of the solutions, may blow up as the inclusions approach to each other, the blow-up rates of the electric field have been studied in [1, 3, 5, 19, 20]. In particular, when there are only two strictly convex inclusions, and let ε be the distance between the two inclusions, then for the perfect conductivity problem, the optimal blow-up rates for the gradients, as ε approaches to zero, were established to be $\varepsilon^{-1/2}$, $(\varepsilon|\ln\varepsilon|)^{-1}$ and ε^{-1} for $n = 2,\ 3$ and $n \geq 4$ respectively. A criteria, in terms of a functional of boundary data, for the situation where blow-up rate is realized was also given. See e.g. the introductions of [5] and [20] for a more detailed description of these results. More recently, Lim and Yun in [15] have obtained further estimates with explicit dependence of the blow-up rates on the size of the inclusions for the perfect conductivity problem (see also [1] for results of this type), and H. Ammari, H. Kang, H. Lee, M. Lim and H. Zribi in [2] have given more refined estimates of the gradient of solutions.

The partial differential equations for the conductivity problems arise also in the study of composite materials. In $\mathbb{R}^2$, as explained in [14], if we use the bounded domain to represent the cross-section of a fiber-reinforced composite and use the inclusions to represent the cross-sections of the embedded fibers, then by a standard anti-plane shear model, the conductivity equations can be derived, in which the electric potential corresponds to the out-of-plane elastic displacement and the electric field corresponds to the stress tensor. Therefore, the gradient estimates for the conductivity problems provide valuable information about the stress intensity inside the composite materials.

When conductivities of the inclusions are away from zero and infinity, the boundedness of the gradients were observed numerically by Babuska, Anderson, Smith and Levin [4]. Bonnetier and Vogelius [6] proved it when the inclusions are two touching balls in $\mathbb{R}^2$. General results were established by Li and Vogelius [14] for second order divergence form elliptic equations with piecewise smooth coefficients,

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and then by Li and Nirenberg [13] for second order divergence form elliptic systems, including linear system of elasticity, with piecewise smooth coefficients. See also [12] and [16] for related studies.

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1 Mathematical set-up and the main results

Let Ω be a domain in $\mathbb{R}^n$ with $C^{2,\alpha}$ boundary, $n \geq 2$, $0 < \alpha < 1$. Let $\{D_i\}$ $(1 \leq i \leq m)$ be m strictly convex open subsets in Ω with $C^{2,\alpha}$ boundaries, $m \geq 2$, satisfying

$$\begin{aligned} &\text{the principal curvature of } \partial D_i \geq \kappa_0, \\ &\varepsilon_{ij} := \mathrm{dist}(D_i, D_j) > 0, \quad (i \neq j) \\ &\mathrm{dist}(D_i, \partial\Omega) > r_0, \quad \mathrm{diam}(\Omega) < \frac{1}{r_0}, \end{aligned} \tag{1.1}$$

where $\kappa_0, r_0 > 0$ are universal constants independent of $\{\varepsilon_{ij}\}$. We also assume that the $C^{2,\alpha}$ norms of ∂D_i are bounded by some universal constant independent of $\{\varepsilon_{ij}\}$. This implies that each D_i contains a ball of radius r_0^* for some universal constant $r_0^* > 0$ independent of $\{\varepsilon_{ij}\}$.

We state more precisely what it means by saying that the boundary of a domain, say Ω, is $C^{2,\alpha}$ for $0 < \alpha < 1$: In a neighborhood of every point of $\partial\Omega$, $\partial\Omega$ is the graph of some $C^{2,\alpha}$ function of $n-1$ variables. We define the $C^{2,\alpha}$ norm of $\partial\Omega$, denoted by $\|\partial\Omega\|_{C^{2,\alpha}}$, as the smallest positive number $\frac{1}{a}$ such that in the $2a-$neighborhood of every point of $\partial\Omega$, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to $\partial\Omega$ at 0, $\partial\Omega$ is given by the graph of a $C^{2,\alpha}$ function, denoted as f, which is defined as $|x'| < a$, the $a-$neighborhood of 0 in the tangent plane. Moreover, $\|f\|_{C^{2,\alpha}(|x'|<a)} \leq \frac{1}{a}$.

Denote

$$\widetilde{\Omega} := \Omega \backslash \overline{\cup_{i=1}^m D_i}.$$

Given $\varphi \in C^{1,\alpha}(\partial\Omega)$, the conductivity problem can be modeled by the following equation:

$$\begin{cases} div(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $k = (k_1, \ldots, k_m)$ and

$$a_k(x) = \begin{cases} k_i \in (0, \infty) & \text{in } D_i, \\ 1 & \text{in } \widetilde{\Omega}. \end{cases} \tag{1.3}$$

The existence and uniqueness of solutions to the above equation is well known. Moreover, we have $\|u_k\|_{H^1(\Omega)} \leq C\|\varphi\|_{C^{1,\alpha}(\partial\Omega)}$ for some constant C independent of k. Therefore, by passing to a subsequence, we have $u_k \rightharpoonup u_\infty$ in $H^1(\Omega)$ as $k_i \to \infty$ for all $1 \leq i \leq m$, where $u_\infty \in H^1(\Omega)$ is the solution to the following perfect conductivity problem,

$$\begin{cases} \Delta u = 0 & in \ \ \widetilde{\Omega}, \\ u|_+ = u|_- & on \ \ \partial D_i, \ (i = 1, 2, \ldots, m), \\ \nabla u \equiv 0 & in \ \ D_i \ (i = 1, 2, \ldots, m), \\ \displaystyle\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_+ = 0 & (i = 1, 2, \ldots, m), \\ u = \varphi & on \ \ \partial\Omega, \end{cases} \tag{1.4}$$

where

$$\frac{\partial u}{\partial \nu}\Big|_+ := \lim_{t \to 0^+} \frac{u(x + t\nu) - u(x)}{t}.$$

Here and throughout this paper ν is the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. For the derivation of the above equation, readers can refer to the Appendix of [5]. Note that the proof there is for $k_1 = k_2 = \cdots = k_m$, but it works also for the general case with modification.
Since the high stress concentration only occurs in the narrow regions between the fibers, we only need to focus on those narrow regions.

For $i \neq j$, denote

$$\mathrm{dist}(x_{ij}^i, x_{ij}^j) = \mathrm{dist}(D_i, D_j) = \varepsilon_{ij} > 0,\ x_{ij}^i \in \partial D_i,\ x_{ij}^j \in \partial D_j,$$

and

$$x_{ij}^0 := \frac{1}{2}(x_{ij}^i + x_{ij}^j).$$

It is easy to see that there exists some positive constant $\delta < \frac{1}{4}$ which depends only on κ_0, r_0 and $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but is independent of $\{\varepsilon_{ij}\}$ such that

$$\text{if } \varepsilon_{ij} < 2\delta,\ B(x_{ij}^0, 2\delta) \text{ only intersects with } D_i \text{ and } D_j. \tag{1.5}$$

Denote

$$\rho_n(\varepsilon) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} & for\ \ n = 2, \\ \dfrac{1}{\varepsilon|\ln\varepsilon|} & for\ \ n = 3, \\ \dfrac{1}{\varepsilon} & for\ \ n \geq 4. \end{cases} \tag{1.6}$$

Then we have the following gradient estimates for the perfect conductivity problem

Theorem 1.1 *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $n \geq 2$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $\varphi \in L^\infty(\partial\Omega)$, δ be the universal constant satisfying (1.5). Suppose $u_\infty \in H^1(\Omega)$ is the solution to equation (1.4), then for any $\varepsilon_{ij} < \delta$, we have*

$$\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega} \cap B(x_{ij}^0, \delta))} \leq C\rho_n(\varepsilon_{ij})\|\varphi\|_{L^\infty(\partial\Omega)}$$

where C is a constant depending only on n, κ_0, r_0, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε_{ij}.

Note that if $\varepsilon_{ij} \geq \delta$, by the maximum principle and the boundary estimates of harmonic functions, we immediately get $\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega} \cap B(x_{ij}^0, \delta))} \leq C\|u\|_{L^\infty(\widetilde{\Omega})} \leq C\|\varphi\|_{L^\infty(\partial\Omega)}$. Here we have used the fact that u_∞ is constant on each ∂D_i. Then by Theorem 1.1 and standard boundary Schauder estimates, see e.g. Theorem 8.33 in [9], we have the global gradient estimates of u_∞ in $\widetilde{\Omega}$.

Corollary 1.1 *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $n \geq 2$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $\varepsilon := \min_{i \neq j} \varepsilon_{ij} > 0$, and $\varphi \in C^{1,\alpha}(\partial\Omega)$, $0 < \alpha < 1$, and let $u_\infty \in H^1(\Omega)$ be the solution to equation (1.4). Then*

$$\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega})} \leq C\rho_n(\varepsilon)\|\varphi\|_{C^{1,\alpha}(\partial\Omega)}.$$

where C is a constant depending only on n, m, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε.

Remark 1.1 *The proof of Theorem 1.1 does not need D_i and D_j to be strictly convex, the strict convexity is only used in a fixed neighborhood of x_{ij}^0 (The size of the neighborhood is independent of $\{\varepsilon_{ij}\}$). In fact, our proofs of Theorem 1.1 also apply, with minor modification, to more general situations where two closely spaced inclusions, D_i and D_j, are not necessarily convex near points on the boundaries where minimal distance ε is realized; see discussions after the proof of Theorem 1.1 in Section 2.*

Next, we study the insulated conductivity problem. Similar to the perfect conductivity problem, the solution to the insulated conductivity problem is also the weak limit of u_k in $H^1(\widetilde{\Omega})$ as k approaches to 0. Here we consider the insulated conductivity problem with anisotropic conductivity.

Let $\Omega, D_i \subset \mathbb{R}^n$, ε_{ij} be defined as in (1.1), $\varphi \in C^{1,\alpha}(\partial\Omega)$, suppose $A(x) := \big(a^{ij}(x)\big)$ is a symmetric matrix function in $\widetilde{\Omega}$, where $a^{ij}(x) \in C^{\alpha}(\overline{\widetilde{\Omega}})$ and for some constants $\Lambda \geq \lambda > 0$,

$$\|a^{ij}\|_{C^{\alpha}(\overline{\widetilde{\Omega}})} \leq \Lambda, \qquad a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in \widetilde{\Omega}.$$

Then the anisotropic insulated conductivity problem can be described by the following equation,

$$\begin{cases} \partial_i(a^{ij}\partial_j u) = 0 & \text{in } \widetilde{\Omega}, \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \partial D_i (i = 1, 2, \ldots, m), \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The existence and uniqueness of solutions to equation (1.7) are elementary, see the Appendix.

As mentioned before, the blow-up can only occur in the narrow regions between two closely spaced inclusions. Therefore, we only derive gradient estimates for the solution to (1.7) in those regions. Without loss of generality, we consider the insulated conductivity problem in the narrow region between D_1 and D_2. Assume

$$\varepsilon = \mathrm{dist}(D_1, D_2)$$

After a possible translation and rotation, we may assume

$$(\varepsilon/2, 0') \in \partial D_1, \ (-\varepsilon/2, 0') \in \partial D_2.$$

Here and throughout this paper by writing $x = (x_1, x')$, we mean x' is the last $n-1$ coordinates of x.

We denote the narrow region between D_1 and D_2 and its boundary on ∂D_1 and ∂D_2 as follows

$$\begin{aligned} \mathcal{O}(r) &:= \widetilde{\Omega} \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \\ \Gamma_+ &:= \partial D_1 \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \\ \Gamma_- &:= \partial D_2 \cap \{x \in \mathbb{R}^n \big| |x'| < r\} \end{aligned} \tag{1.8}$$

where r is some universal constant depending only on $\{\|\partial D_i\|_{C^{2,\alpha}}\}$.

With the above notations, we consider the following problem,

$$\begin{cases} \partial_i(a^{ij}\partial_j u) = 0 & \text{in } \mathcal{O}(r), \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-. \end{cases} \tag{1.9}$$

Then we have:

Theorem 1.2 *If $u_0 \in H^1(\mathcal{O}(r))$ is a weak solution of (1.9), then*

$$|\nabla u_0(x)| \leq \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\sqrt{\varepsilon + |x'|^2}}, \quad \textit{for all} \ \ x \in \mathcal{O}(\frac{r}{2}). \tag{1.10}$$

where C is a constant depending only on n, κ_0, r_0, Λ, λ, r and $\|\partial D_i\|_{C^{2,\alpha}} (i = 1, 2)$, but independent of ε.

Remark 1.2 *Theorem 1.2 also remains true for general second order elliptic systems, its proof is essentially the same as for the equations.*

A consequence of Theorem 1.2 is the following global gradient estimates for the insulated conductivity problem.

Corollary 1.2 *Let $\Omega, \{D_i\} \subset \mathbb{R}^n$, $\{\varepsilon_{ij}\}$ be defined as in (1.1), $\varepsilon := \min\limits_{i \neq j} \varepsilon_{ij} > 0$, and $\varphi \in C^{1,\alpha}(\partial\Omega)$, let $u_0 \in H^1(\widetilde{\Omega})$ be the weak solution to equation (1.7), then*

$$\|\nabla u_0\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C}{\sqrt{\varepsilon}} \|\varphi\|_{C^{1,\alpha}(\partial\Omega)}. \tag{1.11}$$

where C is a constant depending only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of ε.

Note that throughout this paper we often use C to denote different constants, but all these constants are independent of ε.

The paper is organized as follows. In Section 2 we consider the perfect conductivity problem and prove Theorem 1.1. In Section 3 we show Theorem 1.2 for the insulated case. Finally in the Appendix we present some elementary results for the insulated conductivity problem.

2 The perfect conductivity problem with multiple inclusions

In this section, we consider the perfect conductivity problem (1.4). Note that from equation (1.4), we know that $u \equiv C_i$ on $\overline{D}_i,\ 1 \leq i \leq m$, where $\{C_i\}$ are some unknown constants. In order to prove Theorem 1.1, we first estimate $|C_i - C_j|$ for $1 \leq i \neq j \leq m$, which later will allow us to control the gradient of u in the narrow region between D_i and D_j.

2.1 A Matrix Result

To estimate $|C_i - C_j|$, the following proposition plays a crucial role.

Let m be a positive integer, $P = (p_{ij})$ an $m \times m$ real symmetric matrix satisfying,

$$(A1)\ p_{ij} = p_{ji} \leq 0\ (i \neq j);$$
$$(A2)\ 0 < r_1 \leq \bar{p}_i := \sum_{j=1}^{m} p_{ij} \leq r_2,$$

where r_1 and r_2 are some positive constants.

Remark 2.1 *An $m \times m$ matrix P satisfying $|p_{ii}| > \sum_{j\neq i} |p_{ij}|$ is called a diagonally dominant matrix. Such a matrix is nonsingular, see [10]. $(A1)$ and $(A2)$ imply that the matrix P is diagonally dominant.*

Proposition 2.1 *Let $P = (p_{ij})$ be an $m \times m$ real symmetric matrix satisfying $(A1)$ and $(A2)$, $m \geq 1$. For $\beta \in \mathbb{R}^m$, let α be the solution of*

$$P\alpha = \beta, \tag{2.1}$$

then

$$|\alpha_i - \alpha_j| \leq m(m-1)\frac{r_2}{r_1}\frac{|\beta|}{|p_{ij}| + r_1}, \tag{2.2}$$

where $|\beta| = \max_i |\beta_i|$.

Before proving the proposition, we introduce the following lemmas.

Denote

$$\mathcal{I}(l) = \{\text{all } l \times l \text{ diagonal matrices whose diagonal entries are 1 or } -1\},$$
$$\mathcal{I}_e(l) = \{\overline{I} \in \mathcal{I}(l) \big| \overline{I} \text{ has even numbers of } -1 \text{ in its diagonal}\},$$
$$\mathcal{I}_o(l) = \{\overline{I} \in \mathcal{I}(l) \big| \overline{I} \text{ has odd numbers of } -1 \text{ in its diagonal}\}.$$

Lemma 2.1 *For any $x \in \mathbb{R}$ and any $l \times l$ matrix A, $l \geq 1$,*

$$\sum_{\overline{I} \in \mathcal{I}_e(l)} \det(xI + \overline{I}A) \equiv 2^{l-1}(x^l + \det A);$$

$$\sum_{\overline{I} \in \mathcal{I}_o(l)} \det(xI + \overline{I}A) \equiv 2^{l-1}(x^l - \det A).$$

Proof: We prove it by induction. The above identities can be easily checked for $l = 1$. Suppose that the above identities stand for $l = k - 1 \geq 1$, we will prove them for $l = k$. Observe that the above identities hold when $x = 0$. To prove them for all x, it suffices to show that the derivatives with respect to x in both sides of the identities coincide. Since for any $\overline{I} \in \mathcal{I}(k)$,

$$(\det(xI + \overline{I}A))' = \sum_{i=1}^{k} \det(xI + \overline{I}_i A_i)$$

where A_i and $\overline{I}_i$ are the submatrices obtained by eliminating the ith row and the ith column of A and $\overline{I}$ respectively.
Notice that if $\overline{I}$ runs through all the elements of $\mathcal{I}_e(k)$, $\overline{I}_i$ will run through all the elements of $\mathcal{I}(k-1)$ for every fixed $i \in \{1, 2, \ldots, k\}$, so we have

$$\begin{aligned}
&\sum_{\overline{I} \in \mathcal{I}_e(k)} (\det(xI + \overline{I}A))' \\
&= \sum_{i=1}^{k} \Big(\sum_{\overline{I} \in \mathcal{I}_e(k-1)} \det(xI + \overline{I}A_i) + \sum_{\overline{I} \in \mathcal{I}_o(k-1)} \det(xI + \overline{I}A_i) \Big) \\
&= \sum_{i=1}^{k} \big(2^{k-2}(x^{k-1} + \det A_i) + 2^{k-2}(x^{k-1} - \det A_i) \big) \qquad \text{(By induction)} \\
&= k2^{k-1}x^{k-1} = 2^{k-1}(x^k + \det A)'.
\end{aligned}$$

Therefore, we have proved the first identity. The second one follows from the first one by changing the sign of one row of A.
As a consequence of Lemma 2.1, we have

Corollary 2.1 *Let A be an $l \times l$ matrix, if $\det(I + \overline{I}A) \geq 0$ for any $\overline{I} \in \mathcal{I}(l)$, then $|\det A| \leq 1$.*

Lemma 2.2 *Given integers $m > l \geq 1$, let $Q = (q_{ij})$ be an $m \times l$ real matrix which satisfies, for $j = 1, 2, \ldots, l$,*

$$q_{jj} > \sum_{i \neq j} |q_{ij}|. \tag{2.3}$$

Let $\mathcal{A}$ be the set of all $l \times l$ submatrices of the above matrix Q and $S_1 \in \mathcal{A}$ the matrix obtained from the first l rows of Q, then we have

$$\det S_1 = \max_{S \in \mathcal{A}} |\det S|.$$

Proof: For any $S \in \mathcal{A}$, by rearranging the order of its rows we do not change $|\det S|$. Thus we can treat S as a matrix obtained by replacing some rows of S_1 by some other rows of Q. Note that S and S_1 could have no rows in common, which means S is obtained by replacing all the rows of S_1 by some other rows of Q.
Given any $\overline{I} \in \mathcal{I}(l)$, we claim:

$$\det(S_1 + \overline{I}S) \geq 0$$

Proof of the claim: There are two cases between S_1 and S:
Case 1. S_1 and S have no rows in common. Then by (2.3), we know that $S_1 + \overline{I}S$ is diagonally dominant, therefore $\det(S_1 + \overline{I}S) > 0$.
Case 2. S_1 and S have some common rows, denote the order of these rows by $1 \leq i_1 < \cdots < i_s \leq l, 1 \leq s \leq l$. If row i_{s_0} of $\overline{I}S$ is opposite to row i_{s_0} of S for some $1 \leq s_0 \leq s$, then row i_{s_0} of $S_1 + \overline{I}S$ is 0, therefore $\det(S_1 + \overline{I}S) = 0$. Otherwise row i_t of $\overline{I}S$ is the same as that of S and S_1 for any $1 \leq t \leq s$, then we take out the common factors 2 in these rows when we compute $\det(S_1 + \overline{I}S)$, thus we have

$$\det(S_1 + \overline{I}S) = 2^s \det(S_1 + \overline{I}\hat{S}),$$

where $\hat{S}$ is the matrix obtained by replacing row i_t of S by 0 for any $1 \le t \le s$. We know that $S_1 + \overline{I}\hat{S}$ is diagonally dominant according to (2.3), then $\det(S_1 + \overline{I}\hat{S}) > 0$, it yields that $\det(S_1 + \overline{I}S) > 0$. Therefore, the claim is proved.
Since $\det S_1 > 0$ and

$$\det(S_1 + \overline{I}S) = \det(I + \overline{I}SS_1^{-1}) \det S_1$$

we have, by the claim, that for any $\overline{I} \in \mathcal{I}(l)$,

$$\det(I + \overline{I}SS_1^{-1}) \ge 0$$

By Corollary 2.1, we have

$$|\det(SS_1^{-1})| \le 1$$

therefore

$$\det S_1 \ge |\det S|.$$

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1: For $m = 1$ the inequality is automatically true. For $m = 2$, we have, by Cramer's rule,

$$\alpha_1 - \alpha_2 = \frac{\begin{vmatrix} \beta_1 & p_{12} \\ \beta_2 & p_{22} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} - \frac{\begin{vmatrix} p_{11} & \beta_1 \\ p_{21} & \beta_2 \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \beta_1 & \bar{p}_1 \\ \beta_2 & \bar{p}_2 \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}}$$

Since $r_1 \le \bar{p}_i \le r_2$ by Condition (A2),

$$\begin{vmatrix} \beta_1 & \bar{p}_1 \\ \beta_2 & \bar{p}_2 \end{vmatrix} = \beta_1\bar{p}_2 - \beta_2\bar{p}_1 \le 2r_2|\beta|$$

On the other hand, by Condition (A1) and (A2)

$$\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} = \begin{vmatrix} \bar{p}_1 & p_{12} \\ \bar{p}_2 & p_{22} \end{vmatrix} = \bar{p}_1p_{22} - \bar{p}_2p_{12} \ge \bar{p}_1p_{22} \ge r_1(r_1 + |p_{12}|).$$

Therefore, Proposition 2.1 for $m = 2$ follows from the above.

For $m \ge 3$, we only estimate $|\alpha_1 - \alpha_2|$ since the other estimates can be obtained by switching columns of P.

Since α satisfies (2.1), by Cramer's rule, we have:

$$\alpha_1 - \alpha_2 = \frac{\begin{vmatrix} \beta_1 & p_{12} & \cdots & p_{1m} \\ \beta_2 & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & p_{m2} & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}} - \frac{\begin{vmatrix} p_{11} & \beta_1 & \cdots & p_{1m} \\ p_{21} & \beta_2 & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & \beta_m & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \beta_1 & p_{11} + p_{12} & p_{13} & \cdots & p_{1m} \\ \beta_2 & p_{21} + p_{22} & p_{23} & \cdots & p_{2m} \\ \beta_3 & p_{31} + p_{32} & p_{33} & \cdots & p_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_m & p_{m1} + p_{m2} & p_{m3} & \cdots & p_{mm} \end{vmatrix}}{\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}}$$

By adding the last $(m-2)$ columns of the matrix in the numerator to its second column, we have

$$\alpha_1-\alpha_2=\frac{\begin{vmatrix}\beta_1 & \bar{p}_1 & p_{13} & \cdots & p_{1s}\\ \beta_2 & \bar{p}_2 & p_{23} & \cdots & p_{2s}\\ \beta_3 & \bar{p}_3 & p_{33} & \cdots & p_{3s}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \beta_m & \bar{p}_m & p_{m3} & \cdots & p_{mm}\end{vmatrix}}{\begin{vmatrix}p_{11} & p_{12} & \cdots & p_{1m}\\ p_{21} & p_{22} & \cdots & p_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ p_{m1} & p_{m2} & \cdots & p_{mm}\end{vmatrix}}:=\frac{\det\widetilde{P}}{\det P}.$$

Next we estimate the determinants of the above two matrices separately.

Expanding $\det P$ with respect to the first column, we have

$$\det P=\sum_{j=1}^{m}p_{j1}P_{j1}$$

where P_{ji} is the cofactor of p_{j1}.
Applying Lemma 2.2 to the $m\times(m-1)$ matrix obtained by eliminating the first column of P, we know that, among the cofactors P_{j1}, $P_{11}>0$ has the largest absolute value. Since $p_{j1}=p_{1j}\le 0$ $(j\neq 1)$ and $p_{11}>0$ by condition $(A1)$ and $(A2)$, we have

$$\det P\ge\sum_{j=1}^{m}p_{j1}P_{11}=\bar{p}_1P_{11}.$$

For the same reason, we have

$$P_{11}=\begin{vmatrix}p_{22} & \cdots & p_{2m}\\ \vdots & \ddots & \vdots\\ p_{m2} & \cdots & p_{mm}\end{vmatrix}\ge\Big(\sum_{j=2}^{m}p_{2j}\Big)\begin{vmatrix}p_{33} & \cdots & p_{3m}\\ \vdots & \ddots & \vdots\\ p_{m3} & \cdots & p_{mm}\end{vmatrix}.$$

Combining the above two inequalities and using condition $(A1)$ and $(A2)$, we have

$$\begin{aligned}\det P&\ge\bar{p}_1\sum_{j=2}^{m}p_{2j}\begin{vmatrix}p_{33} & \cdots & p_{3m}\\ \vdots & \ddots & \vdots\\ p_{m3} & \cdots & p_{mm}\end{vmatrix}=\bar{p}_1(\bar{p}_2-p_{21})\begin{vmatrix}p_{33} & \cdots & p_{3m}\\ \vdots & \ddots & \vdots\\ p_{m3} & \cdots & p_{mm}\end{vmatrix}\\ &\ge r_1(|p_{12}|+r_1)\begin{vmatrix}p_{33} & \cdots & p_{3m}\\ \vdots & \ddots & \vdots\\ p_{m3} & \cdots & p_{mm}\end{vmatrix}.\end{aligned}\tag{2.4}$$

By Laplace expansion, see e.g. page 130 of [17], we can expand $\det\widetilde{P}$ with respect to the first two columns of P, namely,

$$\det\widetilde{P}=\sum_{i_1,i_2}\begin{vmatrix}\beta_{i_1} & \bar{p}_{i_1}\\ \beta_{i_2} & \bar{p}_{i_2}\end{vmatrix}\widetilde{P}_{i_1i_212},\tag{2.5}$$

where $1\le i_1<i_2\le m$ and $\widetilde{P}_{i_1i_212}$ is the cofactor of the 2nd-order minor in row i_1,i_2 and column $1,2$ of $\widetilde{P}$.
Applying Lemma 2.2 to the $m\times(m-2)$ matrix obtained by eliminating the first 2 columns of $\widetilde{P}$, we know that, among all those cofactors,

$$\begin{vmatrix}p_{33} & \cdots & p_{3m}\\ \vdots & \ddots & \vdots\\ p_{m3} & \cdots & p_{mm}\end{vmatrix}$$

has the largest absolute value. Since $0 < \bar{p}_i \le r_2$ by condition $(A2)$,

$$\begin{vmatrix} \beta_{i_1} & \bar{p}_{i_1} \\ \beta_{i_2} & \bar{p}_{i_2} \end{vmatrix} \le 2r_2|\beta|,$$

then by (2.5), we have

$$\big|\det \widetilde{P}\big| \le m(m-1)r_2|\beta| \begin{vmatrix} p_{33} & \cdots & p_{3m} \\ \vdots & \ddots & \vdots \\ p_{m3} & \cdots & p_{mm} \end{vmatrix}. \tag{2.6}$$

By (2.4) and (2.6), we have

$$|\alpha_1 - \alpha_2| = \frac{|\det \widetilde{P}|}{|\det P|} \le m(m-1)\frac{r_2}{r_1}\frac{|\beta|}{|p_{12}| + r_1}.$$

2.2 Proof of Theorem 1.1

As in [5], we decompose u_∞ into $m+1$ parts:

$$u_\infty = v_0 + \sum_{i=1}^{m} C_i v_i, \tag{2.7}$$

where $v_i \in H^1(\widetilde{\Omega})$ $(i = 0, 1, 2, \ldots, m)$ are determined by the following equations:
for $i = 0$,

$$\begin{cases} \Delta v_0 = 0 & in \;\; \widetilde{\Omega}, \\ v_0 = 0 & on \;\; \partial D_1, \; \partial D_2, \ldots \; \partial D_m, \\ v_0 = \varphi & on \;\; \partial\Omega. \end{cases} \tag{2.8}$$

for $i = 1, 2, \ldots, m$,

$$\begin{cases} \Delta v_i = 0 & in \;\; \widetilde{\Omega}, \\ v_i = 1 & on \;\; \partial D_i, \\ v_i = 0 & on \;\; \partial D_j, \; \text{for } j \ne i, \\ v_i = 0 & on \;\; \partial\Omega. \end{cases} \tag{2.9}$$

Since u_∞ satisfies the integral conditions in equation (1.4), using the decomposition formula (2.7), we know that the vector $(C_1, C_2, \ldots, C_m)$ satisfies the following system of linear equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{2.10}$$

where

$$a_{ij} := \int_{\partial D_j} \frac{\partial v_i}{\partial \nu}, \quad (i, j = 1, 2, \ldots, m), \tag{2.11}$$

$$b_i := -\int_{\partial D_i} \frac{\partial v_0}{\partial \nu}, \qquad (i = 1, 2, \ldots, m). \tag{2.12}$$

Similar to the two inclusions case in [5], we first investigate the properties of v_i $(i = 0, 1, \cdots, m)$, the matrix $A = (a_{ij})$ and the vector b defined by (2.11) and (2.12). Here we state the following lemma, for its proof, readers may refer to Lemma 2.4 in [5].

Lemma 2.3 *For $1 \le i, \; j \le m$, let a_{ij} and b_i be defined by (2.11) and (2.12), then they satisfy the following:*

(1) $a_{ii} < 0, \quad a_{ij} = a_{ji} > 0 \ (i \neq j)$,

(2) $-C \leq \sum_{1\leq j\leq m} a_{ij} \leq -\frac{1}{C}$,

(3) $|b_i| \leq C\|\varphi\|_{L^\infty(\partial\Omega)}$,

where $C > 0$ is a universal constant depending only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of ε_{ij}.

Remark 2.2 *From property (1) and (2) in Lemma 2.3, we know that A is diagonally dominant, therefore it is nonsingular.*

Lemma 2.4 *Let $v_0, v_i (i = 1, \ldots, m)$ be the solutions of equations (2.8) and (2.9) respectively, δ is the constant satisfying (1.5), then there exists a universal constant C depending only on n, m, r_0, κ_0, $\|\partial D_i\|_{C^{2,\alpha}}$ and $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of $\{\varepsilon_{ij}\}$ such that,*

(1) $\|\nabla v_0\|_{L^\infty(\widetilde{\Omega})} \leq C$;

(2) $\|\nabla v_i\|_{L^\infty(B(x^0_{ij},\delta)\cap\widetilde{\Omega})} \leq \frac{C}{\varepsilon_{ij}}$ *if* $\varepsilon_{ij} < \delta$;

(3) $|\nabla v_i| \leq C$ *on* $\widetilde{\Omega} \setminus \big(\bigcup_{j\neq i, \varepsilon_{ij}<\delta} B(x^0_{ij}, \delta)\big)$.

Proof: The proof of (1) is the same as the proof of Lemma 2.3 in [5]. Since $\|v_i\|_{L^\infty(\widetilde{\Omega})} = 1$, δ is the constant satisfying (1.5), then if $\varepsilon_{ij} < \delta$, then by (1.5), we know that $B(x^0_{ij}, \delta)$ only intersects with D_i and D_j, and $B(x^0_{ij}, \delta)$ is at least δ away from other inclusions. Then (2) just follows from the maximum principle and standard boundary estimates for harmonic functions. For the same reason, to prove (3), we only need to prove $\|\nabla v_i\|_{L^\infty(B(x^0_{kl},\delta)\cap\widetilde{\Omega})} \leq C$ if $k, l \neq i$ and $\varepsilon_{kl} < \delta$. Without loss of generality, we assume $k = 1, l = 2, i = 3$. Let $\widetilde{v}_3$ be the solution of the following equation,

$$\begin{cases} \Delta\widetilde{v}_3 = 0 & in \ \ \Omega \setminus \overline{D_1 \cup D_3}, \\ \widetilde{v}_3 = 0 & on \ \ \partial D_1, \\ \widetilde{v}_3 = 1 & on \ \ \partial D_3, \\ \widetilde{v}_3 = 0 & on \ \ \partial\Omega. \end{cases}$$

Then we have $\widetilde{v}_3 \geq v_3$ on $\partial\widetilde{\Omega}$, by the maximum principle, $\widetilde{v}_3 \geq v_3$ in $\widetilde{\Omega}$. Since $\widetilde{v}_3 = v_3 = 0$ on ∂D_1, we have

$$\frac{\partial\widetilde{v}_3}{\partial\nu} \geq \frac{\partial v_3}{\partial\nu} \geq 0.$$

But $|\nabla\widetilde{v}_3| < C$ on $\partial D_1 \cap B(x^0_{12}, \delta)$ by the boundary estimates of harmonic functions, then we have

$$\|\nabla v_3\|_{L^\infty(\partial D_1\cap B(x^0_{12},\delta))} = \|\frac{\partial v_3}{\partial\nu}\|_{L^\infty(\partial D_1\cap B(x^0_{12},\delta))} < C. \tag{2.13}$$

Similarly, we have

$$\|\nabla v_3\|_{L^\infty(\partial D_2\cap B(x^0_{12},\delta))} = \|\frac{\partial v_3}{\partial\nu}\|_{L^\infty(\partial D_2\cap B(x^0_{12},\delta))} < C. \tag{2.14}$$

Furthermore, by gradient estimates and boundary estimates of harmonic functions, we have

$$\|\nabla v_3\|_{L^\infty(\partial B(x^0_{12},\delta)\cap\widetilde{\Omega})} < C. \tag{2.15}$$

Since ∇v_3 is still harmonic function on $B(x^0_{12}, \delta) \cap \widetilde{\Omega}$, by (2.13), (2.14) and (2.15) and the maximum principle, we have

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega}\cap B(x^0_{12},\delta))} < C.$$

Next, we derive some further estimates of $A = (a_{ij})$.

Lemma 2.5 *Let a_{ij} be defined as in (2.11), then there exists a universal constant $C>0$, depending only on n, r_0, κ_0, $\|\partial D_i\|_{C^{2,\alpha}}$ and $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of $\{\varepsilon_{ij}\}$, such that for $1\leq i\neq j\leq m$,*

$$
\begin{aligned}
&-\frac{C}{\sqrt{\min\limits_{k\neq i}\varepsilon_{ik}}}<a_{ii}<-\frac{1}{C\sqrt{\min\limits_{k\neq i}\varepsilon_{ik}}}, && \frac{1}{C\sqrt{\varepsilon_{ij}}}<a_{ij}<\frac{C}{\sqrt{\varepsilon_{ij}}}, && \textit{for } n=2,\\
&-C|\ln(\min_{k\neq i}\varepsilon_{ik})|<a_{ii}<-\frac{1}{C}|\ln(\min_{k\neq i}\varepsilon_{ik})|, && \frac{1}{C}|\ln\varepsilon_{ij}|<a_{ij}<C|\ln\varepsilon_{ij}|, && \textit{for } n=3,\\
&-C<a_{ii}<-\frac{1}{C}, && \frac{1}{C}<a_{ij}<C, && \textit{for } n\geq 4.
\end{aligned}
$$

Proof: Without loss of generality, we assume $i=1, j=2$. The proof of the estimates for a_{11} is the same as that in Lemma 2.5, Lemma 2.6, and Lemma 2.7 in [5]. Here we prove the estimate for a_{12}. In the following, we use C to denote some universal constant depending only on n, r_0, κ_0, $\|\partial D_i\|_{C^{2,\alpha}}$ and $\|\partial\Omega\|_{C^{2,\alpha}}$, but independent of $\{\varepsilon_{ij}\}$.

Notice that if ε_{12} is larger than some universal constant, then the proof is trivial. Therefore, we can assume $\varepsilon_{12}<\delta$, where $\delta<1/4$ is the universal constant satisfying (1.5). By (1.5), we know that $B(x_{12}^0,\delta)$ only intersects with D_1 and D_2.

Denote

$$\Gamma_i:=\partial D_i\cap B(x_{12}^0,\delta)\ (i=1,2),\quad \Gamma_3:=\partial B(x_{12}^0,\delta)\setminus(D_1\cup D_2)$$

Since $B(x_{12}^0,2\delta)$ does not intersect with $D_i (i\geq 3)$ or $\partial\Omega$ by (1.5), then

$$dist(\Gamma_3,\cup_{i=3}^m\partial D_i)>\delta,\quad dist(\Gamma_3,\partial\Omega)>\delta,$$

by standard gradient estimates and boundary estimates for harmonic functions, we have

$$\|\nabla v_1\|_{L^\infty(\Gamma_3)}<C \tag{2.16}$$

By Lemma 2.4, we have $\|\nabla v_1\|_{L^\infty(\partial D_2\setminus\Gamma_2)}<C$.
Therefore, we have

$$a_{12}=\int_{\partial D_2}\frac{\partial v_1}{\partial\nu}=\int_{\Gamma_2}\frac{\partial v_1}{\partial\nu}+\int_{\partial D_2\setminus\Gamma_2}\frac{\partial v_1}{\partial\nu}=\int_{\Gamma_2}\frac{\partial v_1}{\partial\nu}+O(1). \tag{2.17}$$

By the harmonicity of v_1 on $B(x_{12}^0,\delta)\cap\widetilde{\Omega}$ and (2.16), we have

$$0=\int_{\Gamma_1}\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_2}\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_3}\frac{\partial v_1}{\partial\nu}=\int_{\Gamma_1}\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_2}\frac{\partial v_1}{\partial\nu}+O(1). \tag{2.18}$$

Meanwhile, by Green's formula and (2.16), we have

$$
\begin{aligned}
-\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}}|\nabla v_1|^2&=\int_{\Gamma_1}v_1\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_2}v_1\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_3}v_1\frac{\partial v_1}{\partial\nu}\\
&=\int_{\Gamma_1}\frac{\partial v_1}{\partial\nu}+\int_{\Gamma_3}v_1\frac{\partial v_1}{\partial\nu}=\int_{\Gamma_1}\frac{\partial v_1}{\partial\nu}+O(1)
\end{aligned}
\tag{2.19}
$$

Therefore, by combining (2.17), (2.18) and (2.19), we have

$$a_{12}=\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}}|\nabla v_1|^2+O(1).$$

Similar to the energy estimates given in Lemma 1.5, Lemma 1.6, and Lemma 1.7 in [5], we have

$$
\begin{aligned}
\frac{1}{C\sqrt{\varepsilon_{12}}}<\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}}|\nabla v_1|^2<\frac{C}{\sqrt{\varepsilon_{12}}}, &\qquad \text{for } n=2\\
\frac{1}{C}|\ln\varepsilon_{12}|<\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}}|\nabla v_1|^2<C|\ln\varepsilon_{12}|, &\qquad \text{for } n=3\\
\frac{1}{C}<\int_{B(x_{12}^0,\delta)\cap\widetilde{\Omega}}|\nabla v_1|^2<C, &\qquad \text{for } n\geq 4.
\end{aligned}
$$

Therefore,

$$\begin{aligned} \frac{1}{C\sqrt{\varepsilon_{12}}} < a_{12} < \frac{C}{\sqrt{\varepsilon_{12}}}, \qquad & \text{for } n = 2, \\ \frac{1}{C}|\ln \varepsilon_{12}| < a_{12} < C|\ln \varepsilon_{12}|, \qquad & \text{for } n = 3, \\ \frac{1}{C} < a_{12} < C, \qquad & \text{for } n \geq 4. \end{aligned}$$

□

Knowing enough properties of the system of linear equations (2.10) from Lemma 2.3 and Lemma 2.5 , we have

Proposition 2.2 *Let $u_\infty \in H^1(\Omega)$ be the weak solution to equation (1.4) and C_i the value of u_∞ on D_i, then for any $1 \leq i \neq j \leq m$, there exists a universal constant $C > 0$ depending only on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of $\{\varepsilon_{ij}\}$ such that*

$$\begin{aligned} |C_i - C_j| &\leq C\sqrt{\varepsilon_{ij}}\|\varphi\|_{L^\infty(\partial\Omega)} \qquad && for \;\; n = 2, \\ |C_i - C_j| &\leq C\frac{1}{|\ln \varepsilon_{ij}|}\|\varphi\|_{L^\infty(\partial\Omega)} \qquad && for \;\; n = 3, \\ |C_i - C_j| &\leq C\|\varphi\|_{L^\infty(\partial\Omega)} \qquad && for \;\; n \geq 4. \end{aligned} \tag{2.20}$$

Proof: By Lemma 2.3, we know that the matrix $-A$ satisfies condition $(A1)$ and $(A2)$, then applying Proposition 2.1 on (2.10), we have, for any $1 \leq i \neq j \leq m$,

$$|C_i - C_j| \leq \frac{C}{a_{ij}}\|\varphi\|_{L^\infty(\partial\Omega)}$$

where C is some constant depending on n, κ_0, r_0, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\{\|\partial D_i\|_{C^{2,\alpha}}\}$, but independent of $\{\varepsilon_{ij}\}$. By Lemma 2.5, we immediately finish the proof. □

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1: We prove the estimates in dimension 2, the proof for the higher dimensional cases is similar. Without loss of generality, we assume $i = 1$, $j = 2$ and $\varepsilon_{12} < \delta$. Now we need to prove the gradient estimates for u_∞ in the narrow region between D_1 and D_2. For simplicity, we assume $\|\varphi\|_{L^\infty(\partial\Omega)} = 1$.

By the decomposition formula (2.7), we have

$$\nabla u_\infty = (C_1 - C_2)\nabla v_1 + C_2(\nabla(v_1 + v_2)) + \sum_{i=3}^{m} C_i \nabla v_i + \nabla v_0$$

By Lemma 2.4, we have

$$\|\nabla v_1\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < \frac{C}{\varepsilon_{12}}, \qquad \|\nabla v_0\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C \tag{2.21}$$

where C is some universal constant.

For $i = 3, \ldots, m$, we have, by Lemma 2.4,

$$\|\nabla v_i\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C. \tag{2.22}$$

Since $v_1 + v_2 = 1$ on both ∂D_1 and ∂D_2, similar to the proof of Lemma 2.4, we can show that

$$\|\nabla(v_1 + v_2)\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} < C. \tag{2.23}$$

By Proposition 2.2, (2.21), (2.22) and (2.23), we have

$$\begin{aligned}\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))} &\le |C_1-C_2|\|\nabla v_1\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}+|C_2|\|\nabla(v_1+v_2)\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}\\ &\quad+\sum_{i=3}^m|C_i|\|\nabla v_i\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}+\|\nabla v_0\|_{L^\infty(\widetilde{\Omega}\cap B(x_{12}^0,\delta))}\\ &\le C\sqrt{\varepsilon_{12}}\frac{1}{\varepsilon_{12}}+C\\ &\le \frac{C}{\sqrt{\varepsilon_{12}}}.\end{aligned}$$

As we mentioned in Remark 1.1, the strict convexity assumption of the two inclusions can be weakened. In fact, our proof of Theorem 1.1 applies, with minor modification, to more general inclusions as below.

In $\mathbb{R}^n$, $n\ge 2$, for two closely spaced inclusions D_i and D_j which are not necessarily strictly convex, assume $\partial D_i\cap B(0,r)$ and $\partial D_j\cap B(0,r)$ can be represented by the graph of $x_1=f(x')+\frac{\varepsilon_{ij}}{2}$ and $x_1=-g(x')-\frac{\varepsilon_{ij}}{2}$, then $f(0')=g(0')=0$, $\nabla(g+f)(0')=0$. Assume further that

$$\lambda_1|x'|^{2l}\le g(x')+f(x')\le\lambda_2|x'|^{2l},\quad \forall|x'|\le r/2, \tag{2.24}$$

where $\lambda_2\ge\lambda_1>0, l\in\mathbb{Z}^+$.

Under the above assumption, let $u_\infty\in H^1(\Omega)$ be the solution to equation (1.4). Then, for ε_{ij} sufficiently small, we have

$$\begin{aligned}&\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{ij}^0,\delta))}\le C\|\varphi\|_{L^\infty(\partial\Omega)}\varepsilon_{ij}^{-\frac{n-1}{2l}} && \textit{if } n-1<2l,\\ &\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{ij}^0,\delta))}\le C\|\varphi\|_{L^\infty(\partial\Omega)}\frac{1}{\varepsilon_{ij}|\ln\varepsilon_{ij}|} && \textit{if } n-1=2l,\\ &\|\nabla u_\infty\|_{L^\infty(\widetilde{\Omega}\cap B(x_{ij}^0,\delta))}\le C\|\varphi\|_{L^\infty(\partial\Omega)}\frac{1}{\varepsilon_{ij}} && \textit{if } n-1>2l.\end{aligned} \tag{2.25}$$

where C is a constant depending on n, λ_1, λ_2, r_0, $\|\partial D_i\|_{C^{2,\alpha}}$ and $\|\partial D_j\|_{C^{2,\alpha}}$, but independent of ε_{ij}. For the proof, please refer to the corresponding discussion after the proof of Theorem 0.1-0.2 in [5].

3 The insulated conductivity problem

In this section, we consider the anisotropic insulated conductivity problem, which is described by Equation (1.7). As we mentioned in the introduction, the gradient can only blow up when two inclusions are close to each other. In order to establish the gradient estimates for this problem, we first consider the local version of the problem, namely Equation (1.9).

To make the problem easier, we first consider the equation in a strip. In this case, by using a "flipping" technique, we derive the gradient estimates in the strip.

Denote, for any integer l

$$\begin{aligned}\mathcal{Q}_l&:=\{z\in\mathbb{R}^n\big|(2l-1)\delta<z_1<(2l+1)\delta,|z'|\le 1\},\\ \Gamma_l^+&:=\{z\in\mathbb{R}^n\big|z_1=(2l+1)\delta\text{ and }|z'|\le 1\},\\ \Gamma_l^-&:=\{z\in\mathbb{R}^n\big|z_1=(2l-1)\delta\text{ and }|z'|\le 1\},\end{aligned}$$

and

$$\mathcal{Q}=\{z\in\mathbb{R}^n\big||z_1|\le 1\text{ and }|z'|\le 1\}.$$

We consider the following equation in $\mathcal{Q}_0$

$$\begin{cases}\partial_{z_i}\Big(b^{ij}(z)\ \partial_{z_j}w\Big)=0 & \text{in } \mathcal{Q}_0,\\ b^{1j}\partial_{z_j}w=0 & \text{on } \Gamma_0^\pm.\end{cases} \tag{3.1}$$

where $(b^{ij}) \in C^\alpha(\overline{\mathcal{Q}}_0)(0 < \alpha < 1)$ is a symmetric matrix function in $\mathcal{Q}_0$, and there exist constants $\Lambda_2 \geq \lambda_2 > 0$ such that, for all $\xi \in \mathbb{R}^n$,

$$\|b^{ij}(z)\|_{C^\alpha(\overline{\mathcal{Q}}_0)} \leq \Lambda_2, \qquad \lambda_2|\xi|^2 \leq b^{ij}(z)\xi_i\xi_j, \quad \forall z \in \mathcal{Q}_0, \xi \in \mathbb{R}^n.$$

Then we have

Lemma 3.1 *Suppose $w \in H^1(\mathcal{Q}_0) \cap L^\infty(\mathcal{Q}_0)$ is a weak solution of (3.1), then there exists a constant $C > 0$ depending only on $n,\ \lambda_2, \Lambda_2$, but independent of δ, such that*

$$\|\nabla w\|_{L^\infty(\mathcal{Q}_0(\frac{1}{2}))} \leq C\|w\|_{L^\infty(\mathcal{Q}_0)},$$

where $\mathcal{Q}_0(\frac{1}{2}) := \{z \in \mathbb{R}^n \big| |z_1| \leq \delta$ and $|z'| \leq \frac{1}{2}\}$.

Proof: For any integer l, We construct a new function $\widetilde{w}$ by "flipping" w evenly in each $\mathcal{Q}_l$. We define

$$\widetilde{w}(z) = w((-1)^l(z_1 - 2l\delta), z'), \quad \forall z \in \mathcal{Q}_l.$$

Therefore, we have defined $\widetilde{w}$ piecewisely in $\mathcal{Q}$.
We define the corresponding elliptic coefficients as follows
for $\alpha = 2, 3, \ldots, n$,

$$\widetilde{b}^{\alpha 1}(z) = \widetilde{b}^{1\alpha}(z) = (-1)^l b^{1\alpha}((-1)^l(z_1 - 2l\delta), z'), \quad \forall z \in \mathcal{Q}_l.$$

for all other indices

$$\widetilde{b}^{ij}(z) = b^{ij}((-1)^l(z_1 - 2l\delta), z'), \quad \forall z \in \mathcal{Q}_l.$$

Under the above definitions of $\widetilde{w}$ and $\widetilde{b}^{ij}$, we can easily check that, for any integer l,

$$\begin{cases} \partial_{z_i}\Big(\widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\Big) = 0 & \text{in } \mathcal{Q}_l, \\ \widetilde{b}^{1j}\partial_{z_j}\widetilde{w} = 0 & \text{on } \Gamma_l^{\pm}, \end{cases}$$

Then for any test function $\psi \in C_0^\infty(\mathcal{Q})$, we have

$$\begin{aligned} \int_{\mathcal{Q}} \widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\partial_{z_i}\psi &= \sum_l \int_{\mathcal{Q}_l} \widetilde{b}^{ij}(z)\ \partial_{z_j}\widetilde{w}\partial_{z_i}\psi \\ &= 0 \quad \big(\text{by the definition of weak solution}\big) \end{aligned}$$

Therefore $\widetilde{w} \in H^1(\mathcal{Q})$ satisfies

$$\partial_{z_j}\big(\widetilde{b}^{ij}(z)\ \partial_{z_i}\widetilde{w}\big) = 0 \quad \text{in } \mathcal{Q}.$$

Following exactly from [13], we first introduce a new equation

$$\partial_{z_i}\big(\widetilde{B}^{ij}(z)\ \partial_{z_j}u\big) = 0 \quad \text{in } \mathcal{Q}$$

where

$$\widetilde{B}^{ij}(z) = \begin{cases} \lim_{z\in\mathcal{Q}_l,\ z\to((2l-1)\delta,\ 0')} \widetilde{b}^{ij}(z) & z \in \mathcal{Q}_l, l > 0; \\ \widetilde{b}^{ij}(0) & z \in \mathcal{Q}_0 \\ \lim_{z\in\mathcal{Q}_l,\ z\to((2l+1)\delta,\ 0')} \widetilde{b}^{ij}(z) & z \in \mathcal{Q}_l, l < 0; \end{cases}$$

then we define the norm

$$\|F\|_{Y^{s,p}} = \sup_{0<r<1} r^{1-s}\Big(⨍_{r\mathcal{Q}}|F|^p\Big)^{\frac{1}{p}}$$

Since $b^{ij}(z) \in C^\alpha(\overline{\mathcal{Q}}_0)$, $\widetilde{b}^{ij}(z)$ is piecewise C^α continuous in $\mathcal{Q}$, then we can immediately check that

$$\|\widetilde{b}^{ij} - \widetilde{B}^{ij}\|_{Y^{1+\alpha,2}} < C$$

where C is some constant only depending on Λ_2. Using Proposition 4.1 in [13], we have

$$\|\nabla\widetilde{w}\|_{L^\infty(\frac{1}{2}\mathcal{Q})} \leq C\|\widetilde{w}\|_{L^2(\mathcal{Q})} \leq C\|\widetilde{w}\|_{L^\infty(\mathcal{Q})},$$

Then by the definition of $\widetilde{w}$, we have

$$\|\nabla w\|_{L^\infty(\mathcal{Q}_0(\frac{1}{2}))} \leq C\|w\|_{L^\infty(\mathcal{Q}_0)}$$

where $C > 0$ depends on n, λ_2, Λ_2, but is independent of δ. □

Since D_1 and D_2 are strictly convex domains, we can write $\mathcal{O}(r)$, which is defined by (1.8), as follows

$$\mathcal{O}(r) = \{x \in \mathbb{R}^n \big| -g(x') - \varepsilon/2 < x_1 < f(x') + \varepsilon/2,\ |x'| < r\}$$

With the side boundary Γ_+ and Γ_- as

$$\Gamma_+ = \{x \in \mathbb{R}^n \big| x_1 = f(x') + \varepsilon/2, |x'| < r\},\ \Gamma_- = \{x \in \mathbb{R}^n \big| x_1 = -g(x') - \varepsilon/2, |x'| < r\}$$

where $f(x')$ and $g(x')$ are strictly convex functions, moreover they satisfy

$$f(0') = g(0') = 0,\ \nabla f(0') = \nabla g(0') = 0.$$

Under the above notation, we prove Theorem 1.2:

Proof of Theorem 1.2: Fix one point $(0, x_0') \in \mathcal{O}(\frac{r}{2})$ and let $\delta = \sqrt{f(x_0') + g(x_0') + \varepsilon}$, since $f(x')$ and $g(x')$ are strictly convex, then there exists a universal constant C depending only on $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$ such that

$$\frac{1}{C}\sqrt{|x_0'|^2 + \varepsilon} < \delta < C\sqrt{|x_0'|^2 + \varepsilon}. \tag{3.2}$$

We shift the origin to $(0, x_0')$ and rescale the coordinates with δ, then the new coordinates $y = (y_1, y')$ can be written as follows

$$\begin{cases} y_1 = x_1/\delta, \\ y' = (x' - x_0')/\delta. \end{cases} \tag{3.3}$$

Let

$$v(y) = u_0(\delta y_1, x_0' + \delta y'), \qquad \widetilde{a}^{ij}(y) = a^{ij}(\delta y_1, x_0' + \delta y').$$

Denote

$$\widetilde{\mathcal{O}}(\widetilde{r}) := \{y \in \mathbb{R}^n \big| -\frac{\varepsilon}{2} - g(x_0' + \delta y') < \delta y_1 < \frac{\varepsilon}{2} + f(x_0' + \delta y'),\ |y'| < \widetilde{r}\}$$

With its side boundary

$$\widetilde{\Gamma}_+ := \{y \in \mathbb{R}^n \big| \delta y_1 = \frac{\varepsilon}{2} + f(y_0' + \delta y'),\ |y'| < \widetilde{r}\}$$

$$\widetilde{\Gamma}_- := \{y \in \mathbb{R}^n \big| \delta y_1 = -\frac{\varepsilon}{2} - g(y_0' + \delta y'),\ |y'| < \widetilde{r}\}.$$

By (3.2), we can find some universal constant $\widetilde{r}$ depending only on ∂D_1 and ∂D_2, such that $\widetilde{\mathcal{O}}(\widetilde{r})$ is in the image of $\mathcal{O}(r)$ under the above transform. Thus we have

$$\begin{cases} \partial_{y_i}(\widetilde{a}^{ij}\partial_{y_j}v(y)) = 0 & \text{in} \quad \widetilde{\mathcal{O}}(\widetilde{r}), \\ \widetilde{a}^{ij}\partial_{y_j}v\nu_i = 0 & \text{on} \quad \widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_-. \end{cases} \tag{3.4}$$

where the coefficients $\widetilde{a}^{ij}$ satisfy, for some universal constant C,

$$\|\widetilde{a}^{ij}\|_{C^\alpha(\widetilde{\mathcal{O}}(\widetilde{r}))} \leq C\|a^{ij}\|_{C^\alpha(\mathcal{O}(r))} \leq C\Lambda_1,\ \lambda_1|\xi|^2 \leq \widetilde{a}^{ij}(y)\xi_i\xi_j\ (\forall y \in \widetilde{\mathcal{O}}(\widetilde{r}),\ \forall \xi \in \mathbb{R}^n).$$

Next we construct a map $\Phi : \widetilde{\mathcal{O}}(\widetilde{r}) \longmapsto \mathcal{Q}_0$, $\Phi(y) = z$ with

$$\begin{cases} z_1 = 2\delta \dfrac{\delta y_1 + g(x_0' + \delta y') + \varepsilon/2}{f(x_0' + \delta y') + g(x_0' + \delta y') + \varepsilon} - \delta, \\ z' = \dfrac{y'}{\widetilde{r}}. \end{cases} \tag{3.5}$$

It can be verified directly that this map is a diffeomorphism from $\widetilde{\mathcal{O}}(\widetilde{r})$ to $\mathcal{Q}_0$.

Let

$$w(z) = v(\Phi^{-1}(z))$$

Then from the definition of weak solution, we know that $w(z)$ satisfies the following equation

$$\begin{cases} \partial_{z_i}\big(b^{ij}(z)\partial_{z_j} w(z)\big) = 0 & \text{in } \mathcal{Q}_0, \\ b^{1j}(z)\partial_{z_j} w(z) = 0 & \text{on } \Gamma_0^+ \cup \Gamma_0^-. \end{cases} \tag{3.6}$$

where

$$\big(b^{ij}(z)\big) = \frac{(\partial_y z)\big(\widetilde{a}^{ij}(y)\big)(\partial_y z)^t}{|\det(\partial_y z)|}$$

Therefore, we have transferred the original problem into Equation (3.1).

In order to use Lemma 3.1, we have to check that $b^{ij}(z)$ is strictly elliptic and $\|b^{ij}\|_{C^\alpha(\overline{\mathcal{Q}}_0)}$ is bounded by some universal constant. First we show that there exists a universal constant λ_2 such that

$$\xi^t\big(b^{ij}(z)\big)\xi \geq \lambda_2|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \forall z \in \mathcal{Q}_0 \tag{3.7}$$

Notice that the eigenvalues of $(\partial_y z)$ are $\frac{1}{\widetilde{r}}$ with multiplicity $n-1$ and $\partial_{y_1} z_1$. By (3.2), we can prove that

$$\frac{1}{C} < |\partial_{y_1} z_1| = \partial_{y_1} z_1 = \frac{2\delta^2}{f(x_0' + \delta y') + g(x_0' + \delta y') + \varepsilon} < C \tag{3.8}$$

where C is some universal constant.

Based on (3.8),we have

$$\xi^t\big(b^{ij}(z)\big)\xi = \xi^t(\partial_y z)\frac{\big(\widetilde{a}^{ij}(y)\big)}{|\det(\partial_y z)|}(\partial_y z)^t\xi > \lambda_2|\xi|^2, \quad \forall \xi \in \mathbb{R}^n$$

where $\lambda_2 > 0$ is some universal constant

The boundedness of $\|b^{ij}\|_{C^\alpha(\overline{\mathcal{Q}}_0)}$ can be checked similarly.

Now applying Lemma 3.1, we have

$$\|\nabla w\|_{L^\infty(\mathcal{Q}_0(\frac{1}{2}))} \leq C\|w\|_{L^\infty(\mathcal{Q}_0)}$$

Tracing back to u_0 through the transforms, we have, for any point $x \in \mathcal{O}(\frac{r}{2})$,

$$|\nabla u_0(x)| \leq \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\delta} \leq \frac{C\|u_0\|_{L^\infty(\mathcal{O}(r))}}{\sqrt{|x'|^2 + \varepsilon}}.$$

□

4 Appendix

Some elementary results for the insulated conductivity problem

Assume that in $\mathbb{R}^n$, Ω and ω are bounded open sets with $C^{2,\alpha}$ boundaries, $0 < \alpha < 1$, satisfying, for some $m < \infty$,

$$\overline{\omega} = \bigcup_{s=1}^{m} \overline{\omega}_s \subset \Omega,$$

where $\{\omega_s\}$ are connected components of ω. Clearly ω_s is open for all $1 \le s \le m$. Given $\varphi \in C^2(\partial\Omega)$, the conductivity problem we consider is the following transmission problem with Dirichlet boundary condition:

$$\begin{cases} \partial_{x_j}\left\{\left[\left(ka_1^{ij}(x) - a_2^{ij}(x)\right)\chi_\omega + a_2^{ij}(x)\right]\partial_{x_i}u_k\right\} = 0 & in\ \Omega, \\ u_k = \varphi & on\ \partial\Omega, \end{cases} \tag{4.1}$$

where $0 < k < 1$, and χ_ω is the characteristic function of ω.

The $n\times n$ matrixes $A_1(x) := \left(a_1^{ij}(x)\right)$ in ω, $A_2(x) := \left(a_2^{ij}(x)\right)$ in $\Omega\backslash\overline{\omega}$ are symmetric and $\exists$ a constant $\Lambda \ge \lambda > 0$ such that

$$\lambda|\xi|^2 \le a_1^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \ (\forall x \in \omega), \qquad \lambda|\xi|^2 \le a_2^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \ (\forall x \in \Omega\backslash\omega)$$

for all $\xi \in \mathbb{R}^n$ and $a_1^{ij}(x) \in C^2(\overline{\omega})$, $a_2^{ij}(x) \in C^2(\overline{\Omega}\backslash\omega)$.

Equation (4.1) can be rewritten in the following form to emphasize the transmission condition on $\partial\omega$:

$$\begin{cases} \partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_k\right) = 0 & in\ \omega, \\ \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_k\right) = 0 & in\ \Omega\backslash\overline{\omega}, \\ u_k|_+ = u_k|_-, & on\ \partial\omega, \\ a_2^{ij}(x)\partial_{x_i}u_k\nu_j\big|_+ = ka_1^{ij}(x)\partial_{x_i}u_k\nu_j\big|_- & on\ \partial\omega, \\ u_k = \varphi & on\ \partial\Omega. \end{cases} \tag{4.2}$$

It is well known that equation (4.1) has a unique solution u_k in $H^1(\Omega)$, and the solution u_k is in $C^1(\overline{\Omega\backslash\omega})\cap C^1(\overline{\omega})$ and satisfies equation (4.2). On the other hand, if $u_k \in C^1(\overline{\Omega\backslash\omega})\cap C^1(\overline{\omega})$ is a solution of equation (4.2), then $u_k \in H^1(\Omega)$ satisfies equation (4.1).

For $k \in (0,1)$, consider the energy functional

$$I_k[v] := \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}v\partial_{x_j}v + \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{4.3}$$

defined on

$$H^1_\varphi(\Omega) := \{v \in H^1(\Omega)|\ v = \varphi \ \ on\ \partial\Omega\}.$$

It is well known that for $k \in (0,1)$, the solution u_k of (4.1) is the minimizer of the minimization problem:

$$I_k[u_k] = \min_{v\in H^1_\varphi(\Omega)} I_k[v].$$

For $k = 0$, the insulated conducting problem is:

$$\begin{cases} \partial_{x_j}\left(a_2^{ij}(x)\ \partial_{x_i}u_0\right) = 0 & in\ \Omega\backslash\overline{\omega}, \\ a_2^{ij}(x)\partial_{x_i}u_0\nu_j\big|_+ = 0 & on\ \partial\omega, \\ u_0 = \varphi & on\ \partial\Omega, \\ \partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_0\right) = 0 & in\ \omega, \\ u_0|_+ = u_0|_-, & on\ \partial\omega. \end{cases} \tag{4.4}$$

Equation (4.4) has a unique solution $u_0 \in H^1(\Omega)$, which can be solved in $\Omega\setminus\overline{\omega}$ by the first three lines in (4.4), and then, with $u_0|_{\partial\omega}$, be solved in ω using the fourth line in (4.4). It is well known that $u_0 \in C^1(\overline{\Omega}\setminus\omega)\cap C^1(\overline{\omega})$.

Define the energy functional

$$I_0[v] := \frac{1}{2}\int_{\Omega\backslash\overline{\omega}} a_2^{ij}(x)\partial_{x_i}v\partial_{x_j}v, \tag{4.5}$$

where v belongs to the set

$$\mathcal{A}_0 := \{v \in H^1(\Omega \setminus \overline{\omega}) \big| \ v = \varphi \text{ on } \partial\Omega\}.$$

It is well known that there is a unique $v_0 \in \mathcal{A}_0$ which is the minimizer to the minimization problem:

$$I_0[v_0] = \min_{v \in \mathcal{A}_0} I_0[v].$$

Moreover, $v_0 = u_0$ a.e. in $\Omega \setminus \overline{\omega}$, where u_0 is the solution of (4.4).

Now, we give the relationship between u_k and u_0.

Theorem 4.1 *For $0 < k < 1$, let u_k and u_0 in $H^1(\Omega)$ be the solutions of equations (4.2) and (4.4), respectively. Then*

$$u_k \rightharpoonup u_0 \quad \text{in } H^1(\Omega), \quad \text{as } k \to 0, \tag{4.6}$$

and, consequently,

$$\lim_{k \to 0} I_k[u_k] = I_0[u_0]. \tag{4.7}$$

Proof: We will first show that

$$\sup_{0<k<1} \|\nabla u_k\|_{L^2(\Omega)} < \infty. \tag{4.8}$$

Since u_k is the minimizer of I_k in $H^1_\varphi(\Omega)$ and $v_0 := u_0|_{\Omega\setminus\overline{\omega}}$ is the minimizer of I_0 in $\mathcal{A}_0$, we have

$$\begin{aligned}
\frac{\lambda k}{2}\|\nabla u_k\|_{L^2(\omega)} + I_0[v_0] &\le \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k + I_0[v_0] \\
&\le \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_k\partial_{x_j}u_k + I_0[u_k|_{\Omega\setminus\overline{\omega}}] = I_k[u_k] \\
&\le I_k[u_0] = \frac{k}{2}\int_\omega a_1^{ij}(x)\partial_{x_i}u_0\partial_{x_j}u_0 + I_0[v_0], \\
&\le \frac{\Lambda k}{2}\|\nabla u_0\|_{L^2(\omega)} + I_0[v_0].
\end{aligned}$$

Thus

$$\sup_{0<k<1} \|\nabla u_k\|_{L^2(\omega)} < \infty.$$

On the other hand,

$$\frac{\lambda}{2}\|\nabla u_k\|_{L^2(\Omega\setminus\overline{\omega})} \le I_k[u_k] \le I_k[u_0] \le \frac{\Lambda}{2}\|\nabla u_0\|_{L^2(\Omega)}.$$

Estimate (4.8) follows from the above.

Since $u_k = \varphi$ on $\partial\Omega$, we derive from (4.8) that $\sup_{0<k<1}\|u_k\|_{H^1(\Omega)} < \infty$. Let $u_k \rightharpoonup u_0^*$ *in* $H^1_\varphi(\Omega)$ along a subsequence of $k \to 0$ (still denoted as $k \to 0$).

We will show that u_0^* is a solution of equation (4.4). Therefore, $u_0^* = u_0$.

We only need to establish the following three properties:

$$\partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}u_0^*\Big) = 0 \qquad \text{in } \Omega\setminus\overline{\omega}, \tag{4.9}$$

$$\partial_{x_j}\Big(a_1^{ij}(x)\ \partial_{x_i}u_0^*\Big) = 0 \qquad \text{in } \omega, \tag{4.10}$$

$$u_0^* \in C^1(\Omega\setminus\omega), \qquad a_2^{ij}(x)\partial_{x_i}u_0^*\nu_j\big|_+ = 0 \qquad \text{on } \partial\omega. \tag{4.11}$$

(i) For $k \in (0,1)$, we see from equation (4.1) that

$$\partial_{x_j}\Big(a_2^{ij}(x)\ \partial_{x_i}u_k\Big) = 0, \qquad \text{in } \Omega\setminus\overline{\omega},$$

$$\partial_{x_j}\left(a_1^{ij}(x)\ \partial_{x_i}u_k\right) = 0, \qquad \text{in } \omega.$$

Since u_k converges to u_0^* weakly in $H^1(\Omega)$, (4.9) and (4.10) follow from the above.

(ii) For any $w \in \mathcal{A}_0$, we extend it to $\tilde{w} \in H^1_\varphi(\Omega)$ (i.e. $\tilde{w} = w$ in $\Omega \setminus \overline{\omega}$). By the minimality of u_k,

$$I_k(u_k) \le I_k(\tilde{w}).$$

Sending k to 0 leads to

$$I_0(u_0^*|_{\Omega\setminus\omega}) \le I_0(w).$$

Thus $u_0^* = u_0$ a.e. in $\Omega \setminus \omega$. (4.11) follows.

We have proved (4.6). Theorem 4.1 is established.

References

[1] H. Ammari, H. Kang, H. Lee, J. Lee and M. Lim, *Optimal estimates for the Electrical Field in Two Dimensions*, J. Math. Pures Appl. 88 (2007), 307-324.

[2] H. Ammari, H. Kang, H. Lee, M. Lim and H. Zribi, *Decomposition Theorems and Fine Estimates for Electrical Fields in the Presence of Closely Located*, preprint.

[3] H. Ammari, H. Kang and M. Lim, *Gradient estimates for solutions to the conductivity problem*, Math. Ann. 332 (2005), 277-286.

[4] I. Babuska, B. Anderson, P.J. Smith, and K. Levin, *Damage analysis of fiber composites. I. Statistical analysis on fiber scale*, Comput. Methods Appl. Mech. Engrg. 172 (1999), 27-77.

[5] E.S. Bao, Y.Y. Li and B. Yin, *Gradient estimates for the perfect conductivity problem*, Arch. Rational Mech. and Anal. 193 (2009), 195-226.

[6] E. Bonnetier and M. Vogelius, *An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section*, SIAM J. Math. Anal. 31 (2000), 651-677.

[7] B. Budiansky and G.F. Carrier, *High shear stresses in stiff fiber composites*, J. App. Mech. 51 (1984), 733-735.

[8] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.

[9] D.Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[10] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[11] J.B. Keller, *Stresses in narrow regions*, Trans. ASME J. Appl. Mech. 60 (1993), 1054-1056.

[12] D. Kapanadze and B.W. Schulze, *Boundary-contact problems for domains with edge singularities*, J. Differential Equations 234 (2007), 26-53.

[13] Y.Y. Li and L. Nirenberg, *Estimates for elliptic system from composite material*, Comm. Pure Appl. Math. 56 (2003), 892-925.

[14] Y.Y. Li and M. Vogelius, *Gradient estimates for solution to divergence form elliptic equation with discontinuous coefficients*, Arch. Rational Mech. Anal. 153 (2000), 91-151.

[15] M. Lim and K. Yun, *Blow-up of electric fields between closely spaced spherical perfect conductors*, Communications in PDE, to appear.

[16] J. Mateu, J. Orobitg and J. Verdera, *Extra cancellation of even Calderon-Zygmund operators and quasiconformal mappings*, J. Math. Pures Appl. 91 (2009), 402-431.

[17] V.V. Voyevodin, *Linear Algebra*, Mir Publishers, Moscow, 1983.

[18] X. Markenscoff, *Stress amplification in vanishingly small geometries*, Computational Mechanics 19 (1996), 77-83.

[19] K. Yun, *Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape*, SIAM J. on Applied Math. 67 (2007), 714-730.

[20] K. Yun, *Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections*, J. Math. Anal. Appl. 350 (2009), 306–312.