Some remarks on singular solutions of nonlinear elliptic equations. II: symmetry and monotonicity via moving planes

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1 Introduction

Our earlier work [2] contained a result on monotonicity and symmetry of solutions of

$$u > 0, \quad F(x, u, \nabla u, \nabla^2 u) = 0 \tag{1}$$

in $B_1 \setminus \{0\}$, where B_1 is the unit ball in $\mathbb{R}^n$, with

$$u = 0 \quad \text{on} \ \ \partial B_1. \tag{2}$$

Under standard conditions on F, so that the Method of Moving Planes applies, it was shown (Theorem 1.4) that

$$\frac{\partial u}{\partial x_1}(x) < 0 \qquad \text{if } 0 < x_1 < 1,$$

and that u is symmetric in the plane $\{x_1 = 0\}$. The proof assumed the additional condition that $\forall\ 0 < r < \bar{r}$,

$$\inf_{B_r \setminus \{0\}} (u + \text{any linear function}) \text{ occurs on } \partial B_r. \tag{3}$$

An example was also obtained in [2], for $n \geq 2$, showing that condition (3) cannot be dropped.

In November 2008 the last author raised the question whether radial symmetry and monotonicity might hold in the following simple problem:

$$\begin{cases} u > 0, \quad \Delta u + f(u) & = 0 \ \text{ in } B_1 \setminus \{0\}, \\ \qquad\qquad\qquad\quad u & = 0 \qquad \text{on } \partial B_1 \end{cases}$$

in case f is locally Lipschitz. Susanna Terracini then showed a proof; it used an idea in [3] which treats equations with singular coefficients in all of space. Her proof uses a nice variant of the argument in [1] and works also for $f = f(x, u)$ with f nonincreasing in x_1.

The proof of Theorem 1.4 in [2] also worked in case u is defined in a domain Ω in $\mathbb{R}^n$ satisfying

$$\begin{cases} \Omega \text{ is convex in the } \ x_1 - \text{direction, and} \\ \text{if } (x_1, x_2, \cdots, x_n) \in \Omega, x_1 > 0, \text{then } (-x_1, x_2, \cdots, x_n) \in \Omega. \end{cases} \tag{4}$$

In this paper we prove a result on monotonicity and symmetry in x_1 using the Method of Moving Planes for the problem (1), (2) in a domain $\Omega \setminus \{0\}$ in $\mathbb{R}^n$ satisfying (4). u is assumed to be in $C^2(\overline{\Omega} \setminus \{0\})$ but is not assumed to satisfy condition (3), but we require a condition on F.

We first recall the standard conditions on $F(x,u,p,M)$, $p\in\mathbb{R}^n, M\in\mathcal{S}^{n\times n}$, the space of symmetric $n\times n$ matrices.

Notation: $\widehat{M}=\{\widehat{M}_{ij}\}$ with

$$\widehat{M}_{1j}=-M_{1j} \text{ for } 2\le j\le n; \quad \widehat{M}_{i1}=-M_{i1} \text{ for } 2\le i\le n;$$

$$\widehat{M}_{11}=M_{11};\ \widehat{M}_{ij}=M_{ij} \text{ for } i,j>1.$$

$\hat{p}=\{\hat{p}_i\}$ with

$$\hat{p}_1=-p_1; \quad \hat{p}_i=p_i, i>1.$$

Conditions on F. We consider F of a special form:

$$F(x,s,p,M)=G(x,s,p,M)+f(x,u) \tag{5}$$

with $G\in C^1((\overline{\Omega}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^{n\times n})$.

F is elliptic:

$$(\frac{\partial F}{\partial M_{ij}})>0 \quad \forall\ (x,s,p,M)\in(\overline{\Omega}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^{n\times n}. \tag{6}$$

$$\begin{cases} F(\hat{x}_1,x_2,\cdots,x_n,s,\hat{p},\widehat{M})\ge F(x_1,x_2,\cdots,x_n,s,p,M), \\ \qquad \forall\ x\in\Omega, x_1>0, \hat{x}_1\le x_1, x_1+\hat{x}_1>0, \end{cases} \tag{7}$$

$$f(x,u) \text{ is locally Lipschitz in } u \text{ on every compact subset of } \overline{\Omega}\setminus\{0\}. \tag{8}$$

Theorem 1.1 *In Ω satisfying (4), consider F as above and $u\in C^2(\overline{\Omega}\setminus\{0\})$ satisfies (1) in $\Omega\setminus\{0\}$. Assume in addition that*

$$G \textit{ is convex in } (s,p,M). \tag{9}$$

$$\begin{cases} \textit{In case } n=2 \textit{ assume also that } u\in C^{2,\alpha}_{loc}(\overline{\Omega}\setminus\{0\}) \textit{ for some } \alpha>0 \\ \textit{and that } \frac{\partial F}{\partial M_{ij}} \textit{ are locally Hölder continuous in } (\overline{\Omega}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^{n\times n}. \end{cases} \tag{10}$$

Then

$$\frac{\partial u}{\partial x_1}<0 \textit{ if } x_1>0.$$

An immediate consequence is:

Corollary 1.1 *If, in addition, Ω is symmetric about the plane $\{x_1 = 0\}$ and F is symmetric under reflection in the plane, then u is symmetric in x_1.*

Another consequence, as in [1], is radial symmetry. Here $\Omega = B_R(0)$, the ball of radius R centered at 0.

Corollary 1.2 *Let $F(|x|, s, p, M)$ satisfy (5)-(10) and*

$$F_r(r, s, p, M) \leq 0 \text{ for } 0 < r < R, s > 0, p \in \mathbb{R}^n, M \in \mathcal{S}^{n\times n},$$

and

$$F(r, s, Op, O^tMO) \equiv F(r, s, p, M) \text{ for } 0 < r < 1, s > 0, p \in \mathbb{R}^n, M \in \mathcal{S}^{n\times n}, O \in O(n).$$

Suppose $u \in C^2(\overline{B}_R \setminus \{0\})$ and

$$\begin{aligned} u > 0, \quad F(x, u, \nabla u, \nabla^2 u) &= 0 \text{ in } B_R \setminus \{0\}, \\ u &= 0, \text{ on } \partial B_R. \end{aligned}$$

Then

$$u \text{ is radially symmetric about the origin and } u_r < 0 \text{ for } 0 < r < R.$$

Some results for linear elliptic operators will be used in the proof. They are presented in section 2. The proof of Theorem 1.1 is given in section 3.

2 On maximum principles in punctured domains

In this section we give an extension of the following familiar results.

Lemma 2.1 *Let u be C^2 in $\Omega \setminus \{0\}$, Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, satisfying*

$$u \text{ is bounded below, and } \Delta u \leq 0 \text{ in } \Omega \setminus \{0\},$$

$$u \geq 0 \text{ on } \partial\Omega.$$

Then

$$u \geq 0 \text{ in } \Omega \setminus \{0\}.$$

The reason is that the origin has zero Newtonian capacity: the operator Δ does not feel it.

What happens if Δ is replaced by a uniformly elliptic operator

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x) \quad \text{in } \Omega$$

with the $(a_{ij}(x))$ continuous in Ω and uniformly elliptic: For some constant $C > 0$,

$$\frac{|\xi|^2}{C} \leq a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2, \qquad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n,$$

and

$$|b_i(x)|, |c(x)| < C \text{ in } \Omega\ ?$$

Suppose

$$\begin{cases} \Omega \text{ is contained in a small ball } B_R \text{ and that } u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\}), \\ u \text{ is bounded below and } Lu \leq 0 \text{ in } \Omega \setminus \{0\}, \end{cases}$$

and

$$u \geq 0 \text{ on } \partial\Omega.$$

Is it true, for sufficiently small R depending on C, that

$$u \geq 0 \text{ in } \Omega \setminus \{0\}?$$

Proposition 2.1 *In case $n > 2$, the answer is yes. This is not so if $n = 2$, but is true if*

$$\textit{the modulus of continuity of the } \{a_{ij}\} \textit{ at the origin} \leq \frac{\textit{small constant}}{|\log|x||}.$$

In particular, for $n = 2$, if $a_{ij}(0) = \delta_{ij}$, then the answer is yes provided

$$|a_{ij}(x) - a_{ij}(0)| \leq \frac{\alpha}{|\log|x||},\ \ \alpha < \frac{1}{6}. \tag{11}$$

We first present a counter example in case $n = 2$. In the ball $B := B_R$ in $\mathbb{R}^2$, we take

$$u = \frac{1}{\log R} - \frac{1}{\log r}.$$

Then $u < 0$ in $B \setminus \{0\}$, $u = 0$ on ∂B. We have

$$u_i = \frac{1}{\log^2 r} \frac{x_i}{r^2},$$

$$u_{ij} = \frac{1}{\log^2 r} \left(\frac{\delta_{ij}}{r^2} - \frac{2x_i x_j}{r^4} \right) - \frac{2}{\log^3 r} \frac{x_i x_j}{r^4}.$$

We claim that, for R small,

$$Lu := \Delta u - \frac{3x_i x_j}{r^2 \log r} u_{ij} \leq 0, \quad \text{in } B \setminus \{0\},$$

and so we have a counter example. We compute:

$$\begin{aligned} Lu &= -\frac{2}{r^2 \log^3 r} - \frac{3x_i x_j}{r^2 \log r} \cdot \frac{1}{r^2 \log^2 r} \left(\delta_{ij} - \frac{2x_i x_j}{r^2} - \frac{2}{\log r} \frac{x_i x_j}{r^2} \right) \\ &= \frac{1}{r^2 \log^3 r} \left(-2 + 3 + \frac{6}{\log r} \right) \\ &< 0 \qquad \text{for } R \text{ small.} \end{aligned}$$

The claim is proved.

In the computation above, 3 maybe replaced by any positive constant > 2 (for R small).

Proof of Proposition 2.1. It suffices to find a function $h \geq 0$ in $0 < |x| < R$, which tends to ∞ as $|x| \to 0$ and such that

$$Lh \leq 0 \quad \text{in } B_R \setminus \{0\}.$$

Namely, if we have such a function h then, since u is bounded from below, we have for any $\epsilon > 0$

$$L(u + \epsilon h) \leq 0 \quad \text{in } 0 < |x| < R$$

and $u + \epsilon h \geq 0$ on $\partial\Omega$, and tends to $+\infty$ as $x \to 0$. Recall that Ω lies in $|x| < R$. Then, since the operator L is uniformly elliptic, with coefficients bounded, for R small, the minimum principle holds and we may infer that

$$u + \epsilon h \geq 0.$$

Letting $\epsilon \to 0$ we find

$$u \geq 0.$$

For convenience we will suppose that

$$a_{ij}(0) = \delta_{ij}. \tag{12}$$

Take

$$h = (-\log r)^a, \text{ with } 0 < a < 1 \text{ to be chosen.}$$

Compute:

$$h_i = -a(-\log r)^{a-1}\frac{x_i}{r^2},$$

$$h_{ij} = -\frac{a(-\log r)^{a-1}}{r^2}\left(\delta_{ij} - \frac{2x_ix_j}{r^4}\right) + a(a-1)(-\log r)^{a-2}\frac{x_ix_j}{r^4}.$$

Then

$$\begin{aligned} Lh &= -\frac{a(-\log r)^{a-1}}{r^2}\left(\sum_i a_{ii} - 2\frac{a_{ij}x_ix_j}{r^2}\right) + a(a-1)(-\log r)^{a-2}\frac{a_{ij}x_ix_j}{r^4} \\ &\quad -\frac{a(-\log r)^{a-1}}{r^2}b_ix_i + c(-\log r)^a. \end{aligned}$$

Using (12) we find

$$\begin{aligned} Lh &= -\frac{a(-\log r)^{a-1}}{r^2}\left(n - 2 + \sum_i (a_{ii} - 1) - 2(a_{ij} - \delta_{ij})\frac{x_ix_j}{r^2} + O(R)\right) \\ &\quad +a(a-1)(-\log r)^{a-2}\frac{a_{ij}x_ix_j}{r^4}. \end{aligned}$$

In case $n > 2$, for R small, we see that

$$Lh \le 0,$$

the desired result.

Consider now $n = 2$. Then using (11) we find

$$Lh \le \frac{a(-\log r)^{a-1}}{r^2}\left(\frac{6\alpha}{|\log r|} + O(R) + \frac{a-1}{|\log r|} + (a-1)O(\frac{1}{|\log^2 r|})\right) \le 0$$

provided we choose — recall that $\alpha < \frac{1}{6}$ — $0 < a < 1 - 6\alpha$.

3 Proof of Theorem 1.1.

We use the method of moving planes as in [1], and, at some point, we use the idea of S. Terracini.

We may assume that

$$\sup_{x\in\Omega} x_1 = 1.$$

For $0 < \lambda < 1$,

$$x_\lambda := (2\lambda - x_1, x_2, \cdots, x_n)$$

is the reflection of $x = (x_1, \cdots, x_n)$ in the hyperplane $x_1 = \lambda$.

Set

$$\Sigma_\lambda := \{x \mid |x| < 1, \lambda < x_1 < 1\}, \qquad T_\lambda := \{x \mid |x| < 1, x_1 = \lambda\},$$

(Σ_λ may have many components) In Σ_λ, set

$$u_\lambda(x) := u(x_\lambda), \quad \text{and} \quad w_\lambda := u_\lambda - u.$$

We will prove that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \setminus \{0_\lambda\} \qquad \forall\ 0 < \lambda < 1.$$

Step 1. There exists $\frac{1}{2} < \lambda_0 < 1$ such that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \qquad \forall\ \lambda_0 < \lambda < 1.$$

For this step, we just require that F satisfy (7) and need not be of the form (8).

Using (1) and (7), we have in Σ_λ, for all $\frac{3}{4} < \lambda < 1$,

$$\begin{aligned} 0 &= F(x, u(x), \nabla u(x), \nabla^2 u(x)) - F(x_\lambda, u(x_\lambda), \nabla u(x_\lambda), \nabla^2 u(x_\lambda)) \\ &\leq F(x, u(x), \nabla u(x), \nabla^2 u(x)) - F(x, u_\lambda(x), \nabla u_\lambda(x), \nabla^2 u_\lambda(x)). \end{aligned}$$

By (6) and the fact that u is in $C^2(\overline{\Omega} \setminus \{0\})$, there exists some positive constant C_1 independent of λ, and some functions $\{a_{ij}(x)\}$, $\{b_i(x)\}$, $c(x)$ satisfying, with I denoting the $n \times n$ identity matrix,

$$\frac{1}{C_1} I \leq (a_{ij}(x)) \leq C_1 I, \quad |b_i(x)| + |c(x)| \leq C_1, \qquad \text{in } \Sigma_\lambda,$$

such that

$$a_{ij}(x)\partial_{ij} w_\lambda + b_i(x)\partial_i w_\lambda + c(x) w_\lambda \leq 0, \qquad \text{in } \Sigma_\lambda. \tag{13}$$

It is clear that $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$. Using the maximum principle for domains of small measure as in [1], we have $w_\lambda \geq 0$ in Σ_λ if $1-\lambda > 0$ is smaller than some positive constant which depends only on C_1 and n. Step 1 is established.

Define

$$\bar{\lambda} := \inf\{\mu \mid 0 < \mu < 1, w_\lambda \geq 0 \text{ in } \Sigma_\lambda \setminus \{0_\lambda\} \ \forall\ \mu < \lambda < 1\}.$$

Because of Step 1, $0 \leq \bar{\lambda} < 1$.

Step 2. $\bar{\lambda} = 0$.

There are several cases to consider.

Case 1. $\frac{1}{2} < \bar{\lambda} < 1$.

To show that this is impossible, we argue exactly as in [1]. For $\bar{\epsilon} := \frac{1}{4}(\frac{1}{2} + \bar{\lambda})$, $\frac{1}{2} < \frac{1}{2}\bar{\lambda} - \bar{\epsilon} \leq \lambda \leq \bar{\lambda}$, w_λ satisfies a uniformly elliptic inequality (13) in Σ_λ as before, with ellipticity constants independent of λ — though they may depend on $\bar{\epsilon}$ due to the possible singularity of u at $\{0\}$. By continuity, $w_{\bar{\lambda}} \geq 0$ in $\Sigma_{\bar{\lambda}}$. Since $w_{\bar{\lambda}} > 0$ on $\partial\Sigma_{\bar{\lambda}} \cap \partial\Omega$, it follows from the strong maximum principle that $w_{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$. For $0 < \delta$ small, let D_δ be the set of points in $\Sigma_{\bar{\lambda}}$ whose distance to its boundary is $\geq \delta$. Then, in D_δ, $w_{\bar{\lambda}} \geq \alpha$ for some positive constant α.

For $0 < \epsilon < \bar{\epsilon}$ small, $w_{\bar{\lambda}-\epsilon} \geq \alpha/2$ in D_δ. Thus

$$w_{\bar{\lambda}-\epsilon} \geq 0 \qquad \text{on } \partial(\Sigma_{\bar{\lambda}-\epsilon} \setminus D_\delta).$$

For δ and ϵ small we conclude again, by using the maximum principle in domains of small measure as in [1], that

$$w_{\bar{\lambda}-\epsilon} \geq 0 \quad \text{in } (\Sigma_{\bar{\lambda}-\epsilon} \setminus D_\delta) \text{ and hence in } \Sigma_{\bar{\lambda}-\epsilon},$$

contradicting the definition of $\bar{\lambda}$.

Case 2. $\bar{\lambda} = \frac{1}{2}$.

0_λ may lie outside $\overline{\Omega}$. In this case we proceed as in case 1 and see that $\bar{\lambda} = 1/2$ is impossible. So suppose $0_\lambda = (1, 0, \cdots, 0) \in \partial\Omega$.

For $0 < \delta$ small, let D_δ be the set of points in Σ_λ whose distance to its boundary is $\geq \delta$. Then, as before, in D_δ, $w_{\frac{1}{2}} \geq \alpha$, some positive constant. Let A_δ be the set of points in $\overline{\Omega}$ whose distance to $e_1 = (1, 0, \cdots, 0)$ equal to $\delta/2$. Since $u \geq$some positive constant on the set $\{|x| = \delta\}$, we have, making α smaller if necessary,

$$w_{1/2} \geq \alpha \quad \text{on } A_\delta.$$

For $\bar{\epsilon}$ small, and any $0 < \epsilon < \bar{\epsilon}$, using continuity

$$w_{\frac{1}{2}-\epsilon} \geq \alpha/2 \ \text{ in } D_\delta \text{ and on } \ A_\delta.$$

Consequently,

$$w_{\frac{1}{2}-\epsilon} \geq \alpha/2 \quad \text{on } \partial\left(\Sigma_{\frac{1}{2}-\epsilon} \setminus (D_\delta \cup B_{\delta/2}(e_1))\right).$$

For δ, ϵ small, we conclude again, via the maximum principle in domains of small measure, as in [1], that

$$w_{\frac{1}{2}-\epsilon} \geq 0 \quad \text{in } \Sigma_{\frac{1}{2}-\epsilon} \setminus (D_\delta \cup B_{\delta/2}(e_1)$$

and hence in $\Sigma_{\frac{1}{2}-\epsilon} \setminus B_{\delta/2}(e_1)$.

For ϵ small, $0_{\frac{1}{2}-\epsilon}$ lies in $B_{\delta/2}(e_1)$. We have to show that $w_{\frac{1}{2}-\epsilon} \geq 0$ in $B_{\delta/2}(e_1) \cap \Omega$. Then we would have

$$w_{\frac{1}{2}-\epsilon} \geq 0 \quad \text{in } \Sigma_{\frac{1}{2}-\epsilon},$$

contradicting $\bar{\lambda} = \frac{1}{2}$.

We know that $w_{\frac{1}{2}-\epsilon} \geq 0$ on $\partial(\Omega \cap B_{\delta/2}(e_1))$, and, as before, that for $\lambda = \frac{1}{2} - \epsilon$,

$$F(x, u_\lambda(x), \nabla u_\lambda(x), \nabla^2 u_\lambda(x)) \leq F(x, u(x), \nabla u(x), \nabla^2 u(x))$$

in $(\Omega \cap B_{\delta/2}(e_1)) \setminus \{0_\lambda\}$. However we cannot consider w_λ here and use the maximum principle.

Now we use the idea of Terracini. Suppose $w_\lambda < 0$ somewhere in $(\Omega \cap B_{\delta/2}(e_1)) \setminus \{0_\lambda\}$; $\lambda = \frac{1}{2} - \epsilon$. Let G be the set where $w_\lambda < 0$. On its boundary $w_\lambda = 0$. So on this set $u_\lambda < u$ and so is bounded by some constant C. Hence

$$|f(x, u_\lambda) - f(x, u)| \leq \overline{C}|u_\lambda - u| \tag{14}$$

for some $\overline{C}$. We also use (8) and (9). By the convexity condition (9),

$$\begin{aligned} & F(x, u_\lambda(x), \nabla u_\lambda(x), \nabla^2 u_\lambda(x)) - F(x, u(x), \nabla u(x), \nabla^2 u(x)) \\ \geq\ & G_{u_{ij}}(x, u(x), \nabla u(x), \nabla^2 u(x))\partial_{ij} w_\lambda + G_{u_i}(x, u(x), \nabla u(x), \nabla^2 u(x))\partial_i w_\lambda \\ & + G_u(x, u(x), \nabla u(x), \nabla^2 u(x)) w_\lambda + [f(x, u_\lambda) - f(x, u)] \\ =\ & L w_\lambda := a_{ij}(x)\partial_{ij} w_\lambda + b_i(x)\partial_i w_\lambda + c(x) w, \end{aligned}$$

where L is uniformly elliptic with bounded coefficients, and the a_{ij} are continuous; in case $n = 2$, the a_{ij} are Hölder continuous. Here we have used (14) to ensure that $|c|$ is bounded.

We may now apply Proposition 2.1 and infer that $w_\lambda \geq 0$ in G. Thus no such G exists and we conclude that $w_{\frac{1}{2}-\epsilon} \geq 0$ in $(\Omega \cap B_{\delta/2-\epsilon}(e_1)) \setminus \{0_\lambda\}$, as we wished.

We have shown that $\bar{\lambda} = \frac{1}{2}$ is impossible.

Finally we have to treat

Case 3. $0 < \bar{\lambda} < \frac{1}{2}$.

We have to show that this is impossible.

We may argue first as in case 2. In place of $B_{\delta/2}(e_1)$ we use $B_{\delta/2}(0_{\bar{\lambda}})$, and the argument proceeds just as above. We have proved that

$$w_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \setminus \{0_\lambda\}, \ 0 < \lambda < 1.$$

To complete the proof of Theorem 1.1 we have to show

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{if } x_1 > 0.$$

This follows from the Hopf Lemma applied to w_λ in Σ_λ for every $\lambda > 0$. On T_λ, $w_\lambda = 0$. Since w_λ can not be identically zero in $\Sigma_\lambda \setminus \{0_\lambda\}$, we have $w_\lambda > 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$ by the strong maximum principle —- away from $\{0_\lambda\}$, w_λ satisfies a uniformly elliptic inequality. We then apply the Hopf Lemma, and conclude that

$$0 < \frac{\partial}{\partial x_1} w_\lambda = -2u_{x_1} \quad \text{on } T_\lambda.$$

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