Gradient estimates for parabolic systems from composite material

Dedicated to the memory of Professor CHENG MinDe on the occasion of the centenary of his birth

LI HaiGang¹ & LI YanYan^{2,*}

¹*School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China;*

²*Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA*

Email: hgli@bnu.edu.cn, yyli@math.rutgers.edu

Received June 9, 2017; accepted August 15, 2017; published online September 27, 2017

Abstract In this paper, we derive $W^{1,\infty}$ and piecewise $C^{1,\alpha}$ estimates for solutions, and their t -derivatives, of divergence form parabolic systems with coefficients piecewise Hölder continuous in space variables x and smooth in t . This is an extension to parabolic systems of results of Li and Nirenberg [Comm Pure Appl Math, 2003, 56: 892–925] on elliptic systems. These estimates depend on the shape and the size of the surfaces of discontinuity of the coefficients, but are independent of the distance between these surfaces.

Keywords gradient estimates, parabolic system, composite material, linear system of elasticity

MSC(2010) 35K51, 35Q74, 35B65, 74B05

Citation: Li H G, Li Y Y. Gradient estimates for parabolic systems from composite material. Sci China Math, 2017, 60: 2011–2052, doi: 10.1007/s11425-017-9153-0

1 Introduction and main results

The purpose of this paper is to establish gradient estimates for some parabolic systems of divergence form, which arise from the study of composite material. Babuška et al. [2] were interested in elliptic systems arising in elasticity. They observed numerically that, for certain homogeneous isotropic linear systems of elasticity, $|\nabla u|$ stays bounded independently of the distance between the regions. Bonnetier and Vogelius [6] proved the boundedness of $|\nabla u|$ for a scalar elliptic equation in bounded domains with two unit balls touching at a point. This result was extended by Li and Vogelius [22] to general second order elliptic equations with piecewise Hölder coefficients, where stronger $C^{1,\alpha}$ estimates were established. Later, $C^{1,\alpha}$ estimates were obtained by Li and Nirenberg [21] for general second order elliptic systems including systems of elasticity, with an improved Hölder exponent α . A remaining open problem is to determine the optimal regularity (see the open problem on [21, p. 894]). In this paper, we extend the interior $C^{1,\alpha}$ estimates in [21] to parabolic systems. For elliptic problems, some progresses have been made recently for Lamé system with partial infinite coefficients (see [3–5, 20]). For parabolic problems, related results were given by Almgren and Wang [1] and Dong [12].

Let $D \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain that contains L disjoint subdomains $D_1, \dots, D_L$, with $D = (\bigcup \overline{D_m}) \setminus \partial D$. Suppose that their boundaries ∂D and ∂D_m are $C^{1,\alpha}$ for some $0 < \alpha < 1$. Denote $Q_T = D \times (0, T)$, for some $T > 0$.

* Corresponding author

We study interior gradient estimates of solutions of the linear parabolic systems

$$(u^i)_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta u^j) = -D_\alpha g^{\alpha i} + f^i \quad \text{in } Q_T. \quad (1.1)$$

Here, $u(x, t) = (u^1(x, t), \dots, u^N(x, t))$ is a vector-valued function, and we use $D_i u$ for $\partial u / \partial x_i$ while we use u_t (or sometimes $D_t u$) for $\partial u / \partial t$. We also write Du (or sometimes ∇u) for $(D_1 u, \dots, D_N u)$. Throughout this paper, we use the usual summation convention over repeated indices: α and β are summed from 1 to n , while i and j are summed from 1 to N .

The coefficients $A_{ij}^{\alpha\beta}(x, t)$, often also denoted by A ,

(a) are measurable and bounded,

$$|A_{ij}^{\alpha\beta}| \leq \Lambda_0; \quad (1.2)$$

(b) and $A_{ij}^{\alpha\beta}(x, t)$ satisfy the following (weak) parabolic condition: for some constant $\lambda > 0$,

$$\int_D A_{ij}^{\alpha\beta}(x, t) \partial_\alpha \xi^i \partial_\beta \xi^j dx \geq \lambda \int_D |\nabla \xi|^2 dx, \quad \forall \xi \in \mathring{W}_2^{1,0}(Q_T; \mathbb{R}^N), \quad \forall t \in (0, T); \quad (1.3)$$

(c) furthermore, $A_{ij}^{\alpha\beta}(x, t)$ is of class $C^{\mu, \infty}(\overline{D}_m \times (0, T))$, i.e., for some constants $0 < \mu < 1$, and C such that

$$|A(x, t) - A(y, t)| \leq C|x - y|^\mu, \quad \forall (x, t), (y, t) \in D_m \times (0, T), \quad m = 1, 2, \dots, L;$$

and, for every integer $k \geq 1$, there exists Λ_{2k} , depending on k , such that

$$\begin{aligned} \sum_{s=0}^k |D_t^s A(x, t)| &\leq \Lambda_{2k}, \\ \sum_{s=0}^k |D_t^s A(x, t) - D_t^s A(y, t)| &\leq \Lambda_{2k}|x - y|^\mu, \end{aligned} \quad \text{in } D_m \times (0, T), \quad m = 1, \dots, L. \quad (1.4)$$

Finally, we assume that $f \in L^\infty(Q_T)$, and $g \in C^{\mu, 0}(\overline{D}_m \times [0, T])$. Here, $C^{\mu, k}(\overline{D}_m \times [0, T])$ denotes the Banach space of functions $g(x, t)$ that are C^k continuous in t and Hölder continuous in x with exponent $\mu \in [0, 1]$, and have finite norms

$$\|g\|_{C^{\mu, k}} = \sum_{s=0}^k \sup_{\overline{D}_m \times [0, T]} |D_t^s g| + \sup_{\substack{x, y \in D_m \\ t \in [0, T]}} \frac{|g(x, t) - g(y, t)|}{|x - y|^\mu}.$$

Throughout the paper, unless otherwise stated, we will always assume that these hypotheses hold.

We define weak solutions of (1.1) as in [10]. Denote by $V(Q_T; \mathbb{R}^N)$ the set of all $u \in L^2(Q_T; \mathbb{R}^N)$ such that $Du \in L^2(Q_T; \mathbb{R}^N)$, $u(\cdot, t) \in L^2(D; \mathbb{R}^N)$ for a.e. $t \in [0, T]$, and have finite norms

$$\|u\|_{V(Q_T)} := \left(\int_0^T \int_D |Du|^2 dx dt + \text{ess sup}_{0 \leq \tau \leq T} \int_D u^2(x, \tau) dx \right)^{\frac{1}{2}}.$$

$W_2^{1,1}(Q_T; \mathbb{R}^N)$ denotes the Hilbert space with the inner product

$$\langle u, v \rangle_{W_2^{1,1}} = \int_0^T \int_D (uv + D_x u D_x v + D_t u D_t v) dx dt.$$

By $\mathring{V}(Q_T; \mathbb{R}^N)$ and $\mathring{W}_2^{1,1}(Q_T; \mathbb{R}^N)$ we denote the subset of $V(Q_T; \mathbb{R}^N)$ and $W_2^{1,1}(Q_T; \mathbb{R}^N)$, respectively, with the elements satisfying $u(\cdot, t)|_{\partial D} = 0$ a.e. $t \in (0, T)$.

Definition 1.1. For $f \in L^\infty(Q_T)$, $g \in C^{\mu, 0}(\overline{D}_m \times [0, T])$, we say that u is a weak solution of (1.1), if $u \in \mathring{V}(Q_T; \mathbb{R}^N)$ satisfies, for a.e. $\tau \in (0, T)$ and for all $\zeta \in \mathring{W}_2^{1,1}(Q_T; \mathbb{R}^N)$ that vanish at $t = 0$, the identity

$$\int_D (u\zeta)(\cdot, \tau) dx - \int_0^\tau \int_D u \zeta_t dx dt + \int_0^\tau \int_D A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \zeta^i dx dt = \int_0^\tau \int_D (f\zeta + g^\alpha D_\alpha \zeta) dx dt. \quad (1.5)$$

For $\epsilon > 0$ small, set

$$D_\epsilon = \{x \in D \mid \text{dist}(x, \partial D) > \epsilon\}.$$

The first of our main results concerns the Hölder interior estimates for the gradient.

Theorem 1.2. Under the assumptions on D, A, f, g and φ mentioned above, let $u \in V(Q_T; \mathbb{R}^N)$ be a weak solution of (1.1). Then for any $0 < \epsilon < \frac{1}{2}$ and $\alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$,

$$\begin{aligned} & \|u\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_x u\|_{C^{\alpha', 0}((D_\epsilon \cap \bar{D}_m) \times (\epsilon T, T))} \\ & \leq C \left(\|u\|_{L^2(Q_T)} + \|f\|_{L^\infty(Q_T)} + \max_{1 \leq m \leq L} \|g\|_{C^{\alpha', 0}(\bar{D}_m \times [0, T])} \right), \end{aligned} \quad (1.6)$$

where C depends only on $n, N, L, \alpha, \epsilon, \lambda, \Lambda_0, \mu, T, \|A\|_{C^{\alpha', 1}(\bar{D}_m \times [0, T])}$ and the $C^{1, \alpha}$ norm of D_m . In particular,

$$\|D_x u\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} \leq C \left(\|u\|_{L^2(Q_T)} + \|f\|_{L^\infty(Q_T)} + \max_{1 \leq m \leq L} \|g\|_{C^{\alpha', 0}(\bar{D}_m \times [0, T])} \right). \quad (1.7)$$

Remark 1.3. Theorem 1.2 in the case $A(x, t) \equiv A(x)$, independent of t , was included as a part of the thesis of Li [18].

Furthermore, if we suppose that $A_{ij}^{\alpha\beta}$ is of class $C^{\mu, k+1}(\bar{D}_m \times [0, T])$, and $f \in C^{0, k}(\bar{D}_m \times [0, T])$, $g \in C^{\mu, k}(\bar{D}_m \times [0, T])$, then we have the following estimates on u 's higher order derivatives.

Theorem 1.4. Under the assumptions on D, A, f, g and φ mentioned above, let $u \in V(Q_T; \mathbb{R}^N)$ be a weak solution of (1.1). Then for any $l \leq k$, for any $0 < \epsilon < \frac{1}{2}$ and $\alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$,

$$\begin{aligned} & \|D_t^l u\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_t^l D_x u\|_{C^{\alpha', 0}((D_\epsilon \cap \bar{D}_m) \times (\epsilon T, T))} \\ & \leq C \left(\|u\|_{L^2(Q_T)} + \sum_{s=1}^l \|D_t^s f\|_{L^\infty(Q_T)} + \max_{1 \leq m \leq L} \|g\|_{C^{\alpha', l}(\bar{D}_m \times [0, T])} \right), \end{aligned} \quad (1.8)$$

where C depends only on $n, N, L, l, \alpha, \epsilon, \lambda, \Lambda_{2l}, \mu, T, \text{diam}(D)$, the $C^{1, \alpha}$ norm of D_m , and

$$\|A\|_{C^{\alpha', k+1}(\bar{D}_m \times [0, T])}.$$

We draw attention to some closely related results by Fan et al. [13].

This paper is organized as follows. In Section 2, we introduce some notation and present some standard L^2 estimates for readers' convenience. A result of Chipot et al. [11] for laminar elliptic systems is extended to laminar parabolic systems in Section 3. We present a general perturbation result, an extension of Li and Vogelius [22] and Li and Nirenberg [21] for elliptic equations and systems, in Section 4, and apply it to establish uniform gradient estimates in Section 5. Finally, the proofs of main theorems are given in Section 6. This paper has appeared in [19] in preprint form.

2 Preliminary results

In this section, we mainly follow the notation and definitions of [10, 17] and list some standard L^2 estimates for readers' convenience.

2.1 Function spaces

Let D be a bounded open set of $\mathbb{R}^n$. $H^{1, p}(D; \mathbb{R}^N)$ and $H_0^{1, p}(D; \mathbb{R}^N)$ are the usual Sobolev spaces of the vector-valued function $u : D \rightarrow \mathbb{R}^N$; if $p = 2$ we shall write more briefly H^1 and H_0^1 .

We shall also use $V(Q_T)$ and $\dot{V}(Q_T)$ to denote $V(Q_T; \mathbb{R}^N)$ and $\dot{V}(Q_T; \mathbb{R}^N)$, respectively, when there is no ambiguity. Let $W_2^{1, 0}(Q_T)$, $W_2^{1, 1}(Q_T)$ and $W_2^{0, k}(Q_T)$ denote the Hilbert spaces with the inner products

$$\langle u, v \rangle_{W_2^{1, 0}} = \int_0^T \int_D (uv + D_x u D_x v) dx dt,$$

$$\langle u, v \rangle_{W_2^{1,1}} = \int_0^T \int_D (uv + D_x u D_x v + D_t u D_t v) dx dt$$

and

$$\langle u, v \rangle_{W_2^{0,k}} = \int_0^T \int_D \left(\sum_{s \leq k} D_t^s u D_t^s v \right) dx dt,$$

respectively. Denote by $V^{1,0}(Q_T)$ the Banach space consisting of all elements of $V(Q_T)$ satisfying

$$\lim_{h \rightarrow 0} \|u(\cdot, t+h) - u(\cdot, t)\|_{L^2(D)} = 0, \quad \text{uniformly for } t, t+h \in [0, T]. \quad (2.1)$$

In fact,

$$W_2^{1,1}(Q_T) \subset V^{1,0}(Q_T) \subset V(Q_T),$$

and $V^{1,0}(Q_T)$ is the completion of $W_2^{1,1}(Q_T)$ with respect to the norm $\|\cdot\|_{V(Q_T)}$. Similarly, a zero over $W_2^{1,0}(Q_T)$, $W_2^{1,1}(Q_T)$, $W_2^{0,k}(Q_T)$, $V^{1,0}(Q_T)$ and $V(Q_T)$ means that only those elements of the spaces are taken which satisfy $u(\cdot, t)|_{\partial D} = 0$ a.e. $t \in (0, T)$.

For $u(x, t) \in V^{1,0}(Q_T)$, we will consider the Steklov average, i.e., u 's average in t ,

$$u_h(x, t) := \frac{1}{h} \int_t^{t+h} u(x, \tau) d\tau,$$

for $0 < t < t+h < T$. It is clear that if $u \in \mathring{V}^{1,0}(Q_T)$, then for any $0 < h < \delta$ ($0 < \delta < T$), $u_h \in \mathring{W}_2^{1,1}(Q_{T-\delta})$, and

$$\|u_h - u\|_{V(Q_{T-\delta})} \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad (2.2)$$

as shown below. First,

$$\|u_h(\cdot, t) - u(\cdot, t)\|_{L^2(D)}^2 \leq \frac{1}{h} \int_t^{t+h} \|u(x, \tau) - u(\cdot, t)\|_{L^2(D)}^2 d\tau \leq \sup_{t \leq \tau \leq t+h} \|u(x, \tau) - u(\cdot, t)\|_{L^2(D)}^2.$$

Since $u \in V^{1,0}(Q_T)$, it follows that the right-hand side converges to zero as $h \rightarrow 0$. On the other hand,

$$\begin{aligned} \|Du_h(\cdot, t) - Du(\cdot, t)\|_{L^2(Q_{T-\delta})}^2 &\leq \frac{1}{h} \int_0^h \|Du(x, t+s) - Du(x, t)\|_{L^2(Q_{T-\delta})}^2 ds \\ &\leq \sup_{0 \leq s \leq h} \|Du(x, t+s) - Du(x, t)\|_{L^2(Q_{T-\delta})}^2. \end{aligned}$$

Since $L^2(Q_T)$ functions are continuous with respect to translations, this supremum tends to zero as $h \rightarrow 0$.

2.2 L^2 estimates for the initial boundary value problem

Let D and f, g be defined in Section 1. We assume that $A_{ij}^{\alpha\beta}(x, t)$ satisfy (1.2)–(1.4) and f, g are smooth in t . We consider the following initial boundary value problem of the parabolic systems:

$$\begin{cases} (u^i)_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta u^j) = -D_\alpha g^{\alpha i} + f^i & \text{in } D \times (0, T), \\ u = 0 & \text{on } \partial D \times (0, T), \\ u = \varphi(x) & \text{on } D \times \{0\}, \end{cases} \quad (2.3)$$

where $\varphi \in L^2(D)$. We now define a weak solution of (2.3).

Definition 2.1. For $f \in L^\infty(Q_T)$, $g \in C^{\mu,0}(\overline{D}_m \times [0, T])$, and $\varphi \in L^2(D)$, we say that u is a weak solution of (2.3), if $u \in \mathring{V}(Q_T)$ satisfies, for a.e. $\tau \in (0, T)$ and for all $\zeta \in \mathring{W}_2^{1,1}(Q_T)$, the identity

$$\begin{aligned} &\int_D (u\zeta)(\cdot, \tau) dx - \int_D (\varphi\zeta)(\cdot, 0) dx - \int_0^\tau \int_D u\zeta_t dx dt + \int_0^\tau \int_D A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \zeta^i dx dt \\ &= \int_0^\tau \int_D (f\zeta + g^\alpha D_\alpha \zeta) dx dt. \end{aligned} \quad (2.4)$$

The following lemmas and their proofs follow [10, 17]. We present them here for readers' convenience.

Lemma 2.2. Assume the above. Suppose $u \in \mathring{V}(Q_T)$ is a weak solution of (2.3). Then u belongs to $\mathring{V}^{1,0}(Q_T)$.

Remark 2.3. Suppose $u \in V(Q_T)$ is a weak solution of (1.1). Then $u \in V^{1,0}(Q'_T)$, for any $Q'_T = D' \times (0, T)$, $D' \subset\subset D$. The proof of this assertion is similar to the proof of Lemma 2.2.

Proof of Lemma 2.2. By Definition 2.1, we have, for a.e. $\tau \in (0, T)$ and for any $\zeta \in \mathring{W}_2^{1,1}(Q_T)$, u satisfies the identity (2.4). Since $u \in V(Q_T)$, it is obvious that

$$\|u(\cdot, \tau)\|_{L^2(D)} \leq \|u\|_{V(Q_T)}, \quad \text{a.e. } \tau \in (0, T). \quad (2.5)$$

Denote $I = \{\tau \in (0, T) \mid (2.4) \text{ and } (2.5) \text{ hold simultaneously}\}$. Then the measure of $([0, T] \setminus I)$ is zero. For $\tau \in [0, T] \setminus I$, we can redefine $u(x, \tau)$ such that (2.4) and (2.5) still hold. Hence, without loss of generality, we assume that $u(x, \tau)$ satisfies (2.4) and (2.5) for every $\tau \in [0, T]$.

For $\tau \in (0, T)$, and $\Delta t > 0$, denote

$$Q_{\tau, \tau+\Delta t} = D \times (\tau, \tau + \Delta t).$$

By (2.4), we have

$$\begin{aligned} & \int_D (u\zeta)(\cdot, \tau + \Delta t) dx - \int_D (u\zeta)(\cdot, \tau) dx - \int_{\tau}^{\tau+\Delta t} \int_D u \zeta_t dx dt \\ & + \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta} D_{\beta} u^j D_{\alpha} \zeta^i dx dt = \int_{\tau}^{\tau+\Delta t} \int_D (f\zeta + g^{\alpha} D_{\alpha} \zeta) dx dt. \end{aligned} \quad (2.6)$$

Picking the test function $\zeta(x, t) = \zeta(x) \in H_0^1(D)$, the usual Sobolev space, then we have

$$\int_D (u(x, \tau + \Delta t) - u(x, \tau)) \zeta dx = \int_{\tau}^{\tau+\Delta t} \int_D f \zeta dx dt + \int_{\tau}^{\tau+\Delta t} \int_D (g^{\alpha} - A_{ij}^{\alpha\beta} D_{\beta} u) D_{\alpha} \zeta dx dt.$$

It is clear that

$$\lim_{\Delta t \rightarrow 0} \int_D (u(x, \tau + \Delta t) - u(x, \tau)) \zeta(x) dx = 0, \quad \text{for } \zeta \in H_0^1(D). \quad (2.7)$$

For $0 \leq \tau < \tau + \Delta t < T$, consider $\tilde{u}$ (depending on τ),

$$\tilde{u}(x, t) = \begin{cases} u(x, \tau + \Delta t), & t \geq \tau + \Delta t, \\ u(x, t), & 0 \leq t \leq \tau + \Delta t, \\ u(x, \tau), & t \leq \tau. \end{cases}$$

Similar to the Steklov average, for $0 < h < \Delta t$, we define

$$\tilde{u}_h(x, t) := \frac{1}{2h} \int_{t-h}^{t+h} \tilde{u}(x, s) ds.$$

Then $\tilde{u}_h \in \mathring{W}_2^{1,1}(Q_{\tau, \tau+\Delta t})$. Take $\zeta = \tilde{u}_h$ in (2.6), and compute

$$\begin{aligned} & \int_{\tau}^{\tau+\Delta t} \int_D u(\tilde{u}_h)_t dx dt \\ & = \frac{1}{2h} \int_D dx \left\{ \int_{\tau}^{\tau+\Delta t} u(x, t) \tilde{u}(x, t+h) dt - \int_{\tau}^{\tau+\Delta t} u(x, t) \tilde{u}(x, t-h) dt \right\} \\ & = \frac{1}{2h} \int_D dx \left\{ \int_{\tau+\Delta t}^{\tau+\Delta t+h} u(x, t-h) \tilde{u}(x, t) dt - \int_{\tau}^{\tau+h} u(x, t) \tilde{u}(x, t-h) dt \right\} \\ & = \frac{1}{2h} \int_D dx \left\{ \int_{\tau+\Delta t}^{\tau+\Delta t+h} u(x, t-h) u(x, \tau+\Delta t) dt - \int_{\tau}^{\tau+h} u(x, t) u(x, \tau) dt \right\} \end{aligned}$$

$$= \frac{1}{2h} \int_D dx \left\{ \int_{\tau+\Delta t}^{\tau+\Delta t+h} (u(x, t-h) - u(x, \tau+\Delta t)) u(x, \tau+\Delta t) dt \right. \\ \left. - \int_{\tau}^{\tau+h} (u(x, t) - u(x, \tau)) u(x, \tau) dt \right\} + \frac{1}{2} \int_D u^2(x, \tau+\Delta t) dx - \frac{1}{2} \int_D u^2(x, \tau) dx.$$

By (2.7), it follows that

$$\lim_{h \rightarrow 0} \int_{\tau}^{\tau+\Delta t} \int_D u(\tilde{u}_h)_t dx dt = \frac{1}{2} \int_D u^2(x, \tau+\Delta t) dx - \frac{1}{2} \int_D u^2(x, \tau) dx. \quad (2.8)$$

Furthermore, we have

$$\lim_{h \rightarrow 0} \int_D u \tilde{u}_h(x, \tau) dx = \int_D u^2(x, \tau) dx \quad \text{and} \quad \lim_{h \rightarrow 0} \int_D u \tilde{u}_h(x, \tau+\Delta t) dx = \int_D u^2(x, \tau+\Delta t) dx.$$

On the other hand,

$$\lim_{h \rightarrow 0} \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta} D_{\beta} u D_{\alpha} \tilde{u}_h dx dt = \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta} D_{\beta} u D_{\alpha} u dx dt.$$

Indeed, it reduces to show

$$\begin{aligned} |I| &= \left| \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta} D_{\beta} u (D_{\alpha} \tilde{u}_h - D_{\alpha} u) dx dt \right| \\ &= \left| \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta} D_{\beta} u \left(\frac{1}{2h} \int_{t-h}^{t+h} (D_{\alpha} u(x, s) - D_{\alpha} u(x, t)) ds \right) dx dt \right| \\ &= \left| \int_{\tau}^{\tau+\Delta t} \int_D A_{ij}^{\alpha\beta}(x, t) D_{\beta} u(x, t) \left(\frac{1}{2} \int_{-1}^1 (D_{\alpha} u(x, t+hs') - D_{\alpha} u(x, t)) ds' \right) dx dt \right| \\ &\leq C(\Lambda) \|Du\|_{L^2((\tau, \tau+\Delta t) \times D)} \sup_{-h < s < h} \|Du(x, t+s) - Du(x, t)\|_{L^2((\tau, \tau+\Delta t) \times D)} \\ &\rightarrow 0 \quad \text{as } h \rightarrow 0 \quad (\text{since } Du \in L^2(Q_T)). \end{aligned}$$

Similarly, we have

$$\lim_{h \rightarrow 0} \iint g^{\alpha} D_{\alpha} \tilde{u}_h = \iint g^{\alpha} D_{\alpha} u.$$

Then taking $h \rightarrow 0$ in (2.6) with $\zeta = \tilde{u}_h$, we have

$$\frac{1}{2} \int_D u^2(x, \tau+\Delta t) dx - \frac{1}{2} \int_D u^2(x, \tau) dx = \int_{\tau}^{\tau+\Delta t} \int_D (fu + (g^{\alpha} - A_{ij}^{\alpha\beta} D_{\beta} u) D_{\alpha} u) dx dt. \quad (2.9)$$

This implies that

$$\lim_{\Delta t \rightarrow 0} \|u(\cdot, \tau+\Delta t)\|_{L^2(D)}^2 = \|u(\cdot, \tau)\|_{L^2(D)}^2. \quad (2.10)$$

Now we can prove the continuity of $u(\cdot, t)$ in t in the norm of $L^2(D)$. Indeed,

$$\|u(\cdot, \tau+\Delta t) - u(\cdot, t)\|_{L^2(D)}^2 = \|u(\cdot, \tau+\Delta t)\|_{L^2(D)}^2 + \|u(x, \tau)\|_{L^2(D)}^2 - 2 \int_D u(x, \tau+\Delta t) u(x, \tau) dx.$$

Then using (2.7) and (2.10), it follows that

$$\lim_{\Delta t \rightarrow 0} \|u(\cdot, \tau+\Delta t) - u(\cdot, \tau)\|_{L^2(D)}^2 = 0.$$

Hence $u \in \mathring{V}^{1,0}(Q_T)$. □

Lemma 2.4. Assume the above. Suppose $u \in \mathring{V}(Q_T)$ is a weak solution of (2.3). Then we have

$$\|u\|_{V(Q_T)}^2 \leq C(\|\varphi\|_{L^2(D)}^2 + \|f\|_{L^2(Q_T)}^2 + \|g\|_{L^2(Q_T)}^2), \quad (2.11)$$

where C depends only on λ and T .

Proof. By Lemma 2.2, $u(x, t) \in \mathring{V}^{1,0}(Q_T)$. For $0 < \tau < T$, let

$$\bar{u}(x, t) = \begin{cases} u(x, \tau), & t \geq \tau, \\ u(x, t), & 0 \leq t \leq \tau, \\ u(x, 0), & t \leq 0, \end{cases} \quad \text{and} \quad \bar{u}_h(x, t) := \frac{1}{2h} \int_{t-h}^{t+h} \bar{u}(x, s) ds.$$

Replacing ζ by $\bar{u}_h$ in (2.4) gives

$$\begin{aligned} & \int_D (u \bar{u}_h)(\cdot, \tau) dx - \int_D \varphi \bar{u}_h(\cdot, 0) dx - \int_0^\tau \int_D u (\bar{u}_h)_t dx dt \\ & + \int_0^\tau \int_D A_{ij}^{\alpha\beta} D_\beta u D_\alpha (\bar{u}_h) dx dt = \int_0^\tau \int_D (f \bar{u}_h + g^\alpha D_\alpha (\bar{u}_h)) dx dt. \end{aligned} \quad (2.12)$$

Similar to the derivation of (2.8), we obtain

$$\lim_{h \rightarrow 0} \int_0^\tau \int_D u (\bar{u}_h)_t dx dt = \frac{1}{2} \int_D u^2(x, \tau) dx - \frac{1}{2} \int_D u^2(x, 0) dx. \quad (2.13)$$

In fact, the derivation is much easier since we now have $u \in \mathring{V}^{1,0}(Q_T)$. Since $\|\bar{u}_h - u\|_{V(Q_T)} \rightarrow 0$ as $h \rightarrow 0$, and combining with (2.13), then sending $h \rightarrow 0$ in (2.12), it follows that

$$\frac{1}{2} \int_D u^2(x, \tau) dx + \int_0^\tau \int_D A_{ij}^{\alpha\beta} D_\beta u D_\alpha u dx dt = \int_0^\tau \int_D (f u + g^\alpha D_\alpha u) dx dt + \frac{1}{2} \int_D \varphi^2(x) dx.$$

Using the parabolic condition and Cauchy inequality, we have

$$\begin{aligned} & \frac{1}{2} \int_D u^2(x, \tau) dx + \lambda \int_0^\tau \int_D |Du|^2 dx dt \\ & \leq \frac{\lambda}{2} \int_0^\tau \int_D |Du|^2 dx dt + C \int_0^\tau \int_D |g|^2 dx dt \\ & \quad + \frac{1}{2} \int_0^\tau \int_D |u|^2 dx dt + \frac{1}{2} \int_0^\tau \int_D |f|^2 dx dt + \frac{1}{2} \int_D \varphi^2(x) dx, \end{aligned}$$

where C depends only on λ , i.e.,

$$\begin{aligned} & \int_D u^2(x, \tau) dx + \int_0^\tau \int_D |Du|^2 dx dt \\ & \leq C \left(\int_0^\tau \int_D |u|^2 dx dt + \int_D \varphi^2(x) dx + \int_0^\tau \int_D |g|^2 dx dt + \int_0^\tau \int_D |f|^2 dx dt \right), \end{aligned} \quad (2.14)$$

where C depends only on λ . Denote

$$E(\tau) = \int_0^\tau \int_D u^2 dx dt$$

and

$$F(\tau) = \|\varphi\|_{L^2(D)}^2 + \|g\|_{L^2(Q_\tau)} + \|f\|_{L^2(Q_\tau)},$$

where $Q_\tau = D \times (0, \tau)$. By (2.14),

$$\frac{dE(\tau)}{d\tau} \leq C(E(\tau) + F(\tau)), \quad 0 < \tau < T,$$

where C depends only on λ . This implies that

$$E(\tau) \leq e^{C\tau} E(0) + e^{C\tau} F(\tau), \quad 0 < \tau < T.$$

Lemma 2.4 follows from the above and (2.14). $\square$

Now we consider the L^2 estimate for u_t .

Lemma 2.5. Under (1.2)–(1.4), suppose $u(x, t) \in \mathring{V}(Q_T)$ is a weak solution of (2.3). Then for any small $\delta > 0$, we have

$$\sup_{\delta \leq \tau \leq T} \int_D |Du(x, \tau)|^2 dx + \int_\delta^T \int_D |u_t|^2 dx dt \leq C \left(\|\varphi\|_{L^2(D)}^2 + \|f\|_{L^2(Q_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(Q_T)}^2 \right), \quad (2.15)$$

where C depends only on $n, N, \lambda, \Lambda_2, T$ and δ .

Proof. **Step 1.** By Lemma 2.2, $u(x, t) \in \mathring{V}^{1,0}(Q_T)$. Let $\tau_0 \in (0, T)$ and $h \in (0, T - \tau_0)$. Take ζ_h as the test function in (2.4), where ζ is an arbitrary element of $\mathring{W}_2^{1,1}(D \times (0, T + h))$ with $\zeta(x, t) = 0$ if $t \leq h$ and $t \geq T$. A simple calculation shows that $(\zeta_h)_t = (\zeta_t)_h$, and so

$$\int_0^T \int_D u(\zeta_h)_t dx dt = \int_0^T \int_D u(\zeta_t)_h dx dt = \int_0^T \int_D u_{-h} \zeta_t dx dt = - \int_0^T \int_D (u_{-h})_t \zeta dx dt,$$

where the notation

$$\zeta_{-h}(x, t) = \frac{1}{h} \int_{t-h}^t \zeta(x, \tau) d\tau.$$

For all the other terms in (2.4) with $\zeta = \zeta_h$, we also transfer the average $(\cdot)_h$ from ζ to the coefficients, taking into account the permutability of this averaging with differentiation with respect to x . This gives the identity

$$\int_h^T \int_D (u_{-h})_t \zeta dx dt + \int_h^T \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} D_\alpha \zeta dx dt = \int_h^T \int_D (f_{-h} \zeta + g_{-h}^\alpha D_\alpha \zeta) dx dt, \quad (2.16)$$

for any $\zeta \in \mathring{W}_2^{1,1}(Q_{T+h})$ which vanishes for $t \leq h$ and $t \geq T$. This identity is actually valid for any ζ that is equal to zero for $t > \tau$ ($\tau \leq T$) and is equal to some function $\hat{\zeta} \in \mathring{V}^{1,0}(Q_\tau)$ for $t \in [h, \tau]$. Indeed, the set $\mathring{W}_2^{1,1}(Q_T)$ is dense in the space $\mathring{V}^{1,0}(Q_T)$. Thus for any $\hat{\zeta}(x, t) \in \mathring{V}^{1,0}(Q_T)$, there is a sequence of functions $\zeta_m \in \mathring{W}_2^{1,1}(Q_T)$, i.e., strongly convergent to $\hat{\zeta}$ as $m \rightarrow \infty$ in the norm of $\mathring{V}^{1,0}(Q_T)$. We denote $\chi_k(t)$ the continuous piecewise-linear functions

$$\chi_k(t) = \begin{cases} kt, & \text{in } \left(0, \frac{1}{k}\right), \\ 1, & \text{in } \left[\frac{1}{k}, \tau - \frac{1}{k}\right], \\ k(\tau - t), & \text{in } \left(\tau - \frac{1}{k}, \tau\right), \\ 0, & \text{in } (-\infty, 0] \cup [\tau, +\infty). \end{cases}$$

Then (2.16) is established for $\zeta_{m,k} = \zeta_m \chi_k$ for $\tau < T$. One can pass to the limit as $m \rightarrow \infty$ and $k \rightarrow \infty$, thereby

$$\int_0^\tau \int_D (u_{-h})_t \hat{\zeta} dx dt + \int_0^\tau \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} D_\alpha \hat{\zeta} dx dt = \int_0^\tau \int_D f_{-h} \hat{\zeta} + g_{-h}^\alpha D_\alpha \hat{\zeta} dx dt, \quad (2.17)$$

for any $\hat{\zeta} \in \mathring{V}^{1,0}(Q_\tau)$, $\tau \leq T$.

Step 2. For $h < \frac{\delta}{2}$, replacing $\hat{\zeta}$ by $\bar{\eta}^2(u_{-h})_t$ in (2.17), where $\bar{\eta}(x, t)$ is a smooth cutoff function, satisfying $0 \leq \bar{\eta} \leq 1$, and for any $0 < \delta < T$,

$$\bar{\eta}(t) = \begin{cases} 1, & \text{in } (\delta, \tau), \\ 0, & \text{in } (0, T) \setminus \left(\frac{\delta}{2}, \tau\right), \end{cases} \quad |\bar{\eta}_t| \leq \frac{C}{\delta}, \quad (2.18)$$

it follows that

$$\begin{aligned} & \int_0^\tau \int_D (u_{-h})_t \bar{\eta}^2 (u_{-h})_t dx dt + \int_0^\tau \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} \bar{\eta}^2 D_\alpha (u_{-h})_t dx dt \\ &= \int_0^\tau \int_D f_{-h} \bar{\eta}^2 (u_{-h})_t + g_{-h}^\alpha \bar{\eta}^2 D_\alpha (u_{-h})_t dx dt. \end{aligned}$$

Then we have

$$\begin{aligned} \text{LHS} &= \int_0^\tau \int_D |(u_{-h})_t|^2 \bar{\eta}^2 dx dt - \frac{1}{2} \int_0^\tau \int_D ((A_{ij}^{\alpha\beta})_{-h})_t D_\beta u^j(x, t-h) \bar{\eta}^2 D_\alpha u_{-h}^i dx dt \\ &\quad + \frac{1}{2} \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} \bar{\eta}^2 D_\alpha u_{-h}^i dx \Big|_{t=0}^\tau - \int_0^\tau \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} \bar{\eta}_t \bar{\eta} D_\alpha u_{-h}^i \\ &= \int_0^\tau \int_D |u_{ht}|^2 \bar{\eta}^2 dx dt - \frac{1}{2} \int_0^\tau \int_D ((A_{ij}^{\alpha\beta})_{-h})_t D_\beta u^j(x, t-h) \bar{\eta}^2 D_\alpha u_{-h}^i dx dt \\ &\quad + \frac{1}{2} \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_h \bar{\eta}^2 D_\alpha u_h^i(x, \tau) dx dt - \int_0^\tau \int_D (A_{ij}^{\alpha\beta} D_\beta u^j)_{-h} \bar{\eta}_t \bar{\eta} D_\alpha u_{-h}^i \end{aligned}$$

using $\bar{\eta}(0) = 0$, and similarly,

$$\begin{aligned} \text{RHS} &= \int_0^\tau \int_D f_{-h} \bar{\eta}^2 (u_{-h})_t + \int_D (\bar{\eta}^2 g_{-h}^\alpha D_\alpha u_{-h})(x, \tau) dx \\ &\quad - \int_0^\tau \int_D (g_{-h}^\alpha)_t \bar{\eta}^2 D_\alpha u_{-h} dx dt - 2 \int_0^\tau \int_D g_{-h}^\alpha \bar{\eta}_t \bar{\eta} D_\alpha u_{-h} dx dt. \end{aligned}$$

Combining them, by Cauchy inequality, and sending $h \rightarrow 0$, we have

$$\begin{aligned} & \frac{1}{2} \int_0^\tau \int_D \bar{\eta}^2 |u_t|^2 dx dt + \frac{1}{2} \int_D (A_{ij}^{\alpha\beta} \bar{\eta} D_\beta u^j \bar{\eta} D_\alpha u^i)(x, \tau) dx \\ &\leq \int_0^\tau \int_D (D_t A_{ij}^{\alpha\beta})(\bar{\eta} D_\beta u^j)(\bar{\eta} D_\alpha u^i) dx dt + \frac{1}{2} \int_0^\tau \int_D |\bar{\eta} f|^2 dx dt \\ &\quad + \int_D (\bar{\eta}^2 g^\alpha D_\alpha u)(x, \tau) dx - \int_0^\tau \int_D \bar{\eta}^2 (g^\alpha)_t D_\alpha u dx dt - 2 \int_0^\tau \int_D \bar{\eta}_t \bar{\eta} g^\alpha D_\alpha u dx dt. \end{aligned}$$

Then using the parabolic condition and Cauchy inequality, we obtain

$$\begin{aligned} & \int_\delta^\tau \int_D |u_t|^2 dx dt + \lambda \int_D |Du(x, \tau)|^2 dx \\ &\leq C(n, N, \Lambda_2) \int_0^\tau \int_D |Du|^2 dx dt + \int_0^\tau \int_D |f|^2 dx dt \\ &\quad + C(\lambda) \int_D |g(x, \tau)|^2 dx + C \int_0^\tau \int_D |g|^2 dx dt + C \int_0^\tau \int_D |g_t|^2 dx dt. \end{aligned} \quad (2.19)$$

Since

$$\begin{aligned} \|g(x, \tau)\|_{L^2(D)}^2 - \frac{1}{\tau} \int_0^\tau \|g(x, t)\|_{L^2(D)}^2 dt &= \frac{1}{\tau} \int_0^\tau \int_D |g(x, \tau)|^2 - |g(x, t)|^2 dx dt \\ &= \frac{1}{\tau} \int_0^\tau \int_D \int_t^\tau D_s(g(x, s))^2 ds dx dt \\ &\leq 2 \int_0^\tau \int_D |g(x, t) g_t(x, t)| dx dt \\ &\leq \int_0^\tau \int_D |g(x, t)|^2 dx dt + \int_0^\tau \int_D |g_t(x, t)|^2 dx dt, \end{aligned}$$

it follows that

$$\|g(x, \tau)\|_{L^2(D)}^2 \leq C \left(\int_0^\tau \int_D |g(x, t)|^2 dx dt + \int_0^\tau \int_D |g_t(x, t)|^2 dx dt \right),$$

where C depends only on δ . Substituting it into (2.19) gives

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \int_D |Du(x, \tau)|^2 dx + \int_\delta^\tau \int_D |u_t|^2 dx dt \\ & \leq C \left(\int_0^\tau \int_D |Du|^2 dx dt + \int_0^\tau \int_D |f|^2 dx dt + \int_0^\tau \int_D (|g|^2 + |g_t|^2) dx dt \right), \end{aligned} \quad (2.20)$$

where C depends on $n, N, \lambda, \Lambda_2, \delta$ and τ . In combination with Lemma 2.4, (2.15) is established. $\square$

Since A, f and g are smooth in t , it is standard to use difference quotients in t to estimate higher derivatives.

Lemma 2.6. Under (1.2)–(1.4), if $u \in V(Q_T)$ is a weak solution of (2.3), then for any $0 < \delta < T$ and $0 < \epsilon < 1$, we have

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \int_{D_\epsilon} |u_t(x, \tau)|^2 dx + \int_\delta^T \int_{D_\epsilon} |Du_t|^2 dx dt \\ & \leq C \left(\|\varphi\|_{L^2(D)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(Q_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(D_m \times (0, T))}^2 \right), \end{aligned} \quad (2.21)$$

where C depends only on $n, N, \lambda, \Lambda_2, T, \epsilon$ and δ .

Proof. For $h < \delta$, we apply Δ_t^h to (2.3), thus

$$\Delta_t^h u_t - D_\alpha (A_{ij}^{\alpha\beta} D_\beta \Delta_t^h u^j) = D_\alpha ((\Delta_t^h A_{ij}^{\alpha\beta})(D_\beta u^j) - \Delta_t^h g^{\alpha i}) + \Delta_t^h f^i, \quad (2.22)$$

where

$$\Delta_t^h u^i = \frac{u^i(x, t) - u^i(x, t - h)}{h}.$$

Similar to that in Lemma 2.4, multiplying $\bar{\eta}^2 \Delta_t^h u$ on both sides of (2.22), where $\bar{\eta}$ is defined by (2.18), integrating by parts, and using the property of difference quotients, we have

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \int_D |u_t(x, \tau)|^2 dx + \int_\delta^T \int_D |Du_t|^2 dx dt \\ & \leq C (\|Du\|_{L^2(Q_T)}^2 + \|u_t\|_{L^2(Q_T)}^2 + \|D_t f\|_{L^2(Q_T)}^2 + \|D_t g\|_{L^2(Q_T)}^2), \end{aligned}$$

where C depends only on n, N, λ, Λ_2 and δ . In combination with (2.11) and (2.15), we have established (2.21). $\square$

3 Estimates for laminar systems

In this section, we extend results for laminar elliptic systems due to Chipot et al. [11] to laminar parabolic systems. In Subsection 3.1, we first consider the initial boundary value problem of laminar parabolic systems. Then in Subsection 3.2, we give the interior estimates for (1.1) in general domain $D \times (0, T)$, similar to that in Section 2. In Subsection 3.3, we consider the laminar parabolic systems in $\omega \times (0, T)$.

3.1 Estimates for the initial boundary value problem

Let D be the unit cube ω ,

$$\omega = \left\{ x \in \mathbb{R}^n : |x_i| < \frac{1}{2}, 1 \leq i \leq n \right\},$$

divided into ω_m . However, the ω_m are different; they are strips

$$\omega_m = \{x \in \omega : c_{m-1} < x_n < c_m\},$$

where the c_m are increasing constants lying between $-\frac{1}{2}$ and $\frac{1}{2}$. There may be infinitely many strips; if so, we set $c_{-\infty} = -\frac{1}{2}$ and $c_{\infty} = \frac{1}{2}$. We consider the following initial boundary value problem of the laminar parabolic systems:

$$\begin{cases} (u^i)_t - D_{\alpha}(A_{ij}^{\alpha\beta}(x, t)D_{\beta}u^j) = -D_{\alpha}g^{\alpha i} + f^i & \text{in } \omega \times (0, T), \\ u = 0 & \text{on } \partial\omega \times (0, T), \\ u = \varphi(x) & \text{on } \omega \times \{0\}. \end{cases} \quad (3.1)$$

Here, g and f are smooth in each $\bar{\omega}_m \times [0, T]$, and $\varphi \in C^{\infty}(\bar{\omega}_m)$. We assume that $A_{ij}^{\alpha\beta}(x, t)$ satisfy (1.2) and (1.3), and furthermore, for any non-negative integer r and s ,

$$\sum_{r+2s \leq l} |D_x^r D_t^s A_{ij}^{\alpha\beta}(x, t)| \leq \Lambda_l, \quad \forall (x, t) \in \omega_m \times (0, T), \quad (3.2)$$

where $\Lambda = \Lambda_0 \leq \Lambda_1 \leq \dots \leq \Lambda_l \leq \dots$.

Proposition 3.1. Assume the above. Let $u \in \mathring{V}(\Omega_T)$ be a weak solution of (3.1). Then for $0 < \delta < T$ and for all γ' , $D_{x'}^{\gamma'} u \in C^0(\omega \times (\delta, T))$, and for each m , and for any $0 < \epsilon < 1$, $u \in C^{\infty}((\omega_{\epsilon} \cap \bar{\omega}_m) \times (\delta, T))$. Moreover for any non-negative integer k , and for any m ,

$$\begin{aligned} & \sum_{r+2s \leq k} \|D_x^r D_t^s u\|_{L^{\infty}((\bar{\omega}_m \cap \omega_{\epsilon}) \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq \bar{k} + k + 1, s \leq \frac{k}{2} + 1} (\|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)}) \right), \end{aligned} \quad (3.3)$$

where

$$\bar{k} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

and C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and $\Lambda_{\bar{k}+k+2}$.

Our proof of Proposition 3.1 adapts the alternative proof of Li and Nirenberg [21] in the elliptic case to the parabolic systems. For laminar systems, we could establish the estimates in $\omega \times (\delta, T)$, as well as in Lemmas 2.4–2.6, just replacing the domain D by ω . Next, we will establish the interior estimates for higher derivatives of weak solutions.

Denote

$$\Omega_T = \omega \times (0, T), \quad \text{and} \quad \omega_{\epsilon} = \{x \in \omega \mid \text{dist}(x, \partial\omega) > \epsilon\}, \quad \text{for } 0 < \epsilon < 1.$$

Then we have the following lemma.

Lemma 3.2. Under (1.2), (1.3) and (3.2), if $u \in V(\Omega_T)$ is a weak solution of (3.1), then for any $0 < \epsilon < 1$ and integer $k \geq 0$, for $0 \leq |\gamma'| \leq k$, $D_{x'}^{\gamma'} u \in W_2^{1,0}(\Omega_T)$, and we have

$$\begin{aligned} & \sup_{0 \leq \tau \leq T} \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k} \int_0^T \int_{\omega_{\epsilon}} |DD_{x'}^{\gamma'} u(x, t)|^2 dx dt \\ & \leq C \left\{ \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} \varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} g\|_{L^2(\Omega_T)}^2 \right\}, \end{aligned} \quad (3.4)$$

where C depends only on $n, N, \lambda, \Lambda_k, T, k$ and ϵ .

In order to estimate higher derivative, it is customary to differentiate the equation, to multiply by a suitable derivative of u and by a cutoff function, and to integrate by parts. Clearly, we are not allowed to apply D_n across $\{x_n = c_m\}$ since the coefficients are smooth only in $x' = (x_1, \dots, x_{n-1})$. Furthermore, we do not know yet that u has additional derivatives in the x' directions. So in place of taking derivatives, it is standard to use difference quotients in these directions.

Proof of Lemma 3.2. For $k = 0$, (3.4) is done by Lemma 2.4 for $D = \omega$. For $|\gamma'| = 1$, denote the difference quotient in x_s -direction ($s = 1, \dots, n-1$) as

$$\Delta_\iota^s u(x, t) = \frac{u(x + \iota e_s, t) - u(x, t)}{\iota}.$$

Taking the difference quotient Δ_ι^s to the equation in (3.1), we obtain

$$\Delta_\iota^s u_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t) D_\beta \Delta_\iota^s u^j) = -D_\alpha(\Delta_\iota^s g^{\alpha i} - (\Delta_\iota^s A_{ij}^{\alpha\beta})(x + \iota e_s, t) D_\beta u^j) + \Delta_\iota^s f^i. \quad (3.5)$$

Multiply by $\zeta = (\tilde{\eta}^2 \Delta_\iota^s u)_h$, where $\tilde{\eta}(x) \in C_0^\infty(\omega)$ is a cutoff function satisfying $0 \leq \tilde{\eta}(x) \leq 1$, and for any $\epsilon > 0$,

$$\tilde{\eta}(x) = \begin{cases} 1 & \text{in } \omega_\epsilon, \\ 0 & \text{outside } \omega_{\frac{\epsilon}{2}}, \end{cases} \quad |D\tilde{\eta}| \leq \frac{C(n)}{\epsilon}. \quad (3.6)$$

Then, for $0 \leq |\iota| \leq \frac{\epsilon}{4}$, integrating by parts, we obtain

$$\begin{aligned} & \int_\omega \Delta_\iota^s u (\tilde{\eta}^2 \Delta_\iota^s u)_h(\cdot, \tau) dx - \int_\omega \Delta_\iota^s u (\tilde{\eta}^2 \Delta_\iota^s u)_h(\cdot, 0) dx \\ & - \int_0^\tau \int_\omega (\Delta_\iota^s u) ((\tilde{\eta}^2 \Delta_\iota^s u)_h)_t dx dt + \int_0^\tau \int_\omega A_{ij}^{\alpha\beta}(x, t) D_\beta \Delta_\iota^s u^j D_\alpha (\tilde{\eta}^2 \Delta_\iota^s u^i)_h dx dt \\ & = \int_0^\tau \int_\omega \Delta_\iota^s f^i (\tilde{\eta}^2 \Delta_\iota^s u^i)_h dx dt \\ & + \int_0^\tau \int_\omega (\Delta_\iota^s g^{\alpha i} - (\Delta_\iota^s A_{ij}^{\alpha\beta})(x + \iota e_s, t) D_\beta u^j) D_\alpha (\tilde{\eta}^2 \Delta_\iota^s u^i)_h dx dt. \end{aligned}$$

Similar to that in the proof of Lemma 2.2, sending $h \rightarrow 0$, and making use of (1.2), (1.3) and (3.2), we have

$$\begin{aligned} & \sup_{0 \leq \tau \leq T} \int_\omega \tilde{\eta}^2 |\Delta_\iota^s u(x, \tau)|^2 dx + \int_0^T \int_\omega \tilde{\eta}^2 |D \Delta_\iota^s u(x, t)|^2 dx dt \\ & \leq C \{ \|Du\|_{L^2(\Omega_T)}^2 + \|\Delta_\iota^s \varphi\|_{L^2(\omega)}^2 + \|\Delta_\iota^s f\|_{L^2(\Omega_T)}^2 + \|\Delta_\iota^s g\|_{L^2(\Omega_T)}^2 \}, \end{aligned}$$

where C depends only on n, N, λ, Λ_1 and ϵ . So

$$\begin{aligned} & \sup_{0 \leq \tau \leq T} \int_\omega \tilde{\eta}^2 |D_{x_s} u(x, \tau)|^2 dx + \int_0^T \int_\omega \tilde{\eta}^2 |DD_{x_s} u(x, t)|^2 dx dt \\ & \leq C \{ \|Du\|_{L^2(\Omega_T)}^2 + \|D_{x_s} \varphi\|_{L^2(\omega)}^2 + \|D_{x_s} f\|_{L^2(\Omega_T)}^2 + \|D_{x_s} g\|_{L^2(\Omega_T)}^2 \}. \end{aligned} \quad (3.7)$$

Hence, for any γ' , $|\gamma'| = 1$, combining with Lemma 2.4, we obtain (3.4) for $k = 1$. For the general k , we can make use of further differentiation in the x' direction, and obtain (3.4) by induction. $\square$

Lemma 3.3. Under (1.2), (1.3) and (3.2), if $u \in V(\Omega_T)$ is a weak solution of (3.1), then for any $0 < \epsilon < 1$, $0 < \delta < T$, and integer $k \geq 0$, for $0 \leq |\gamma'| \leq k$, $D_{x'}^{\gamma'} u \in W_2^{1,0}(\omega_\epsilon \times (\delta, T))$, and we have

$$\begin{aligned} & \sum_{|\gamma'| \leq k} \sup_{\delta \leq \tau \leq T} \int_{\omega_\epsilon} |D_{x'}^{\gamma'} u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k} \int_\delta^T \int_{\omega_\epsilon} |DD_{x'}^{\gamma'} u|^2 dx dt \\ & \leq C \left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} g\|_{L^2(\Omega_T)}^2 \right), \end{aligned} \quad (3.8)$$

where C depends only on $n, N, \lambda, \Lambda_k, T, k, \epsilon$ and δ .

Proof. For $k = 0$, (3.8) has been established in Lemma 2.4 with $D = \omega$. For any $0 < \delta < T$, by Lemma 2.4, we have, for $|\gamma'| = 1$,

$$\int_{\frac{\delta}{10}}^{\frac{\delta}{9}} \int_\omega |D_{x'} u(x, t)|^2 dx dt \leq C (\|\varphi\|_{L^2(\omega)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \|g\|_{L^2(\Omega_T)}^2).$$

Then by Fubini's theorem, there exists $\bar{t}_1 \in (\frac{\delta}{10}, \frac{\delta}{9})$ such that

$$\int_{\omega} |D_{x'} u(x, \bar{t}_1)|^2 dx \leq C(\|\varphi\|_{L^2(\omega)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \|g\|_{L^2(\Omega_T)}^2). \quad (3.9)$$

Consider (3.5) in $\omega \times (\bar{t}_1, T)$, by the same process, we obtain, similar to (3.7),

$$\begin{aligned} & \sup_{\bar{t}_1 \leq \tau \leq T} \int_{\omega} \tilde{\eta}^2 |D_{x_s} u(x, \tau)|^2 dx + \int_{\bar{t}_1}^T \int_{\omega} \tilde{\eta}^2 |DD_{x_s} u(x, t)|^2 dx dt \\ & \leq C\{\|Du\|_{L^2(\Omega_T)}^2 + \|D_{x_s} u(x, \bar{t}_1)\|_{L^2(\omega)}^2 + \|D_{x_s} f\|_{L^2(\Omega_T)}^2 + \|D_{x_s} g\|_{L^2(\Omega_T)}^2\}, \end{aligned}$$

where C depends only on $n, N, \lambda, \Lambda_1, \epsilon$ and δ . Then combining with (3.9) and (2.11), we prove Lemma 3.3 for $k = 1$, and

$$\begin{aligned} & \sum_{|\gamma'| \leq 1} \sup_{\bar{t}_1 \leq \tau \leq T} \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq 1} \int_{\bar{t}_1}^T \int_{\omega_{\epsilon}} |DD_{x'}^{\gamma'} u|^2 dx dt \\ & \leq C\left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq 1} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq 1} \|D_{x'}^{\gamma'} g\|_{L^2(\Omega_T)}^2\right), \end{aligned} \quad (3.10)$$

where C depends only on $n, N, \lambda, \Lambda_1, T, \epsilon$ and δ . Then by (3.10), we have for $|\gamma'| = 2$,

$$\int_{\frac{\delta}{9}}^{\frac{\delta}{8}} \int_{\omega} |D_{x'}^{\gamma'} u(x, t)|^2 dx dt \leq C\left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq 1} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq 1} \|D_{x'}^{\gamma'} g\|_{L^2(\Omega_T)}^2\right).$$

By Fubini's theorem, there exists $\bar{t}_2 \in (\frac{\delta}{9}, \frac{\delta}{8})$. Repeating the above process in $\omega \times (\bar{t}_2, T)$, we obtain (3.8) for $k = 2$. For further $k > 2$, we can establish (3.8) by induction. $\square$

To save space and the reader's patience, in the following we shall simply differentiate the equation in place of taking difference quotients in x' to obtain higher regularity.

Lemma 3.4. Under (1.2), (1.3) and (3.2), if $u \in V(\Omega_T)$ is a weak solution of (3.1), then for any $0 < \delta < T$ and $0 < \epsilon < 1$, we have, for $|\gamma'| \leq k$, and $l \geq 0$,

$$\begin{aligned} & \sum_{|\gamma'| \leq k, s \leq l-1} \sup_{\delta \leq \tau \leq T} \int_{\omega_{\epsilon}} |DD_{x'}^{\gamma'} D_t^s u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k, s \leq l} \int_{\delta}^T \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} D_t^s u|^2 dx dt \\ & \leq C\left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq k, s \leq l-1} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)}^2\right), \end{aligned} \quad (3.11)$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+2}, T, k, \epsilon$ and δ , and

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \sum_{|\gamma'| \leq k, s \leq l} \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} D_t^s u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k, s \leq l} \int_{\delta}^T \int_{\omega_{\epsilon}} |DD_{x'}^{\gamma'} D_t^s u|^2 dx dt \\ & \leq C\left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\omega_m \times (0, T))}^2\right), \end{aligned} \quad (3.12)$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+2}, T, k, \epsilon$ and δ .

Proof. For any γ' , $|\gamma'| = 1$, applying $D_{x'}^{\gamma'}$ on both sides of the equation in (3.1), we have

$$D_{x'}^{\gamma'} u_t - D_{\alpha}(A_{ij}^{\alpha\beta} D_{\beta} D_{x'}^{\gamma'} u^j) = D_{\alpha}(D_{x'}^{\gamma'} A_{ij}^{\alpha\beta} D_{\beta} u^j - g^{\alpha i}) + D_{x'}^{\gamma'} f.$$

Similar to the process to prove Lemma 2.5, in virtue of a cutoff function, defined by (3.6), instead of (2.20), we obtain

$$\sum_{|\gamma'| \leq k} \sup_{\delta \leq \tau \leq T} \int_{\omega_{\epsilon}} |DD_{x'}^{\gamma'} u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k} \int_{\delta}^T \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} u_t|^2 dx dt$$

$$\leq C \left(\sum_{|\gamma'| \leq k} \int_{\frac{\delta}{2}}^T \int_{\omega_{\frac{\epsilon}{2}}} |DD_{x'}^{\gamma'} u| dx dt + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)}^2 \right),$$

where C depends only on $n, N, \lambda, \Lambda_{k+2}, k, T, \epsilon$ and δ . Combining with Lemma 3.3, we know

$$D_{x'}^{\gamma'} u \in W_2^{1,1}(\omega_m \cap \omega_{\epsilon} \times (0, T)),$$

and we have (3.11).

Further applying $D_{x'}^{\gamma'}$ to (2.22), we have

$$\begin{aligned} D_{x'}^{\gamma'} \Delta_t^h u_t - D_{\alpha}((A_{ij}^{\alpha\beta}) D_{\beta} D_{x'}^{\gamma'} \Delta_t^h u^j) &= D_{\alpha}((\Delta_t^h A_{ij}^{\alpha\beta}) D_{\beta} D_{x'}^{\gamma'} u^j) + D_{\alpha}((\Delta_t^h D_{x'}^{\gamma'} A_{ij}^{\alpha\beta}) D_{\beta} u^j) \\ &\quad + (D_{x'}^{\gamma'} A_{ij}^{\alpha\beta})(\Delta_t^h D_{\beta} u^j) - \Delta_t^h g^{\alpha i} + D_{x'}^{\gamma'} \Delta_t^h f^i. \end{aligned} \quad (3.13)$$

Multiplying $\tilde{\eta}^2 D_{x'}^{\gamma'} \Delta_t^h u$ on both sides of (3.13), integrating by parts, we have (3.12). For general k and l , we can obtain (3.11) and (3.12) by induction. $\square$

Now denoting

$$w = (w^i) = (A_{ij}^{\alpha\beta} D_{\beta} u^j - g^{\alpha i}), \quad (3.14)$$

then we have the following lemma.

Lemma 3.5. For $0 < \delta < T$, $0 < \epsilon < 1$, and for $|\gamma'| \leq k$, $s \leq l$, $D_{x'}^{\gamma'} D_t^s w$, $D_{x'}^{\gamma'} D_t^s \partial_n w \in L_{\text{loc}}^2((\omega_{\epsilon} \cap \omega_m) \times (\delta, T))$,

$$\begin{aligned} &\sum_{|\gamma'| \leq k, s \leq l} \int_{\delta}^T \int_{\omega_{\epsilon}} |D_{x'}^{\gamma'} D_t^s w|^2 dx dt + \sum_{|\gamma'| \leq k-1, s \leq l} \int_{\delta}^T \int_{\omega_{\epsilon} \cap \omega_m} |D_{x'}^{\gamma'} D_t^s \partial_n w|^2 dx dt \\ &\leq C \left(\|\varphi\|_{L^2(\omega)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l+1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)}^2 \right), \end{aligned} \quad (3.15)$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+1}, T, k, \epsilon$ and δ .

Proof. Rewrite the equation in (3.1) as

$$\partial_n w = u_t^i + \sum_{\alpha \leq n-1} \partial_{\alpha}(g^{\alpha i} - A_{ij}^{\alpha\beta} D_{\beta} u^j) - f^i. \quad (3.16)$$

For any γ' , $|\gamma'| = 1$, applying $D_{x'}^{\gamma'}$ to (3.14) and (3.16), by virtue of (3.8) and (3.11) with $l = 1$, we obtain, $D_{x'}^{\gamma'} \partial_n w \in L_{\text{loc}}^2(\Omega_T)$ for $k = 2$. Further applying ∂_t , similar to the above, by virtue of (3.11) and (3.12), we obtain (3.15) for $k = 2$ and $l = 1$. For general k and l , we will obtain (3.15) by induction. $\square$

Here, we need the following embedding inequality, which is a variation of well-known Sobolev's inequality. The proof also could be found in [21].

Lemma 3.6. Let f be a real function in a bounded domain $\omega \subset \mathbb{R}^n$ with $D_{x'}^{\gamma'} f \in L^2(\omega)$ and $D_{x'}^{\gamma'} \partial_n f \in L^2(\omega)$ for all $0 \leq |\gamma'| \leq [\frac{n-1}{2}] + 1 =: \bar{k}$. Then $f \in C^0(\bar{\omega})$, and

$$\|f\|_{L^{\infty}(\omega)} \leq C(n) \sum_{|\gamma'| \leq \bar{k}} (\|D_{x'}^{\gamma'} f\|_{L^2(\omega)} + \|D_{x'}^{\gamma'} \partial_n f\|_{L^2(\omega)}).$$

Furthermore, if $f(x, t) \in L^2(0, T; C^0(\omega))$ and $f_t(x, t) \in L^2(0, T; C^0(\omega))$, then $f \in C^0(\bar{\Omega}_T)$, and

$$\|f\|_{L^{\infty}(\Omega_T)} \leq C(n) \sum_{|\gamma'| \leq \bar{k}} (\|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)} + \|D_{x'}^{\gamma'} \partial_n f\|_{L^2(\Omega_T)} + \|D_{x'}^{\gamma'} f_t\|_{L^2(\Omega_T)} + \|D_{x'}^{\gamma'} \partial_n f_t\|_{L^2(\Omega_T)}).$$

Proof of Proposition 3.1. It is well known that for each m , $u \in C^{\infty}((\omega_m \cap \omega_{\epsilon}) \times (\delta, T))$. For

$$k \geq \bar{k} = \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

and $|\gamma'| \leq k - \bar{k}$, by Lemmas 3.3 and 3.4 and application of Lemma 3.6 with $f = D_{x'}^{\gamma'} u$, we have $D_{x'}^{\gamma'} u \in C^0(\omega_\epsilon \times (\delta, T))$, and

$$\begin{aligned} & \sum_{|\gamma'| \leq k - \bar{k}} \|D_{x'}^{\gamma'} u\|_{L^\infty(\omega_\epsilon \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned} \quad (3.17)$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+2} . Similarly, for $k \geq \bar{k} + 1$ and $|\gamma'| \leq k - \bar{k} - 1$, by Lemmas 3.5 and 3.6 with $f = D_{x'}^{\gamma'} w$, we have $D_{x'}^{\gamma'} w \in C^0(\omega_\epsilon \times (\delta, T))$, and

$$\begin{aligned} & \sum_{|\gamma'| \leq k - \bar{k} - 1} \|D_{x'}^{\gamma'} w\|_{L^\infty(\omega_\epsilon \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq 2} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned} \quad (3.18)$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+3} . Consequently, $DD_{x'}^{\gamma'} u \in L_{\text{loc}}^\infty(\omega \times (\delta, T))$, and

$$\begin{aligned} & \sum_{|\gamma'| \leq k - \bar{k} - 1} \|DD_{x'}^{\gamma'} u\|_{L^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq 2} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned} \quad (3.19)$$

where C has the same dependence as in (3.18). Indeed, by (3.17), we only need to show that

$$\partial_n D_{x'}^{\gamma'} u \in L_{\text{loc}}^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T)),$$

for $|\gamma'| \leq k - \bar{k} - 1$. Since

$$A_{ij}^{nn} \partial_n D_{x'}^{\gamma'} u^j = D_{x'}^{\gamma'} w - \left((D_{x'}^{\gamma'} A_{ij}^{n\beta}) \partial_\beta u^j + \sum_{\beta \leq n-1} A_{ij}^{n\beta} (\partial_\beta D_{x'}^{\gamma'} u^j) - D_{x'}^{\gamma'} g^{ni} \right),$$

it follows, by (3.17) and (3.18), that $A_{ij}^{nn} \partial_n D_{x'}^{\gamma'} u^j \in L_{\text{loc}}^\infty(\omega \times (\delta, T))$ and

$$\begin{aligned} & \sum_{|\gamma'| \leq k - \bar{k} - 1} \|A_{ij}^{nn} \partial_n D_{x'}^{\gamma'} u\|_{L^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq 2} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned}$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+3} . Because of (1.2) and (1.3), (A_{ij}^{nn}) is a positive definite $N \times N$ matrix with eigenvalues in $[\lambda, \Lambda_0]$. Consequently, $\partial_n D_{x'}^{\gamma'} u \in L_{\text{loc}}^\infty(\omega \times (\delta, T))$ and

$$\begin{aligned} & \sum_{|\gamma'| \leq k - \bar{k} - 1} \|\partial_n D_{x'}^{\gamma'} u\|_{L^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq 2} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq 1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned}$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+3} .

On the other hand, by Lemmas 3.4 and 3.6, we have

$$\sum_{|\gamma'| \leq k - \bar{k} - 1} \|D_{x'}^{\gamma'} u_t\|_{L^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T))}$$

$$\leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k-1, s \leq 2} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k-1, s \leq 2} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right),$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+3} .

Inequality (3.19) gives us the desired bound for tangential derivatives in spatial space of u and $\partial_n u$. To estimate derivatives involving $\partial_n^k u$ for $k > 1$, we simply observe that these can be derived recursively from those already established. Indeed, according to the equation in (3.1), and by the definition of weak solution,

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j - g^{\alpha i}) + f^i \quad \text{in } \omega \times (0, T), \quad \text{in the sense of distribution.}$$

So in every $\omega_m \times (0, T)$, the equation could be rewritten piecewise as

$$A_{ij}^{nn} D_{nn} u^j = u_t^i + D_\alpha g^{\alpha i} - \sum_{\alpha+\beta \leq 2n-1} D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) - (D_n A_{ij}^{nn}) D_n u^j - f^i. \quad (3.20)$$

Since the matrix A_{ij}^{nn} has a bounded inverse, we can estimate $D_{x'}^{\gamma'} \partial_n^2 u$ pointwise for each open strip. Applying ∂_n and D_t to (3.20), we can then estimate higher derivatives. We thus obtain

$$\begin{aligned} & \sum_{r+2s \leq k-\bar{k}+1} \|D_x^r D_t^s u\|_{L^\infty((\omega_m \cap \omega_\epsilon) \times (\delta, T))} \\ & \leq C \left(\|\varphi\|_{L^2(\omega)} + \sum_{|\gamma'| \leq k, s \leq \frac{k-\bar{k}-1}{2}+1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq k, s \leq \frac{k-\bar{k}-1}{2}+1} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)} \right), \end{aligned}$$

where C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and Λ_{k+3} . Hence, $u \in C^\infty((\bar{\omega}_m \cap \omega_\epsilon) \times [\delta, T])$. Proposition 3.1 is established. $\square$

3.2 Interior estimates for the equations in domain $D \times (0, T)$

In this subsection, we consider (1.1) in $D \times (0, T)$, defined as in Section 2. Here, we still assume that $f, g \in C^\infty(\bar{D}_m \times [0, T])$, and $A_{ij}^{\alpha\beta}$ satisfies (1.2)–(1.4). We will establish the interior estimates for a weak solution of (1.1).

Lemma 3.7. Under (1.2)–(1.4), suppose $u(x, t) \in V(Q_T)$ is a weak solution of (1.1), then, for $0 < \epsilon < 1$, $0 < \delta < T$, we have

$$\|u\|_{V(D_\epsilon \times (\delta, T))}^2 \leq C(\|u\|_{L^2(Q_T)}^2 + \|f\|_{L^2(Q_T)}^2 + \|g\|_{L^2(Q_T)}^2), \quad (3.21)$$

where C depends only on λ, ϵ and δ .

Proof. By Remark 2.3, for $0 < \epsilon < 1$, $0 < \delta < T$, we have $u \in V^{1,0}(D_{\frac{\epsilon}{2}} \times (\frac{\delta}{2}, T))$. Take $\zeta = \bar{u}_h \eta^2(x, t)$ in (1.5), where $\eta(x, t)$ is a suitable cutoff function, satisfying $0 \leq \eta \leq 1$, and

$$\eta(x, t) = \begin{cases} 1 & \text{in } D_\epsilon \times (\delta, T), \\ 0 & \text{in } Q_T \setminus D_{\frac{\epsilon}{2}} \times \left(\frac{\delta}{2}, T\right), \end{cases} \quad |\eta_t| \leq \frac{C}{\delta}, \quad |D\eta| \leq \frac{C}{\epsilon}. \quad (3.22)$$

Then, instead of (2.13), we have

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_0^\tau \int_D u(\bar{u}_h \eta^2)_t dx dt \\ & = \frac{1}{2} \int_D (\eta^2 u^2)(x, \tau) dx - \frac{1}{2} \int_D (\eta^2 u^2)(x, 0) dx + 2 \int_0^\tau \int_D \eta \eta_t u^2 dx dt \\ & = \frac{1}{2} \int_D (\eta^2 u^2)(x, \tau) dx + 2 \int_0^\tau \int_D \eta \eta_t u^2 dx dt. \end{aligned}$$

In addition, passing to the limit as $h \rightarrow 0$ in other terms, it leads to

$$\begin{aligned} & \frac{1}{2} \int_D (\eta^2 u^2)(x, \tau) dx + \int_0^\tau \int_D A_{ij}^{\alpha\beta} \eta D_\beta u \eta D_\alpha u dx dt \\ & = -2 \int_0^\tau \int_D \eta u A_{ij}^{\alpha\beta} D_\beta u D_\alpha \eta dx dt + \int_0^\tau \int_D (f u \eta^2 + \eta^2 g^\alpha D_\alpha u + 2\eta \eta_t u^2 + 2u \eta g^\alpha D_\alpha \eta) dx dt. \end{aligned}$$

Estimate (3.21) follows from the above. $\square$

For the derivatives in t , we have the following lemma.

Lemma 3.8. Under (1.2), (1.3) and (3.2), if $u \in V(Q_T)$ is a weak solution of (1.1), then for any $0 < \epsilon < 1$, $0 < \delta < T$, we have

$$\sup_{\delta \leq \tau \leq T} \int_{D_\epsilon} |Du(x, \tau)|^2 dx + \int_\delta^T \int_{D_\epsilon} |u_t|^2 dx dt \leq C \left(\|u\|_{L^2(Q_T)}^2 + \|f\|_{L^2(Q_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(Q_T)}^2 \right), \quad (3.23)$$

where C depends only on $n, N, \lambda, \Lambda_2, T, \epsilon$ and δ .

Lemma 3.9. Under (1.2), (1.3) and (3.2), if $u \in V(Q_T)$ is a weak solution of (1.1), then for any $0 < \epsilon < 1$ and $0 < \delta < T$, we have

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \sum_{|\gamma'| \leq k} \int_{D_\epsilon} |u_t(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k} \int_\delta^T \int_{D_\epsilon} |Du_t|^2 dx dt \\ & \leq C \left(\|u\|_{L^2(Q_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(Q_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(D_m \times (0, T))}^2 \right), \end{aligned} \quad (3.24)$$

where C depends only on $n, N, \lambda, \Lambda_2, T, \epsilon$ and δ .

3.3 Interior estimates for laminar systems in domain $\omega \times (0, T)$

In this subsection, we consider the laminar systems (1.1) in $\omega \times (0, T)$, defined as in Subsection 3.1. Here, we still assume that $f, g \in C^\infty(\bar{\omega}_m \times [0, T])$, and $A_{ij}^{\alpha\beta}$ satisfies (1.2), (1.3) and (3.2). We have the following proposition.

Proposition 3.10. Under (1.2), (1.3) and (3.2), let $u \in V(\Omega_T)$ be a weak solution of (1.1). Then, for $0 < \delta < T$ and for all $\gamma', D_{x'}^{\gamma'} u \in C^0(\omega \times (\delta, T))$, and for each m , $u \in C^\infty((\omega \cap \bar{\omega}_m) \times (\delta, T))$. Moreover for any small $\epsilon > 0$, $k \geq 0$, and for any m ,

$$\begin{aligned} & \sum_{r+2s \leq k} \|D_x^r D_t^s u\|_{L^\infty(\bar{\omega}_m \cap \omega_\epsilon \times (\delta, T))} \\ & \leq C \left(\|u\|_{L^2(\Omega_T)} + \sum_{|\gamma'| \leq \bar{k} + k + 1, s \leq \frac{k}{2} + 1} (\|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)} + \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)}) \right), \end{aligned} \quad (3.25)$$

where

$$\bar{k} = \left\lceil \frac{n-1}{2} \right\rceil + 1$$

and C depends only on $\epsilon, \delta, k, n, N, \lambda, T$ and $\Lambda_{\bar{k}+k+2}$.

Corollary 3.11. Suppose A is constant in x and smooth in t in $\omega_m \times (0, T)$, satisfying (1.2), (1.3), and f and g are constant in $\omega_m \times (0, T)$. If $u \in V(\Omega_T)$ is a weak solution of (1.1), then, for $0 < \delta < T$ and for all $\gamma', D_{x'}^{\gamma'} u \in C^0(\omega \times (\delta, T))$, and for any $0 < \epsilon < 1$, $k \geq 0$, we have

$$\|u\|_{C^{k, k/2}((\omega_m \cap \omega_\epsilon) \times (\delta, T))} \leq C (\|u\|_{L^2(\Omega_T)} + \|f\|_{L^\infty(\Omega_T)} + \|g\|_{L^\infty(\Omega_T)}),$$

where $C = C(\epsilon, k, n, N, \lambda, \Lambda, T, A)$.

The proof of Proposition 3.10 will need the following lemmas, analogically as in Subsection 3.1. From the arguments similar to those in the derivation of Lemma 3.3–3.5, just replacing (2.11) by (3.21) with $D = \omega$, we have the following higher interior estimates for the solution of (1.1).

Lemma 3.12. Under (1.2), (1.3) and (3.2), if $u \in V(\Omega_T)$ is a weak solution of (1.1), then for any $0 < \epsilon < 1$, $0 < \delta < T$ and positive integer k , for $1 \leq |\gamma'| \leq k$, $D_{x'}^{\gamma'} u \in W_2^{1,0}(\omega_\epsilon \times (\delta, T))$, and we have

$$\begin{aligned} & \sum_{|\gamma'| \leq k} \sup_{\delta \leq \tau \leq T} \int_{\omega_\epsilon(\tau)} |D_{x'}^{\gamma'} u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k} \int_\delta^T \int_{\omega_\epsilon} |DD_{x'}^{\gamma'} u|^2 dx dt \\ & \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k} \|D_{x'}^{\gamma'} g\|_{L^2(\Omega_T)}^2 \right), \end{aligned}$$

where C depends only on $n, N, \lambda, \Lambda_k, k, \epsilon$ and δ .

Lemma 3.13. Under (1.2), (1.3) and (3.2), if $u \in V(\Omega_T)$ is a weak solution of (3.1), then for any $0 < \delta < T$ and $0 < \epsilon < 1$, we have, for $|\gamma'| \leq k$, and for $l \geq 0$,

$$\begin{aligned} & \sum_{|\gamma'| \leq k, s \leq l} \sup_{\delta \leq \tau \leq T} \int_{\omega_\epsilon} |DD_{x'}^{\gamma'} D_t^s u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k, s \leq l+1} \int_\delta^T \int_{\omega_\epsilon} |D_{x'}^{\gamma'} D_t^s u|^2 dx dt \\ & \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l+1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)}^2 \right), \end{aligned}$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+2}, T, k, \epsilon$ and δ , and

$$\begin{aligned} & \sup_{\delta \leq \tau \leq T} \sum_{|\gamma'| \leq k, s \leq l} \int_{\omega_\epsilon} |D_{x'}^{\gamma'} D_t^s u(x, \tau)|^2 dx + \sum_{|\gamma'| \leq k, s \leq l} \int_\delta^T \int_{\omega_\epsilon} |DD_{x'}^{\gamma'} D_t^s u|^2 dx dt \\ & \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} D_t^s f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\omega_m \times (0, T))}^2 \right), \end{aligned}$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+2}, T, k, \epsilon$ and δ .

Lemma 3.14. For $0 < \delta < T$, $0 < \epsilon < 1$, and for $|\gamma'| \leq k$, $s \leq l$, $D_{x'}^{\gamma'} D_t^s w$, $D_{x'}^{\gamma'} D_t^s \partial_n w \in L_{\text{loc}}^2((\omega_\epsilon \cap \omega_m) \times (\delta, T))$,

$$\begin{aligned} & \sum_{|\gamma'| \leq k, s \leq l} \int_\delta^T \int_{\omega_\epsilon} |D_{x'}^{\gamma'} D_t^s w|^2 dx dt + \sum_{|\gamma'| \leq k-1, s \leq l} \int_\delta^T \int_{\omega_\epsilon \cap \omega_m} |D_{x'}^{\gamma'} D_t^s \partial_n w|^2 dx dt \\ & \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l} \|D_{x'}^{\gamma'} f\|_{L^2(\Omega_T)}^2 + \sum_{|\gamma'| \leq k, s \leq l+1} \|D_{x'}^{\gamma'} D_t^s g\|_{L^2(\Omega_T)}^2 \right), \end{aligned}$$

where C depends only on $n, N, \lambda, \Lambda_{k+2l+1}, T, k, \epsilon$ and δ .

The proof of Proposition 3.10 can be established by Lemmas 3.6 and 3.12–3.14.

4 A perturbation lemma

In this section, we will give a perturbation lemma for parabolic systems. For simplicity, here we still suppose that ω is the unit cube. For $0 < \lambda < \Lambda_0 < \infty$, $\mathcal{A}(\lambda, \Lambda_0)$ denotes the class of measurable vector-valued functions $(A_{ij}^{\alpha\beta}(x, t))$ satisfying (1.2) and (1.3). Denote

$$\Omega_T^{(\sigma)} = (1 - \sigma)\omega \times (\sigma T, T).$$

In this section we use, unless otherwise stated, C to denote various positive constants whose values may change from line to line and which depend only on n, N, λ, Λ_2 and T .

Lemma 4.1. For $0 < \epsilon < 1$, suppose $A, B \in \mathcal{A}(\lambda, \Lambda_0)$ satisfies

$$\int_0^T \int_{\omega} |A - B| < \epsilon. \quad (4.1)$$

If A is C^1 in t , then for any $f \in W_2^{0,1}(\omega \times (0, T))$, $g \in W_2^{0,1}(\omega \times (0, T))$ and the solution $u \in V(\omega \times (0, T))$ of

$$(u^i)_t - D_{\alpha}(A_{ij}^{\alpha\beta}(x, t)D_{\beta}u^j) = -D_{\alpha}g^{\alpha i} + f^i \quad \text{in } \Omega_T = \omega \times (0, T),$$

there exists some solution $v \in V(\Omega_T^{(\frac{1}{4})})$ of

$$(v^i)_t - D_{\alpha}(B_{ij}^{\alpha\beta}(x, t)D_{\beta}v^j) = 0 \quad \text{in } \Omega_T^{(\frac{1}{4})},$$

and we have

$$\|u - v\|_{V(\Omega_T^{(\frac{1}{2})})} \leq C(\|g\|_{L^2(\Omega_T)} + \|f\|_{L^2(\Omega_T)} + \epsilon^{\gamma}(\|u\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Omega_T)} + \|f\|_{L^2(\Omega_T)}^2)),$$

where C and γ depend on n, N, λ, Λ_2 and T .

In our proof of Lemma 4.1, we make use of Theorem A.1, an extension of a classical result of Campanato [9] for strongly parabolic systems to parabolic systems with coefficients satisfying only (1.2) and (1.3). See Appendix A.

Proof of Lemma 4.1. **Step 1.** By Lemmas 3.7 and 3.9 with $D = \omega$, we have

$$\|u\|_{W_2^{1,1}(\Omega_T^{(\frac{1}{2})})}^2 \leq C\left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\omega \times (0, T))}^2 + \|f\|_{L^2(\Omega_T)}^2\right).$$

By Fubini's theorem, there exists $\frac{1}{8} < \sigma < \frac{1}{4}$ such that

$$\int_{\partial((1-\sigma)\omega)} \left(\int_{\frac{1}{5}T}^T (|u|^2 + |u_t|^2 + |\nabla u|^2) dt \right) dS \leq C\|u\|_{W_2^{1,1}(\Omega_T^{(\frac{1}{5})})}^2. \quad (4.2)$$

By Lemmas 3.7, 3.9 and 3.12 with $D = \omega$, for any $\frac{T}{10} \leq \tau \leq T$, we have

$$\begin{aligned} \int_{(1-\sigma)\omega} |u(x, \tau)|^2 dx &\leq C(\|u\|_{L^2(\Omega_T)}^2 + \|g\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2), \\ \int_{(1-\sigma)\omega} |Du(x, \tau)|^2 dx &\leq C\left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2\right), \\ \int_{(1-\sigma)\omega} |u_t(x, \tau)|^2 dx &\leq C\left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{(1-\sigma)\omega} (|u|^2 + |u_t|^2 + |\nabla u|^2)(x, \sigma T) dx + \int_{(1-\sigma)\omega} (|u|^2 + |u_t|^2 + |\nabla u|^2)(x, T) dx \\ \leq C\left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2\right). \end{aligned}$$

Then combining with (4.2), we have

$$\int_{\partial(\Omega_T^{(\sigma)})} (|u|^2 + |u_t|^2 + |\nabla u|^2) \leq C\left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2\right).$$

Fix some $0 < \delta < 1$, then take $U \in H^{\frac{3}{2}-\delta}(\Omega_T^{(\sigma)})$ (the usual fractional Sobolev space in $n+1$ dimensions) as an extension of u on $\partial\Omega_T^{(\sigma)}$ such that

$$\|U\|_{H^{\frac{3}{2}-\delta}(\Omega_T^{(\sigma)})}^2 \leq C \int_{\partial(\Omega_T^{(\sigma)})} (|u|^2 + |u_t|^2 + |\nabla u|^2).$$

By the Sobolev embedding theorem,

$$\|U_t\|_{L^{\bar{p}}(\Omega_T^{(\sigma)})} + \|\nabla U\|_{L^{\bar{p}}(\Omega_T^{(\sigma)})} \leq C\|U\|_{H^{3/2-\delta}(\Omega_T^{(\sigma)})},$$

where

$$\bar{p} = \frac{2n+2}{n+2\delta} \in \left(2, \frac{2n+2}{n}\right).$$

It implies that

$$\|U_t\|_{L^{\bar{p}}(\Omega_T^{(\sigma)})}^2 + \|\nabla U\|_{L^{\bar{p}}(\Omega_T^{(\sigma)})}^2 \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2 \right). \quad (4.3)$$

Step 2. There exists $v \in V(\Omega_T^{(\sigma)})$ satisfying

$$\begin{cases} -(v^i)_t + D_\alpha(B_{ij}^{\alpha\beta}(x, t)D_\beta v^j) = 0 & \text{in } \Omega_T^{(\sigma)}, \\ v(x, t) = u(x, t) & \text{on } \partial((1-\sigma)\omega) \times (\sigma T, T), \\ v(x, t) = u(x, t) & \text{on } (1-\sigma)\omega \times \{\sigma T\}. \end{cases}$$

Since $U \in W_2^{1,1}(\Omega_T^{(\sigma)}) \subset V(\Omega_T^{(\sigma)})$, then $v - U \in \dot{V}(\Omega_T^{(\sigma)})$ and there holds

$$-(v^i - U^i)_t + D_\alpha(B_{ij}^{\alpha\beta}(x, t)D_\beta(v^j - U^j)) = U_t^i - D_\alpha(B_{ij}^{\alpha\beta}(x, t)D_\beta U^j) \quad \text{in } \Omega_T^{(\sigma)}.$$

Applying Theorem A.1, we have for all $2 \leq p < p_0$, where p_0 is the one in Theorem A.1 (to determine the above constant δ) depending only on n, λ and Λ_0 , such that

$$\|\nabla(v - U)\|_{L^p(\Omega_T^{(\sigma)})} \leq C(\|U_t\|_{L^p(\Omega_T^{(\sigma)})} + \|\nabla U\|_{L^p((1-\sigma)\omega \times (\sigma T, T))}).$$

Now we choose δ such that $2 < \bar{p} < p_0$ with $\bar{p} = p$. Then,

$$\|\nabla(v - U)\|_{L^p(\Omega_T^{(\sigma)})}^2 \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2 \right).$$

Recalling (4.3), we have

$$\|\nabla v\|_{L^p(\Omega_T^{(\sigma)})}^2 \leq C \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2 \right).$$

Step 3. Combining the equations of u 's and v 's, we obtain that

$$-(u^i - v^i)_t + D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta(u^j - v^j)) = D_\alpha g^{\alpha i} + f^i + D_\alpha((B_{ij}^{\alpha\beta} - A_{ij}^{\alpha\beta})D_\beta v^j) \quad \text{in } \Omega_T^{(\sigma)}.$$

Since $u - v = 0$ on $\partial_p(\Omega_T^{(\sigma)})$, the parabolic boundary of $\Omega_T^{(\sigma)}$, it follows from Lemma 2.4 that

$$\begin{aligned} & \sup_{\sigma T \leq \tau \leq T} \int_{(1-\sigma)\omega} |(u - v)(x, \tau)|^2 dx + \|\nabla(u - v)\|_{L^2(\Omega_T^{(\sigma)})}^2 \\ & \leq C \left(\|g\|_{L^2(\Omega_T^{(\sigma)})}^2 + \|f\|_{L^2(\Omega_T^{(\sigma)})}^2 + \int_{\sigma T}^T \int_{(1-\sigma)\omega} |(B - A)\nabla v|^2 dx dt \right). \end{aligned}$$

By Hölder's inequality and (4.1), we have

$$\begin{aligned} & \sup_{\frac{1}{2}T \leq \tau \leq T} \int_{\frac{1}{2}\omega} |(u - v)(x, \tau)|^2 dx + \|\nabla(u - v)\|_{L^2(\Omega_T^{(\frac{1}{2})})}^2 \\ & \leq C \left(\|g\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \left(\int_{\sigma T}^T \int_{(1-\sigma)\omega} |B - A|^{\frac{2p}{p-2}} dx dt \right)^{\frac{p-2}{p}} \left(\int_{\sigma T}^T \int_{(1-\sigma)\omega} |\nabla v|^p dx dt \right)^{\frac{2}{p}} \right) \\ & \leq C(\|g\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \epsilon^{\frac{p-2}{p}} \|\nabla v\|_{L^p(\Omega_T^{(\sigma)})}^2) \\ & \leq C \left(\|g\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \epsilon^{\frac{p-2}{p}} \left(\|u\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s g\|_{L^2(\Omega_T)}^2 + \sum_{s \leq 1} \|D_t^s f\|_{L^2(\Omega_T)}^2 \right) \right). \end{aligned}$$

Taking $\gamma = \frac{p-2}{2p}$, Lemma 4.1 is proved. $\square$

5 Estimates of $|u|$ and preliminaries for estimates of $|\nabla u|$

Before proving Theorem 1.2, we derive uniform L^∞ and gradient estimates in this section. In order to estimate $|u|$ and $|\nabla u|$ at a point (x, t) in some $D_m \times (0, T)$, we need only consider the case that for some m_0 , $X^0 = (x^0, t^0)$ is in $D_{m_0} \times (0, T)$ and close to the lateral boundary $\partial D_{m_0} \times (0, T)$, and estimate $|u|$ and $|\nabla u|$ at X^0 . In that case we approximate the problem by laminar systems with a finite number of strips.

We shall focus on a neighborhood of X^0 in this section. Without loss of generality, we take x^0 as the origin in $\mathbb{R}^n$ and $t^0 = 1 < T$, i.e., $X^0 = (0, 1)$. By suitable rotation and scaling, we suppose there lie a finite number of the ∂D_m in the cube $\omega = (-\frac{1}{2}, \frac{1}{2})^n$, and these hypersurfaces take the form

$$x_n = f_m(x'), \quad x' \in \left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1}, \quad m = 1, \dots, l,$$

and

$$-\frac{1}{2} = f_0(x') < f_1(x') < \dots < f_l(x') < f_{l+1} = \frac{1}{2},$$

where $f_m \in C^{1,\alpha}([-\frac{1}{2}, \frac{1}{2}]^{n-1})$, thus we have $l+1$ regions

$$D_m = \left\{ x = (x', x_n) \in \omega \mid f_m(x') < x_n < f_{m+1}(x'), x' \in \left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1} \right\}, \quad 0 \leq m \leq l.$$

We suppose that $f_{m_0}(0') < 0 < f_{m_0+1}(0')$, and $(0', f_{m_0+1}(0'))$ is the closest point on ∂D_{m_0} to the origin. So that

$$\nabla' f_{m_0+1}(0') = 0.$$

After rotation and scaling, (1.1) still have the same form, and the coefficient conditions (1.2) and (1.3) still hold. We now consider the equation in $\Omega_1 = \omega \times (0, 1)$. Denote the cylinder

$$Q(0, r) = r\omega \times (1 - r^2, 1),$$

where

$$r\omega = \left\{ x \in \mathbb{R}^n \mid |x_i| < \frac{r}{2}, i = 1, \dots, n \right\},$$

then $Q(0, 1) = \Omega_1 = \omega \times (0, 1)$. Our desired estimate for $\nabla u(0, 1)$ is given by the following proposition.

Proposition 5.1. Suppose the coefficients $A(x, t) \in C^{\mu, 1}(\overline{D_m} \times (0, 1))$ ($0 < \mu < 1$) satisfy (1.2) and (1.3), with $\{D_m\}_{m=0}^l$ as above. If $u \in V(\Omega_1)$ is a weak solution of (1.1), then for $0 < \alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$,

$$|u(0, 1)| + |\nabla u(0, 1)| \leq C \left(\|u\|_{L^2(\Omega_1)} + \|f\|_{L^\infty(\Omega_1)} + \max_{1 \leq m \leq l+1} \|g\|_{C^{\alpha', 0}(\overline{D_m} \times (0, 1))} \right),$$

where C depends only on $n, N, l, \alpha, \mu, \lambda, \Lambda_0, T, \max_{0 \leq m \leq l} \|f_m\|_{C^{1,\alpha}([-\frac{1}{2}, \frac{1}{2}]^{n-1})}$ and

$$\max_{1 \leq m \leq l} \|A\|_{C^{\alpha', 1}(\overline{D_m} \times [0, 1])}.$$

The L^∞ estimate of $|u|$ in (1.6) and (1.7) in Theorem 1.2 follows from Proposition 5.1. The proof of Proposition 5.1 will use the perturbation Lemma 4.1 in $Q(0, 1)$. We approximate the system “A” by a laminar system with coefficients $\overline{A}$ that are piecewise smooth functions. Precisely, we introduce strips in ω ,

$$\omega_m = \{x \in \omega : f_m(0') < x_n < f_{m+1}(0')\},$$

and define the coefficients $\overline{A}$ as

$$\overline{A}(x, t) = \begin{cases} \lim_{\substack{y \in D_m \\ y \rightarrow (0', f_m(0'))}} A_{ij}^{\alpha\beta}(y, t), & x \in \omega_m \times (0, 1), \quad m > m_0, \\ A_{ij}^{\alpha\beta}(0, t), & x \in \omega_{m_0} \times (0, 1), \\ \lim_{\substack{y \in D_m \\ y \rightarrow (0', f_{m+1}(0'))}} A_{ij}^{\alpha\beta}(y, t), & x \in \omega_m \times (0, 1), \quad m < m_0. \end{cases}$$

Using f and g , we similarly define $\tilde{F}$ and $\tilde{G}$, respectively. We measure the difference $A - \bar{A}$ in terms of a norm $\|\cdot\|_{Y^{s,p}}$ defined below on $Q(0,1)$.

Definition 5.2. For $s > 0$, $1 \leq p < \infty$, and any vector- or matrix-valued function F , we introduce the norm

$$\|F\|_{Y^{s,p}} = \sup_{0 < r < 1} r^{1-s} \left(\frac{1}{|Q(0,r)|} \int_{Q(0,r)} |F|^p dx dt \right)^{1/p}.$$

Lemma 5.3. Take

$$0 < \alpha' < \min \left\{ \mu, \frac{\alpha}{2(\alpha+1)} \right\},$$

and A and $\bar{A}$ defined as above. Then there exists a positive constant E , depending only on $n, l, \alpha, \alpha', \lambda, \Lambda_0$, $\max_{0 \leq m \leq l+1} \|A\|_{C^{\alpha',0}(\bar{D}_m \times (0,1))}$ and $\max_{0 \leq m \leq l} \|f_m\|_{C^{1,\alpha}([- \frac{1}{2}, \frac{1}{2}]^{n-1})}$, such that

$$\|A - \bar{A}\|_{Y^{1+\alpha',2}} \leq E.$$

It can be proved in the same way as [22, Lemma 5.2]. For reader's convenience, we present the proof here.

Proof of Lemma 5.3. Due to the definition of D_m and ω_m , and [22, Lemma 5.1], we have

$$r^{-n} |(\omega_m \cap r\omega) \setminus D_m| \leq Cr^{\frac{\alpha}{1+\alpha}}.$$

Then by the definition of A and $\bar{A}$,

$$\begin{aligned} & \left(\frac{1}{|Q(0,r)|} \int_{Q(0,r)} |A - \bar{A}|^2 dx dt \right)^{1/2} \\ &= \left(\frac{1}{r^{n+2}} \sum_m \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A - \bar{A}|^2 dx dt + \frac{1}{r^{n+2}} \sum_m \int_{1-r^2}^1 \int_{(\omega_m \cap r\omega) \setminus D_m} |A - \bar{A}|^2 dx dt \right)^{1/2} \\ &\leq \left(\frac{1}{r^{n+2}} \sum_m \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A - \bar{A}|^2 dx dt \right)^{1/2} + Cr^{\frac{\alpha}{2(1+\alpha)}}, \end{aligned} \quad (5.1)$$

where C depends only on Λ_0, l and the $C^{1,\alpha}$ norm of f_m , $1 \leq m \leq l$. The first term in the right-hand side of (5.1) requires a slightly different estimate, depending on whether $m < m_0$, $m = m_0$ or $m > m_0$. For $m < m_0$,

$$\begin{aligned} & \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A - \bar{A}|^2 dx dt \right)^{1/2} \\ &= \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A(x,t) - A(0', f_m(0'), t)|^2 dx dt \right)^{1/2} \\ &\leq C \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |x - (0', f_m(0'))|^{2\alpha'} dx dt \right)^{1/2} \leq Cr^{\alpha'}, \end{aligned}$$

where C depends only on $\|A\|_{C^{\alpha',0}(\bar{D}_m \times (0,1))}$ ($\alpha' < \mu$). For $m = m_0$,

$$\begin{aligned} & \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_{m_0} \cap r\omega \cap D_{m_0}} |A - \bar{A}|^2 dx dt \right)^{1/2} \\ &= \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_{m_0} \cap r\omega \cap D_{m_0}} |A(x,t) - A(0,t)|^2 dx dt \right)^{1/2} \leq Cr^{\alpha'}, \end{aligned}$$

and for $m > m_0$,

$$\left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A - \bar{A}|^2 dx dt \right)^{1/2}$$

$$\begin{aligned}
&= \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |A(x, t) - A(0', f_{m-1}(0'), t)|^2 dx dt \right)^{1/2} \\
&\leq C \left(\frac{1}{r^{n+2}} \int_{1-r^2}^1 \int_{\omega_m \cap r\omega \cap D_m} |x - (0', f_{m-1}(0'))|^{2\alpha'} dx dt \right)^{1/2} \leq Cr^{\alpha'}.
\end{aligned}$$

In either case we therefore conclude from (5.1) that

$$\left(\frac{1}{|Q(0, r)|} \int_{Q(0, r)} |A - \bar{A}|^2 dx dt \right)^{1/2} \leq C(r^{\alpha'} + r^{\frac{\alpha}{2(1+\alpha)}}) \leq Cr^{\alpha'}.$$

We now choose $E = C$, then the lemma follows. $\square$

We now prove Proposition 5.1. The method we use here is an adaption of that of Li and Nirenberg [21] for the elliptic case. See also related papers of Caffarelli [7] and Caffarelli and Cabré [8].

Proof of Proposition 5.1. For simplicity, we treat the case $f^i = g^{\alpha i} \equiv 0$. We will show that

$$|u(0, 1)| + |\nabla u(0, 1)| \leq C\|u\|_{L^2(\Omega_1)}. \quad (5.2)$$

By Lemma 5.3,

$$\|A - \bar{A}\|_{Y^{1+\alpha', 2}} \leq E.$$

In fact, we can further assume that

$$\|A - \bar{A}\|_{Y^{1+\alpha', 2}} \leq \epsilon_0, \quad (5.3)$$

for some small enough $\epsilon_0 > 0$ (depending only on $n, N, \lambda, \Lambda, \alpha'$ and E). Indeed, we pick r_0 satisfying $r_0^{\alpha'}(1 + E) = \epsilon_0$ and let

$$\widehat{A}(x, t) = A(r_0 x, r_0^2(t-1) + 1), \quad \widehat{\bar{A}}(x, t) = \bar{A}(r_0 x, r_0^2(t-1) + 1),$$

and

$$\widehat{u}(x, t) = u(r_0 x, r_0^2(t-1) + 1).$$

A simple calculation yields

$$\begin{aligned}
&r^{-\alpha'} \left(\frac{1}{|Q(0, r)|} \int_{Q(0, r)} |\widehat{A} - \widehat{\bar{A}}|^2 dx dt \right)^{1/2} \\
&= r^{-\alpha'} \left(\frac{1}{r^{n+2}} \int_{\substack{|x| < r \\ 1-r^2 < t < 1}} |A(r_0 x, r_0^2(t-1) + 1) - \bar{A}(r_0 x, r_0^2(t-1) + 1)|^2 dx dt \right)^{1/2} \\
&= r^{-\alpha'} \left(\frac{1}{(r_0 r)^{n+2}} \int_{Q(0, r_0 r)} |A - \bar{A}|^2(y, s) dy ds \right)^{1/2},
\end{aligned}$$

so

$$\|\widehat{A} - \widehat{\bar{A}}\|_{Y^{1+\alpha', 2}} \leq r_0^{\alpha'} \|A - \bar{A}\|_{Y^{1+\alpha', 2}} \leq \epsilon_0,$$

and, since $f^i = g^{\alpha i} \equiv 0$,

$$(\widehat{u})_t - \partial(\widehat{A}\partial\widehat{u}) = 0 \quad \text{in } \Omega_1.$$

In the following, we will always assume that for sufficiently small ϵ_0 , (5.3) holds and u is normalized to satisfy $\|u\|_{L^2(\Omega_1)} \leq 1$. We will find

$$w_k \in V\left(Q\left(0, \frac{3}{4^{k+1}}\right)\right), \quad k \geq 0,$$

such that for all $k \geq 0$,

$$(w_k)_t - \partial(\bar{A}\partial w_k) = 0, \quad Q\left(0, \frac{3}{4^{k+1}}\right), \quad (5.4)$$

$$\|w_k\|_{L^2(Q(0, \frac{2}{4^{k+1}}))} \leq C' 4^{-\frac{k(n+4+2\alpha')}{2}}, \quad \|Dw_k\|_{L^\infty(Q(0, \frac{1}{4^{k+1}}))} \leq C' 4^{-k\alpha'}, \quad (5.5)$$

$$\|D_t w_k\|_{L^\infty(Q(0, \frac{1}{4^{k+1}}))} \leq C' 4^{-k(\alpha'-1)}, \quad \|DD_t w_k\|_{L^\infty(Q(0, \frac{1}{4^{k+1}}))} \leq C' 4^{-k(\alpha'-2)}, \quad (5.6)$$

$$\left\| u - \sum_{j=0}^k w_j \right\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} \leq 4^{-\frac{k(n+4+2\alpha')}{2}}. \quad (5.7)$$

In the proof of (5.4)–(5.6), C , C' and ϵ_0 denote various constants that depend only on parameters specified in the proposition. In particular, they are independent of k . C will be chosen first and will be large, then C' (much larger than C), and finally $\epsilon_0 \in (0, 1)$ (much smaller than $1/CC'$).

By Lemma 4.1, we can find $w_0 \in V(Q(0, \frac{3}{4}))$, such that

$$(w_0)_t - \partial(\bar{A}\partial w_0) = 0, \quad (x, t) \in Q\left(0, \frac{3}{4}\right),$$

with

$$\|u - w_0\|_{V(Q(0, \frac{1}{2}))} \leq C\epsilon_0^\gamma,$$

so

$$\|u - w_0\|_{L^2(Q(0, \frac{1}{2}))} \leq \frac{1}{2}C\epsilon_0^\gamma,$$

and

$$\|w_0\|_{L^2(Q(0, \frac{1}{2}))} \leq \|u\|_{L^2(Q(0, \frac{1}{2}))} + \|u - w_0\|_{L^2(Q(0, \frac{1}{2}))} \leq C (\leq C').$$

Recalling the definition of $\bar{A}$, it follows from Corollary 3.11 that

$$\|Dw_0\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C \leq C', \quad \|D_t w_0\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C', \quad \|DD_t w_0\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C'.$$

So far, we have verified (5.4)–(5.7) for $k = 0$. Suppose that (5.4)–(5.7) hold up to k ($k \geq 0$), and we will prove them for $k + 1$. Let

$$\begin{aligned} W_{k+1}(x, t) &= \left(u - \sum_{j=0}^k w_j\right) \left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \\ A_{k+1}(x, t) &= A\left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \quad \bar{A}_{k+1}(x, t) = \bar{A}\left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \\ g_{k+1}(x, t) &= -\frac{1}{4^{k+1}}(A_{k+1} - \bar{A}_{k+1})(x, t) \sum_{j=0}^k \partial w_j \left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right). \end{aligned}$$

Then W_{k+1} satisfies

$$(W_{k+1})_t - \partial(A_{k+1}\partial W_{k+1}) = -\partial g_{k+1}, \quad (x, t) \in Q(0, 1). \quad (5.8)$$

By a simple calculation, using the fact that $|Q(0, 1)| = 1$ and (5.3), we obtain

$$\begin{aligned} \|A_{k+1} - \bar{A}_{k+1}\|_{L^2(Q(0, 1))} &= \left(\frac{1}{|Q(0, \frac{1}{4^{k+1}})|} \int_{Q(0, \frac{1}{4^{k+1}})} |A - \bar{A}|^2\right)^{1/2} \\ &\leq \frac{1}{4^{(k+1)\alpha'}} \|A - \bar{A}\|_{Y^{1+\alpha', 2}} \leq \frac{\epsilon_0}{4^{(k+1)\alpha'}}. \end{aligned} \quad (5.9)$$

By Lemma 4.1, there exists $v_{k+1} \in V(Q(0, \frac{3}{4}))$ such that

$$-(v_{k+1})_t + \partial(\bar{A}_{k+1}\partial v_{k+1}) = 0 \quad \text{in } Q\left(0, \frac{3}{4}\right),$$

and

$$\|W_{k+1} - v_{k+1}\|_{V(Q(0, \frac{1}{2}))}$$

$$\leq C \left(\|g_{k+1}\|_{L^2(Q(0,1))} + \left(\frac{\epsilon_0}{4^{(k+1)\alpha'}} \right)^\gamma (\|W_{k+1}\|_{L^2(Q(0,1))} + \|D_t g_{k+1}\|_{L^2(Q(0,1))}) \right). \quad (5.10)$$

In the following, we will estimate these three terms on the right-hand side of (5.10). Making a change of variable and using (5.7), we first have

$$\begin{aligned} \|W_{k+1}\|_{L^2(Q(0,1))} &= \left(4^{(k+1)(n+2)} \iint_{Q(0, \frac{1}{4^{k+1}})} \left| u - \sum_{j=0}^k w_j \right|^2(y, s) dy ds \right)^{1/2} \\ &\leq 4^{\frac{(k+1)(n+2)}{2}} 4^{-\frac{k(n+4+2\alpha')}{2}} \\ &\leq \frac{C}{4^{(k+1)(1+\alpha')}}. \end{aligned} \quad (5.11)$$

Furthermore, we need to estimate the L^2 norm of g_{k+1} and $D_t g_{k+1}$. In fact, by the induction hypothesis (5.5) and (5.6), we have

$$\|g_{k+1}\|_{L^2(Q(0,1))} \leq \frac{1}{4^{k+1}} \|A_{k+1} - \bar{A}_{k+1}\|_{L^2(Q(0,1))} \sum_{j=0}^k C' 4^{-j\alpha'}, \quad (5.12)$$

and

$$\begin{aligned} \|D_t g_{k+1}\|_{L^2(Q(0,1))} &\leq \frac{1}{4^{k+1}} \|D_t A_{k+1} - D_t \bar{A}_{k+1}\|_{L^2(Q(0,1))} \sum_{j=0}^k C' 4^{-j\alpha'} \\ &\quad + \frac{1}{4^{k+1}} \|A_{k+1} - \bar{A}_{k+1}\|_{L^2(Q(0,1))} \sum_{j=0}^k \frac{1}{4^{2(k+1)}} \cdot C' 4^{-j(\alpha'-2)}. \end{aligned} \quad (5.13)$$

Recalling the definition of A_{k+1} and $\bar{A}_{k+1}$, and using the smoothness of A , we have

$$\begin{aligned} \left(\int_{Q(0,1)} |D_t(A_{k+1} - \bar{A}_{k+1})|^2 dx dt \right)^{1/2} &\leq \frac{1}{4^{2(k+1)}} \left(\frac{1}{|Q(0, \frac{1}{4^{k+1}})|} \int_{Q(0, \frac{1}{4^{k+1}})} |D_t(A - \bar{A})|^2 dx dt \right)^{1/2} \\ &\leq \frac{2}{4^{2(k+1)}} \|D_t A\|_{L^\infty(Q(0, \frac{1}{4^{k+1}}))} \\ &\leq \frac{C}{4^{2(k+1)}}. \end{aligned}$$

Therefore, combining these estimates with (5.9), (5.12) and (5.13), we have

$$\|g_{k+1}\|_{L^2(Q(0,1))} \leq \frac{1}{4^{k+1}} \cdot \frac{\epsilon_0}{4^{(k+1)\alpha'}} \cdot \frac{4^{\alpha'}}{4^{\alpha'} - 1} C' \leq \frac{CC'\epsilon_0}{4^{(k+1)(1+\alpha')}}, \quad (5.14)$$

and

$$\|D_t g_{k+1}\|_{L^2(Q(0,1))} \leq \frac{CC'}{4^{3(k+1)}}. \quad (5.15)$$

So substituting (5.11), (5.14) and (5.15) into (5.10), we have

$$\|W_{k+1} - v_{k+1}\|_{V(Q(0, \frac{1}{2}))} \leq CC' \max\{\epsilon_0^\gamma, \epsilon_0\} \cdot \frac{1}{4^{(k+1)(1+\alpha')}}, \quad (5.16)$$

and

$$\begin{aligned} \|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} &\leq \|W_{k+1} - v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} + \|W_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \\ &\leq \max\{\epsilon_0^\gamma, \epsilon_0\} \cdot \frac{CC'}{4^{(k+1)(1+\alpha')}} + \frac{C}{4^{(k+1)(1+\alpha')}}. \end{aligned}$$

Let

$$w_{k+1}(x, t) = v_{k+1}(4^{k+1}x, 4^{2(k+1)}(t-1) + 1), \quad (x, t) \in Q\left(0, \frac{3}{4^{k+2}}\right).$$

A change of variables yields (5.4), and

$$\begin{aligned} & \|w_{k+1}\|_{L^2(Q(0, \frac{1}{4^{k+2}}))} \\ &= \left(\int_{Q(0, \frac{1}{4^{k+2}})} |v_{k+1}|^2(4^{k+1}x, 4^{2(k+1)}(t-1) + 1) dx dt \right)^{1/2} \\ &= \left(4^{-(k+1)(n+2)} \int_{Q(0, \frac{1}{4})} |v_{k+1}|^2(y, s) dy ds \right)^{1/2} \\ &= \frac{1}{4^{\frac{(k+1)(n+2)}{2}}} \|v_{k+1}\|_{L^2(Q(0, \frac{1}{4}))} \\ &\leq C' 4^{-\frac{(k+1)(n+4+2\alpha')}{2}}, \end{aligned}$$

i.e., (5.5) is obtained for $k+1$;

$$\begin{aligned} & \left\| u - \sum_{j=0}^{k+1} w_j \right\|_{L^2(Q(0, \frac{1}{4^{k+2}}))} \\ &= \left(4^{-(k+1)(n+2)} \int_{Q(0, \frac{1}{4})} |W_{k+1} - v_{k+1}|^2(x, t) dx dt \right)^{1/2} \\ &= \frac{1}{4^{\frac{(k+1)(n+2)}{2}}} \|W_{k+1} - v_{k+1}\|_{L^2(Q(0, \frac{1}{4}))} \\ &\leq 4^{-\frac{(k+1)(n+4+2\alpha')}{2}}, \end{aligned}$$

i.e., (5.7) holds for $k+1$.

Combining the above and Corollary 3.11, we have

$$\begin{aligned} \|Dv_{k+1}\|_{L^\infty(Q(0, \frac{1}{4}))} &\leq C \|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \leq \frac{2C}{4^{(k+1)(1+\alpha')}}, \\ \|D_tv_{k+1}\|_{L^\infty(Q(0, \frac{1}{4}))} &\leq C \|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \leq \frac{2C}{4^{(k+1)(1+\alpha')}}, \end{aligned}$$

and

$$\|DD_tv_{k+1}\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C \|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \leq \frac{2C}{4^{(k+1)(1+\alpha')}}.$$

By a change of variables, we have

$$\|Dw_{k+1}\|_{L^\infty(Q(0, \frac{1}{4^{k+2}}))} \leq \frac{C'}{4^{(k+1)\alpha'}}, \quad \|D_tw_{k+1}\|_{L^\infty(Q(0, \frac{1}{4^{k+2}}))} \leq \frac{C'}{4^{(k+1)(\alpha'-1)}},$$

and

$$\|DD_tw_{k+1}\|_{L^\infty(Q(0, \frac{1}{4^{k+2}}))} \leq \frac{C'}{4^{(k+1)(\alpha'-2)}}.$$

Estimates (5.5) and (5.6) for $k+1$ follow from the above estimates. Thus we have established (5.4)–(5.7) for all k .

As an easy consequence of (5.5) and (5.6), we have

$$\|w_k(\cdot, t)\|_{L^\infty(Q(0, \frac{1}{4^{k+1}}))} \leq \frac{C}{4^{k(1+\alpha')}}. \quad (5.17)$$

For $|x| < \frac{1}{4^{k+1}}$ and $1 - \frac{1}{4^{2(k+1)}} < t \leq 1$, using (5.6) and (5.17), we have

$$\left| \sum_{j=0}^k w_j(x, t) - \sum_{j=0}^\infty w_j(0, 1) \right| \leq \left| \sum_{j=0}^k w_j(x, t) - \sum_{j=0}^k w_j(0, 1) \right| + \sum_{j=k+1}^\infty |w_j(0, 1)|$$

$$\begin{aligned}
&\leq C \left(\sum_{j=0}^k \frac{|x|}{4^{j\alpha'}} + \sum_{j=0}^k \frac{|1-t|}{4^{j(\alpha'-1)}} \right) + C \sum_{j=k+1}^{\infty} \frac{1}{4^{j(1+\alpha')}} \\
&\leq C \left(\sum_{j=0}^k \frac{|x|}{4^{j\alpha'}} + \sum_{j=0}^k \frac{\sqrt{|1-t|}}{4^{j\alpha'}} \right) + C 4^{-k(1+\alpha')} \\
&\leq C(|x| + \sqrt{|1-t|}) + C 4^{-k}.
\end{aligned}$$

Then

$$\left\| \sum_{j=0}^k w_j - \sum_{j=0}^{\infty} w_j(0, 1) \right\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} \leq \frac{C}{4^{\frac{k(n+4)}{2}}}.$$

So, in view of (5.7), we have

$$\begin{aligned}
&\left\| u - \sum_{j=0}^{\infty} w_j(0, 1) \right\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} \\
&\leq \left\| u - \sum_{j=0}^k w_j \right\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} + \left\| \sum_{j=0}^k w_j - \sum_{j=0}^{\infty} w_j(0, 1) \right\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} \\
&\leq \frac{C}{4^{\frac{k(n+4)}{2}}}.
\end{aligned} \tag{5.18}$$

Thus, after sending $k \rightarrow \infty$,

$$u(0, 1) = \sum_{j=0}^{\infty} w_j(0, 1). \tag{5.19}$$

The estimate of $|u(0, 1)|$ in (5.2) is established. By Taylor expansion,

$$u(x, t) - u(0, 1) = \nabla_x u(0, 1) \cdot x + O(|x|^2 + |1-t|).$$

Using (5.18) and (5.19), we have

$$\begin{aligned}
\|\nabla_x u(0, 1) \cdot x\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} &\leq \|u - u(0, 1)\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} + C\|x^2 + |1-t|\|_{L^2(Q(0, \frac{1}{4^{k+1}}))} \\
&\leq \frac{C}{4^{\frac{k(n+4)}{2}}}.
\end{aligned} \tag{5.20}$$

Let

$$e = \frac{\nabla_x u(0, 1)}{|\nabla_x u(0, 1)|}.$$

Then

$$\begin{aligned}
\left(\iint_{\substack{x \cdot e > \frac{1}{2}|x| \\ (x, t) \in Q(0, \frac{1}{4^{k+1}})}} |\nabla_x u(0, 1) \cdot x|^2 dx dt \right)^{1/2} &\geq \left(\iint_{\substack{x \cdot e > \frac{1}{2}|x| \\ (x, t) \in Q(0, \frac{1}{4^{k+1}})}} \left(\frac{1}{2} |\nabla_x u(0, 1)| \cdot |x| \right)^2 dx dt \right)^{1/2} \\
&\geq \frac{1}{C} |\nabla_x u(0, 1)| \left(\iint_{Q(0, \frac{1}{4^{k+1}})} |x|^2 dx dt \right)^{1/2} \\
&= \frac{|\nabla_x u(0, 1)|}{C} \frac{1}{4^{\frac{k(n+2)}{2}}}.
\end{aligned} \tag{5.21}$$

Combining (5.20) and (5.21), it implies that

$$|\nabla u(0, 1)| \leq C.$$

Estimate (5.2) is established. We have completed the proof of Proposition 5.1 for $f^i = g^{\alpha i} \equiv 0$. For the general case, by the method used in [22, Proposition 5.3], suppose

$$\|u\|_{L^2(\Omega_1)} + \|f\|_{L^\infty(\Omega_1)} + \max_{1 \leq m \leq l+1} \|g\|_{C^{\alpha, 0}(\overline{D}_m \times (0, 1))} \leq 1,$$

and we can obtain the same assertion. We leave the details to the interested readers. $\square$

6 Hölder estimates of the gradient

Theorem 1.2 can be deduced from the following proposition. We use the notation of Section 5.

Proposition 6.1. *Let A be as in Section 5, and let $u \in V(\Omega_1)$ be a solution of*

$$(u^i)_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta u^j) = 0 \quad \text{in } \Omega_1. \quad (6.1)$$

Then for all $x \in D_{m_0} \cap \frac{1}{2}\omega$,

$$|\nabla u(x, 1) - \nabla u(0, 1)| \leq C \|u\|_{L^2(\Omega_1)} |x|^{\alpha'},$$

where

$$\alpha' < \min \left\{ \mu, \frac{\alpha}{2(1+\alpha)} \right\},$$

and C depends only on $n, N, l, \alpha, \epsilon, \lambda, \Lambda_2, \mu$, the $C^{1,\alpha}$ norm of ω_m and $\|A\|_{C^{\alpha',1}(\overline{D}_m \times [0,T])}$.

6.1 Beginning of the proof of Proposition 6.1

As explained in Section 5, we may assume without loss of generality that

$$\|u\|_{L^2(\Omega_1)} \leq 1 \quad \text{and} \quad \|A - \overline{A}\|_{Y^{1+\alpha',2}} \leq \epsilon_0,$$

where ϵ_0 is the small constant in Section 5. To prove the $C^{1,\alpha}$ estimate, we slightly strengthen (5.4)–(5.7). Namely, we show that we can find $\{w_k\}_{k=0}^\infty$ in $V(Q(0, \frac{3}{4^{k+1}}))$ such that for $k \geq 0$, w_k satisfy, in addition to (5.4)–(5.7), for any $1 - \frac{1}{4^{2(k+1)}} \leq t \leq 1$,

$$\left\| u(\cdot, t) - \sum_{j=0}^{k+1} w_j(\cdot, t) \right\|_{L^2(\frac{1}{4^{k+1}}\omega)} \leq 4^{-\frac{(k+1)(n+2+2\alpha')}{2}} \quad (6.2)$$

and

$$\|D^2 w_k\|_{L^\infty((\omega_m \cap \frac{1}{4^{k+1}}\omega) \times (1 - \frac{1}{4^{2(k+1)}}, 1))} \leq C 4^{k(1-\alpha')}. \quad (6.3)$$

These estimates will be used in the proof of Proposition 6.1.

Proof of (6.2) and (6.3). We will prove those $\{w_k\}_{k=0}^\infty$, found in Proposition 5.1, also satisfy (6.2) and (6.3). First, for $k = 0$, by Lemma 4.1, we can find $w_0 \in V(Q(0, \frac{3}{4}))$, such that

$$(w_0)_t - \partial(\overline{A}\partial w_0) = 0, \quad (x, t) \in Q\left(0, \frac{3}{4}\right),$$

with

$$\|u - w_0\|_{V(Q(0, \frac{1}{2}))} \leq C\epsilon_0^\gamma,$$

and by Corollary 3.11, we have

$$\|D^2 w_0\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C \|w_0\|_{L^2(Q(0, \frac{1}{2}))} \leq C \leq C'.$$

Suppose that (5.4)–(5.7), (6.2) and (6.3) hold up to k ($k \geq 0$), and we will prove them for $k+1$. Let

$$\begin{aligned} W_{k+1}(x, t) &= \left(u - \sum_{j=0}^k w_j\right) \left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \\ A_{k+1}(x, t) &= A\left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \quad \overline{A}_{k+1}(x, t) = \overline{A}\left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right), \\ g_{k+1}(x, t) &= -\frac{1}{4^{k+1}}(A_{k+1} - \overline{A}_{k+1})(x, t) \sum_{j=0}^k \partial w_j \left(\frac{x}{4^{k+1}}, \frac{t-1}{4^{2(k+1)}} + 1\right). \end{aligned}$$

Then W_{k+1} satisfies

$$(W_{k+1})_t - \partial(A_{k+1}\partial W_{k+1}) = -\partial g_{k+1}, \quad (x, t) \in Q(0, 1).$$

There exists $v_{k+1} \in V(Q(0, \frac{3}{4}))$ such that

$$-(v_{k+1})_t + \partial(\bar{A}_{k+1}\partial v_{k+1}) = 0 \quad \text{in } Q\left(0, \frac{3}{4}\right),$$

with

$$\begin{aligned} & \|W_{k+1} - v_{k+1}\|_{V(Q(0, \frac{1}{2}))} \\ & \leq C \left(\|g_{k+1}\|_{L^2(Q(0, 1))} + \left(\frac{\epsilon_0}{4^{(k+1)\alpha'}} \right)^\gamma (\|W_{k+1}\|_{L^2(Q(0, 1))} + \|D_t g_{k+1}\|_{L^2(Q(0, 1))}) \right) \\ & \leq CC' \max\{\epsilon_0^\gamma, \epsilon_0\} \cdot \frac{1}{4^{(k+1)(1+\alpha')}} \end{aligned}$$

and

$$\|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \leq \max\{\epsilon_0^\gamma, \epsilon_0\} \cdot \frac{CC'}{4^{(k+1)(1+\alpha')}} + \frac{C}{4^{(k+1)(1+\alpha')}}.$$

Let

$$w_{k+1}(x, t) = v_{k+1}(4^{k+1}x, 4^{2(k+1)}(t-1) + 1), \quad (x, t) \in Q\left(0, \frac{3}{4^{k+2}}\right).$$

Then we have, for $1 - \frac{1}{4^{2(k+2)}} \leq \tau \leq 1$,

$$\begin{aligned} & \left(\int_{\frac{1}{4^{k+2}}\omega} \left| u(x, \tau) - \sum_{j=0}^{k+1} w_j(x, \tau) \right|^2 dx \right)^{1/2} \\ & = \left(4^{-(k+1)n} \int_{\frac{1}{4}\omega} |W_{k+1} - v_{k+1}|^2(x, \tau) dx \right)^{1/2} \\ & = \frac{1}{4^{\frac{(k+1)n}{2}}} \|W_{k+1} - v_{k+1}\|_{V(Q(0, \frac{1}{4}))} \\ & \leq 4^{-\frac{(k+1)(n+2+2\alpha')}{2}}. \end{aligned}$$

Similar to the proof of (5.5) and (5.6), in view of Corollary 3.11, we have

$$\|D^2 v_{k+1}\|_{L^\infty(Q(0, \frac{1}{4}))} \leq C \|v_{k+1}\|_{L^2(Q(0, \frac{1}{2}))} \leq \frac{2C}{4^{(k+1)(1+\alpha')}}.$$

By a change of variables, we have (6.3). □

Similar to that in [21], associated with

$$\bar{A}^{(m)} := \bar{A}|_{\omega_m \times (0, T)},$$

we introduce a linear transformation $N^{(m)} : \mathbb{R}^{nN} \rightarrow \mathbb{R}^{nN}$ as follows: For $b = (b_\alpha^i) \in \mathbb{R}^{nN}$ ($1 \leq \alpha \leq n, 1 \leq i \leq N$),

$$\begin{aligned} (N^{(m)}b)_\alpha^i &= b_\alpha^i, \quad 1 \leq \alpha \leq n-1, \quad 1 \leq i \leq N, \\ (N^{(m)}b)_n^i &= \bar{A}_{ij}^{(m)n\beta} b_\beta^i, \quad 1 \leq i \leq N. \end{aligned}$$

Since $(\bar{A}_{ij}^{(m)n\beta})$ is a positive definite $N \times N$ matrix with eigenvalues in $[\lambda, \Lambda_0]$, it is clear that $N^{(m)}$ is invertible and

$$\|N^{(m)}\| \leq C(n, N, \lambda, \Lambda_0), \quad \|(N^{(m)})^{-1}\| \leq C(n, N, \lambda, \Lambda_0). \quad (6.4)$$

We also define linear transformations $T^{(m)} : \mathbb{R}^{nN} \rightarrow \mathbb{R}^{nN}$ by setting

$$T^{(m)} = (N^{(m)})^{-1} N^{(m_0)}.$$

Lemma 6.2. *It holds that*

$$\nabla u(0, 1) = \sum_{j=0}^{\infty} \nabla w_j(0, 1), \quad (6.5)$$

and for $x \in (\frac{1}{4^{k+1}}\omega \cap \omega_m) \setminus \frac{1}{4^{k+2}}\omega$,

$$\left| \sum_{j=0}^k \nabla w_j(x, 1) - \sum_{j=0}^k T^{(m)} \nabla w_j(0, 1) \right| \leq C|x|^{\alpha'}. \quad (6.6)$$

Proof. We first prove (6.5). For

$$\frac{1}{4^{k+1}}\omega \subset \omega_{m_0},$$

it follows from (6.3) that

$$|w_j(x, 1) - [w_j(0, 1) + \nabla w_j(0, 1)x]| \leq 4^{j(1-\alpha')}|x|^2, \quad j \leq k, \quad x \in \frac{1}{4^{k+1}}\omega.$$

This, and (6.2), yield

$$\begin{aligned} & \left\| u(x, 1) - \left[\sum_{j=0}^k w_j(0, 1) + \nabla w_j(0, 1)x \right] \right\|_{L^2(\frac{1}{4^{k+1}}\omega)} \\ & \leq \left\| u(x, 1) - \sum_{j=0}^k w_j(x, 1) \right\|_{L^2(\frac{1}{4^{k+1}}\omega)} + \left\| \sum_{j=0}^k [w_j(x, 1) - w_j(0, 1) - \nabla w_j(0, 1)x] \right\|_{L^2(\frac{1}{4^{k+1}}\omega)} \\ & \leq 4^{-\frac{k(n+2+2\alpha')}{2}} + C \sum_{j=0}^k 4^{j(1-\alpha')} \| |x|^2 \|_{L^2(\frac{1}{4^{k+1}}\omega)} \\ & \leq C 4^{-\frac{k(n+2)}{2}}. \end{aligned} \quad (6.7)$$

From (5.17) and (5.5), we know that $\sum_{j=0}^{\infty} w_j(0, 1)$ and $\sum_{j=0}^{\infty} \nabla w_j(0, 1)$ are convergent and

$$\left| \sum_{j=0}^{\infty} w_j(0, 1) - \sum_{j=0}^k w_j(0, 1) \right| \leq C \frac{1}{4^{k(1+\alpha')}}, \quad (6.8)$$

$$\left| \sum_{j=0}^{\infty} \nabla w_j(0, 1) - \sum_{j=0}^k \nabla w_j(0, 1) \right| \leq C \frac{1}{4^{k\alpha'}}. \quad (6.9)$$

Combining (6.7)–(6.9), we have

$$\left\| u(x, 1) - \left[\sum_{j=0}^{\infty} w_j(0, 1) + \sum_{j=0}^{\infty} \nabla w_j(0, 1)x \right] \right\|_{L^2(\frac{1}{4^{k+1}}\omega)} \leq \frac{C}{4^{k(n+2)/2}}.$$

Equation (6.5) follows from the above.

Next, we prove (6.6). The matching condition of w_j at $x_n = c_{m-1}$ is, for all $x' \in (-\frac{1}{2}, \frac{1}{2})^{n-1}$,

$$N^{(m)} \nabla w_j^{(m)}(x', c_{m-1}, 1) = N^{(m-1)} \nabla w_j^{(m-1)}(x', c_{m-1}, 1), \quad (6.10)$$

where $w_j^{(m)}(\cdot, 1) = w_j(\cdot, 1)|_{\omega_m}$.

For $m = m_0$, (6.6) follows from (6.3). We will only show (6.6) for $m \geq m_0 + 1$ since the proof is the same for $m \leq m_0 - 1$. For

$$x = (x', x_n) \in \frac{1}{4^{k+1}}\omega \cap \omega_m \setminus \frac{1}{4^{k+2}}\omega, \quad m \geq m_0 + 1,$$

we have

$$\begin{aligned} & \sum_{j=0}^k |\nabla w_j^{(m)}(x, 1) - T^{(m)} \nabla w_j(0, 1)| \\ & \leq \sum_{j=0}^k |\nabla w_j^{(m)}(x, 1) - \nabla w_j^{(m)}(0', c_{m-1}, 1)| + \sum_{j=0}^k |\nabla w_j^{(m)}(0', c_{m-1}, 1) - T^{(m)} \nabla w_j(0, 1)|. \end{aligned}$$

By (6.3),

$$|\nabla w_j^{(m)}(x, 1) - \nabla w_j^{(m)}(0', c_{m-1}, 1)| \leq C4^{j(1-\alpha')}(|x'| + x_n - c_{m-1}) \leq C4^{j(1-\alpha')}|x|.$$

By (6.4), (6.10) and (6.3),

$$\begin{aligned} & |\nabla w_j^{(m)}(0', c_{m-1}, 1) - T^{(m)} \nabla w_j(0, 1)| \\ & \leq C|N^{(m)} \nabla w_j^{(m)}(0', c_{m-1}, 1) - N^{(m_0)} \nabla w_j^{(m_0)}(0, 1)| \\ & \leq C \sum_{i=m_0+2}^m |N^{(i)} \nabla w_j^{(i)}(0', c_{i-1}, 1) - N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-2}, 1)| \\ & \quad + C|N^{(m_0+1)} \nabla w_j^{(m_0+1)}(0', c_{m_0}, 1) - N^{(m_0)} \nabla w_j^{(m)}(0, 1)| \\ & \leq C \sum_{i=m_0+2}^m |N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-1}, 1) - N^{(i-1)} \nabla w_j^{(i-1)}(0', c_{i-2}, 1)| \\ & \quad + C|N^{(m_0)} \nabla w_j^{(m_0)}(0', c_{m_0}, 1) - N^{(m_0)} \nabla w_j^{(m)}(0, 1)| \\ & \leq C \left(\sum_{i=m_0+2}^m 4^{j(1-\alpha')}(c_{i-1} - c_{i-2}) + 4^{j(1-\alpha')}(c_{m_0} - 0) \right) \\ & = C4^{j(1-\alpha')}c_{m-1} \\ & \leq C4^{j(1-\alpha')}|x|. \end{aligned}$$

It follows that

$$\sum_{j=0}^k |\nabla w_j^{(m)}(x, 1) - T^{(m)} \nabla w_j(0, 1)| \leq C4^{k(1-\alpha')}|x| \leq C4|x|^{\alpha'}.$$

Estimate (6.6) is established; so is Lemma 6.2. $\square$

Lemma 6.3. Let $\bar{x}$ be on the x_n -axis and $\bar{x} + a|\bar{x}|\omega \subset (D_{m+1} \cap \omega_{m+1})$ for some $a > 0$. Then

$$\left| \nabla u(y, 1) - \sum_{j=0}^k \nabla w_j(y, 1) \right| \leq C(a)|\bar{x}|^{\alpha'}, \quad y \in \bar{x} + \frac{a}{2}|\bar{x}|\omega, \quad (6.11)$$

where k satisfies

$$4^{-(k+2)} \leq |\bar{x}| < 4^{-(k+1)};$$

consequently,

$$|\nabla u(y, 1) - \nabla u(z, 1)| \leq C(a)|\bar{x}|^{\alpha'}, \quad y, z \in \bar{x} + \frac{a}{2}|\bar{x}|\omega. \quad (6.12)$$

Proof. Let

$$\widehat{w}(y, t) = u(\bar{x} + a|\bar{x}|y, (a|\bar{x}|)^2(t-1) + 1) - \sum_{j=0}^k w_j(\bar{x} + a|\bar{x}|y, (a|\bar{x}|)^2(t-1) + 1), \quad y \in \omega.$$

Then $\widehat{w}$ satisfies the following systems:

$$\widehat{w}_t - \partial(A(\bar{x} + a|\bar{x}| \cdot, (a|\bar{x}|)^2(t-1) + 1)\partial\widehat{w}) = \partial\widehat{g} \quad \text{in } \omega \times (0, 1),$$

where

$$\begin{aligned}\widehat{g} = & -a|\bar{x}| \sum_{j=0}^k (A^{(m+1)}(\bar{x} + a|\bar{x}|y, (a|\bar{x}|)^2(t-1) + 1) \\ & - A^{(m+1)}(0', c_m, (a|\bar{x}|)^2(t-1) + 1)) \cdot \partial w_j(\bar{x} + a|\bar{x}|y, (a|\bar{x}|)^2(t-1) + 1),\end{aligned}$$

with

$$A^{(m+1)}(\cdot, (a|\bar{x}|)^2(t-1) + 1) := A(\cdot, (a|\bar{x}|)^2(t-1) + 1)|_{D_{m+1} \times (0,1)}.$$

Since $\bar{x} + a|\bar{x}|\omega \in (D_{m+1} \cap \omega_{m+1})$, the $C^\mu(\omega)$ -seminorm of $A^{(m+1)}(\bar{x} + a|\bar{x}| \cdot, (a|\bar{x}|)^2(t-1) + 1)$ is bounded by $C(a)|\bar{x}|^\mu$. Thus, by (5.6) and (6.3),

$$\|\widehat{g}\|_{C^\mu(\omega)} \leq C(a)|\bar{x}|^{1+\mu}.$$

We also deduce from (6.2) that, for $1 - \frac{1}{4^{2(k+1)}} < t \leq 1$,

$$\|\widehat{w}(\cdot, t)\|_{L^2(\omega)} \leq C(a)|\bar{x}|^{1+\mu}.$$

By the Schauder theory,

$$\|\nabla \widehat{w}(\cdot, 1)\|_{L^\infty(\frac{1}{2}\omega)} \leq C(a)|\bar{x}|^{1+\alpha'}.$$

Estimate (6.11) follows from the above. Estimate (6.12) follows from (6.11) and (6.3). $\square$

6.2 Completion of the proof of Proposition 6.1

For some small r_1 , depending only on the parameters specified in Proposition 6.1, if x satisfies $|x| \geq r_1$, the desired estimate in Proposition 6.1 follows from the gradient estimates in Proposition 5.1. So we always assume that $x \in D_{m_0} \setminus \{0\}$ and $|x| < r_1$. In the following, we repeatedly use the smallness of $|x|$. We select an $\bar{x}$ as follows. If $c_{m_0} > 80|x|$, set $\bar{x} = (0', 10|x|)$ (and $m = m_0$), otherwise let $m \geq m_0 + 1$ be the smallest index for which $c_{m+1} - c_m > 80|x|$, and set $\bar{x} = (0', c_m + 10|x|)$. Clearly, $10|x| \leq |\bar{x}| \leq 100(l+1)|x|$ and $\bar{x} + a|x|\omega \subset D_{m+1} \cap \omega_{m+1}$, with $a = 8$. With this choice of $\bar{x}$, let k satisfy

$$\frac{1}{4^{k+2}} \leq |\bar{x}| \leq \frac{1}{4^{k+1}}.$$

Then by (6.5), (6.6) and (6.11), we have

$$\begin{aligned}& |\nabla u(\bar{x}, 1) - T^{(m)} \nabla u(0, 1)| \\ & \leq \left| \nabla u(\bar{x}, 1) - \sum_{j=0}^k \nabla w_j(\bar{x}, 1) \right| + \left| \sum_{j=0}^k \nabla w_j(\bar{x}, 1) - \sum_{j=0}^\infty T^{(m)} \nabla w_j(0, 1) \right| \\ & \leq C|\bar{x}|^{\alpha'} \leq C|x|^{\alpha'}.\end{aligned}\tag{6.13}$$

Let z be on either the graph of f_{m_0} or f_{m_0+1} , so that the distance of x to z is the least distance of x to the union of graphs of $\{f_i\}$. Let L be the line passing through z that is normal to this graph. Clearly $x \in L$. Let $z^{(j)}$ denote the intersection of L with the graph of f_i for $m_0 \leq j \leq m+1$. Using the smallness of $|x|$ and the $C^{\alpha'}$ property of $\{f_i\}$, it is not difficult to see that

$$|z^{(j)} - (0', f_j(0'))| \leq 4|x|, \quad m_0 \leq j \leq m,\tag{6.14}$$

and

$$|z^{(m+1)} - z^{(m)}| \geq 40|x|.$$

Here, m is as defined before, and we have used the fact that the point $(0', f_{m_0}(0'))$ is the projection of the origin onto the graph of the function f_{m_0} . The same argument shows that we can find $\bar{z}$ on the segment determined by $z^{(m)}$ and $z^{(m+1)}$ with $|\bar{z} - z^{(m)}| = 10|x|$ such that

$$|\nabla u(\bar{z}, 1) - \tilde{T}^{(m)} \nabla u(x, 1)| \leq C|x|^{\alpha'},$$

where the $\{\tilde{T}^{(m)}\}$ are defined in the natural way. Due to (6.14) and the Hölder continuity of $A^{(j)}$, we have

$$|T^{(m)} - \tilde{T}^{(m)}| \leq C|x|^\mu,$$

so

$$|\nabla u(\bar{z}, 1) - T^{(m)} \nabla u(x, 1)| \leq C|x|^{\alpha'}. \quad (6.15)$$

It is easy to see, by the smallness of r_1 and the Hölder continuity of $\{\nabla f_j\}$, that

$$|\bar{x} - \bar{z}| \leq 2|x|.$$

By (6.12),

$$|\nabla u(\bar{x}, 1) - \nabla u(\bar{z}, 1)| \leq C|\bar{x}|^{\alpha'} \leq C|x|^{\alpha'}. \quad (6.16)$$

A combination of (6.13), (6.15), (6.16) and (6.4) yields

$$|\nabla u(x, 1) - \nabla u(0, 1)| \leq C|T^{(m)}[\nabla u(x, 1) - \nabla u(0, 1)]| \leq C|x|^{\alpha'}.$$

Proposition 6.1 is established.

Similarly, we can prove the following more general proposition; we leave the details to the interested reader.

Proposition 6.4. *Let A be as in Section 5, and let $u \in V(\Omega_1)$ be a solution of*

$$(u^i)_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta u^j) = -D_\alpha g^{\alpha i} + f^i \quad \text{in } \Omega_1.$$

Then for all $x \in D_{m_0} \cap \frac{1}{2}\omega$,

$$|\nabla u(x, 1) - u(0, 1)| \leq C \left(\|u\|_{L^2(\Omega_T)} + \|f\|_{L^\infty(\Omega_T)} + \max_{1 \leq m \leq L} \|g\|_{C^{\alpha', 0}(\overline{\omega_m} \times [0, T])} \right) |x|^{\alpha'},$$

where $\alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$, and C depends only on $n, N, l, \alpha, \epsilon, \lambda, \Lambda_0, \mu, \|A\|_{C^{\alpha', 1}(\overline{\omega_m} \times [0, T])}$ and the $C^{1, \alpha}$ norm of ω_m .

6.3 Proof of Theorem 1.4

Here, for simplicity, we still only treat the case $f^i = g^{\alpha i} \equiv 0$.

Proof of Theorem 1.4. Since $A_{ij}^{\alpha\beta} \in C^{\mu, k}(\overline{D_m} \times [0, T])$, applying D_t to (6.1) and denoting $v = u_t$, we have

$$(v^i)_t - D_\alpha(A_{ij}^{\alpha\beta}(x, t)D_\beta v^j) = D_\alpha(D_t A_{ij}^{\alpha\beta}(x, t)D_\beta u^j) \quad \text{in } Q_T. \quad (6.17)$$

Since $A_t \in C^{\mu, k-1}(\overline{D_m} \times [0, T])$ and $\partial u \in C^{\alpha', 0}((D_\epsilon \cap \overline{D_m}) \times (\epsilon T, T))$, it follows that $A_t \partial u \in C^{\alpha', 0}((D_\epsilon \cap \overline{D_m}) \times (\epsilon T, T))$. Then for $l = 1$, applying Theorem 1.2 to (6.17), and in view of Lemma 3.8, we have, for any $0 < \epsilon < \frac{1}{2}$ and $\alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$,

$$\begin{aligned} & \|v\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_x v\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_x v\|_{C^{\alpha', 0}((D_\epsilon \cap \overline{D_m}) \times (\epsilon T, T))} \\ & \leq C \left(\|v\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2} T, T))} + \max_{1 \leq m \leq L} \|A_t D u\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2} T, T])} \right) \\ & \leq C \left(\|u_t\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2} T, T))} + \max_{1 \leq m \leq L} \|D u\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2} T, T])} \right) \\ & \leq C \|u\|_{L^2(Q_T)}, \end{aligned} \quad (6.18)$$

where C depends only on $n, N, L, \alpha, \epsilon, \lambda, \Lambda_2, \mu, T, \|A\|_{C^{\alpha', 2}(\overline{D_m} \times [0, T])}$ and the $C^{1, \alpha}$ norm of D_m . Estimate (1.8) for $l = 1$ is proved. To prove (1.8) for $l = 2$, we apply Lemma 3.8 to (6.17) to obtain

$$\begin{aligned} & \|v_t\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2} T, T))} \\ & \leq C \left(\|v\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4} T, T))} + \|(D_t A) \nabla u\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4} T, T))} + \|D_t[(\partial_t A) \nabla u]\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4} T, T))} \right), \end{aligned}$$

where C depends only on $n, N, \epsilon, \lambda, \Lambda_2, T$. Since

$$D_t[(D_t A)\nabla u] = (D_{tt} A)\nabla u + (D_t A)D_t(\nabla u)$$

it follows, from Lemmas 3.7 and 3.8, that

$$\begin{aligned} \|(D_t A)D_t(\nabla u)\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4}T, T))} &= \|(D_t A)\nabla v\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4}T, T))} \\ &\leq C\|\nabla v\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4}T, T))} \\ &\leq C(\|v\|_{L^2(D_{\frac{\epsilon}{8}} \times (\frac{\epsilon}{8}T, T))} + \|(D_t A)\nabla u\|_{L^2(D_{\frac{\epsilon}{8}} \times (\frac{\epsilon}{8}T, T))}) \\ &\leq C(\|u_t\|_{L^2(D_{\frac{\epsilon}{8}} \times (\frac{\epsilon}{8}T, T))} + \|\nabla u\|_{L^2(D_{\frac{\epsilon}{8}} \times (\frac{\epsilon}{8}T, T))}) \\ &\leq C\|u\|_{L^2(Q_T)}, \end{aligned}$$

where C depends only on $n, N, \epsilon, \lambda, \Lambda_2, T$. Thus we have shown

$$\begin{aligned} \|v_t\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2}T, T))} &\leq C(\|u_t\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4}T, T))} + \|\nabla u\|_{L^2(D_{\frac{\epsilon}{4}} \times (\frac{\epsilon}{4}T, T))} + \|u\|_{L^2(Q_T)}) \\ &\leq C\|u\|_{L^2(Q_T)}, \end{aligned} \quad (6.19)$$

where C depends only on $n, N, \epsilon, \lambda, \Lambda_4, T$.

Applying D_t to (6.17) and writing $w = v_t (= u_{tt})$, we have

$$(w^i)_t - D_\alpha(A_{ij}^{\alpha\beta} D_\beta w^j) = D_\alpha(D_{tt} A_{ij}^{\alpha\beta} D_\beta w^j + 2D_t A_{ij}^{\alpha\beta} D_\beta v^j) \quad \text{in } Q_T. \quad (6.20)$$

Applying Theorem 1.2 to (6.20), and in combination with (6.18) and (6.19), we have, for any $0 < \epsilon < \frac{1}{2}$ and $\alpha' < \min\{\mu, \frac{\alpha}{2(1+\alpha)}\}$,

$$\begin{aligned} &\|w\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_x w\|_{L^\infty(D_\epsilon \times (\epsilon T, T))} + \|D_x w\|_{C^{\alpha', 0}((D_\epsilon \cap \overline{D_m}) \times (\epsilon T, T))} \\ &\leq C\left(\|w\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2}T, T))} + \max_{1 \leq m \leq L} (\|A_{tt} Du\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2}T, T])} + \|A_t Dv\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2}T, T])})\right) \\ &\leq C\left(\|v_t\|_{L^2(D_{\frac{\epsilon}{2}} \times (\frac{\epsilon}{2}T, T))} + \max_{1 \leq m \leq L} (\|Du\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2}T, T])} + \|Dv\|_{C^{\alpha', 0}((D_{\frac{\epsilon}{2}} \cap \overline{D_m}) \times [\frac{\epsilon}{2}T, T])})\right) \\ &\leq C\|u\|_{L^2(Q_T)}, \end{aligned}$$

where C depends only on $n, N, L, \alpha, \epsilon, \lambda, \Lambda_4, \mu, T$, $\|A\|_{C^{\alpha', 3}(\overline{D_m} \times [0, T])}$ and the $C^{1, \alpha}$ norm of D_m . Estimate (1.8) for $l = 2$ is proved. For general k and $l > 2$, repeating this process by induction, we proved Theorem 1.4. $\square$

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11571042, 11371060 and 11631002), Fok Ying Tung Education Foundation (Grant No. 151003) and National Science Foundation of USA (Grant No. DMS-0701545). The first author is grateful to Professor Jiguang Bao for helpful comments and encouragement. Part of the work was completed while the first author was visiting Rutgers University. The first author also thanks the Department of Mathematics and the Nonlinear Analysis Center for the hospitality.

References

- 1 Almgren F, Wang L H. Mathematical existence of crystal growth with Gibbs-Thomson curvature effects. *J Geom Anal*, 2000, 10: 1–100
- 2 Babuška I, Andersson B, Smith P, et al. Damage analysis of fiber composites, I: Statistical analysis on fiber scale. *Comput Methods Appl Mech Engrg*, 1999, 172: 27–77
- 3 Bao J G, Ju H J, Li H G. Optimal boundary gradient estimates for Lamé systems with partially infinite coefficients. *Adv Math*, 2017, 314: 583–629
- 4 Bao J G, Li H G, Li Y Y. Gradient estimates for solutions of the Lamé system with partially infinite coefficients. *Arch Ration Mech Anal*, 2015, 215: 307–351

- 5 Bao J G, Li H G, Li Y Y. Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two. *Adv Math*, 2017, 305: 298–338
- 6 Bonnetier E, Vogelius M. An elliptic regularity result for a composite medium with touching fibers of circular cross-section. *SIAM J Math Anal*, 2000, 31: 651–677
- 7 Caffarelli L A. Interior a priori estimates for solutions of fully nonlinear equations. *Ann of Math (2)*, 1989, 130: 189–213
- 8 Caffarelli L A, Cabré X. *Fully Nonlinear Elliptic Equations*. American Mathematical Society Colloquium Publications, vol. 43. Providence: Amer Math Soc, 1995
- 9 Campanato S. L^p regularity for weak solutions of parabolic systems. *Ann Sc Norm Super Pisa Cl Sci (5)*, 1980, 7: 65–85
- 10 Chen Y Z. *Second Order Parabolic Partial Differential Equations*. Beijing: Peking University Press, 2002
- 11 Chipot M, Kinderlehrer D, Vergara-Caffarelli G. Smoothness of linear laminates. *Arch Ration Mech Anal*, 1986, 96: 81–96
- 12 Dong H J. Gradient estimates for parabolic and elliptic systems from linear laminates. *Arch Ration Mech Anal*, 2012, 205: 119–149
- 13 Fan J S, Kim K, Nagayasu S, et al. A gradient estimate for solutions to parabolic equations with discontinuous coefficients. *Electron J Differential Equations*, 2013, 2013: 1–24
- 14 Giaquinta M. Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. *Annals of Mathematics Studies*, vol. 105. Princeton: Princeton University, 1983
- 15 Giaquinta M, Modica G. Regularity results for some classes of higher order nonlinear elliptic systems. *J Reine Angew Math*, 1979, 311/312: 145–169
- 16 Giaquinta M, Struwe M. On the partial regularity of weak solutions of nonlinear parabolic systems. *Math Z*, 1982, 179: 437–451
- 17 Ladyzenskaja O A, Solonnikov V A, Uralceva N N. *Linear and Quasi-Linear Equations of Parabolic Type*. Providence: Amer Math Soc, 1968
- 18 Li H G. Several problems for partial differential equations from astrophysics and physics: Existence of compressible rotating Newtonian stars and linear elasticity of composite material. PhD Thesis. Beijing: Beijing Normal University, 2009
- 19 Li H G, Li Y Y. Gradient estimates for parabolic systems from composite material. *ArXiv:1105.1437v2*, 2012
- 20 Li H G, Li Y Y, Bao E S, et al. Derivative estimates of solutions of elliptic systems in narrow regions. *Quart Appl Math*, 2014, 72: 589–596
- 21 Li Y Y, Nirenberg L. Estimates for elliptic system from composite material. *Comm Pure Appl Math*, 2003, 56: 892–925
- 22 Li Y Y, Vogelius M. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch Ration Mech Anal*, 2000, 153: 91–151

Appendix A

Let $\omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary $\partial\omega$. For $0 < \lambda < \Lambda < \infty$, $T > 0$, $\mathcal{A}(\lambda, \Lambda)$ denotes the class of measurable vector-valued functions $(A_{ij}^{\alpha\beta}(x, t))$ satisfying (1.2) and (1.3), with $\Lambda_0 = \Lambda$ and $Q_T = \omega \times (0, T)$. Consider, for $T > 0$, $g \in L^2(\omega \times (0, T), \mathbb{R}^N)$ and $f \in L^2(0, T, L^2(\omega, \mathbb{R}^N))$,

$$\begin{cases} u_t^i - \partial_{x_\alpha}(A_{ij}^{\alpha\beta}(x, t)\partial_{x_\beta}u^j) = \partial_{x_\alpha}g_\alpha^i(z) + f^i(z) & \text{in } \omega \times (0, T), \quad \forall i, \\ u = 0 & \text{on } (\partial\omega \times (0, T)) \cup (\omega \times \{0\}). \end{cases} \quad (\text{A.1})$$

In the following we use the notation $2^* = \frac{2n}{n-2}$ if $n \geq 3$, $2^* = \infty$ if $n = 1, 2$.

Theorem A.1. For $n \geq 1$, $N \geq 1$, $0 < \lambda \leq \Lambda < \infty$, $T > 0$, and $A \in \mathcal{A}(\lambda, \Lambda)$, let $u \in \overset{\circ}{V}(\omega \times (0, T), \mathbb{R}^N)$ be a weak solution of (A.1). Then there exists a $2 < p_0 < 2^*$, depending only on n, N, λ and Λ , such that if $p \in [2, p_0)$, $g \in L^p(\omega \times (0, T), \mathbb{R}^N)$ and $f \in L^p(0, T, L^2(\omega, \mathbb{R}^N))$, then $u \in L^p(0, T, W_0^{1,p}(\omega, \mathbb{R}^N))$. Moreover,

$$\int_0^T \int_\omega (|u|^p + |\nabla u|^p) dx dt \leq C \left(\int_0^T \left(\int_\omega |f|^2 dx \right)^{\frac{p}{2}} dt + \int_0^T \int_\omega |g|^p dx dt \right),$$

where C depends only on $\lambda, \Lambda, n, N, \omega$ and T .

Remark A.2. The above theorem was proved by Campanato [9] for strongly parabolic systems, i.e., under

$$\lambda|\xi|^2 \leq A_{ij}^{\alpha\beta}(x, t)\xi_\alpha^i\xi_\beta^j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^{nN}. \quad (\text{A.2})$$

The method in [9] does not apply under the weaker hypotheses (1.2) and (1.3).

We introduce some standard notation:

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

$$z_0 = (x_0, t_0), \quad Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0).$$

When no confusion may arise, we shall omit x_0 and z_0 in the notation. In the following, we use Q to denote some parabolic cube, i.e., for some $(\bar{x}_0, \bar{t}_0) \in \mathbb{R}^{n+1}$ and $\bar{R} > 0$,

$$Q = \{(x, t) \in \mathbb{R}^{n+1} : \bar{t}_0 - \bar{R}^2 < t < \bar{t}_0, |x_i - \bar{x}_{0i}| < \bar{R}, i = 1, 2, \dots, n\}.$$

Appendix A.1 Theorem A.1 when $f \equiv 0$

In this subsection, we prove Theorem A.1 when $f \equiv 0$. The proof relies on the following result that can be proved exactly the same way as for the analogous elliptic one in [14, Proposition 1.1 in Chapter V] by simply changing Euclidean cubes to parabolic cubes. For $\theta > 0$ small, the elliptic one was proved in [15].

Proposition A.3. *Let Q be a parabolic cube, $q > 1$, $0 < \theta < 1$, and let h and H be two non-negative functions in $L^q(Q)$. Suppose*

$$\begin{aligned} \frac{1}{|Q_R(z_0)|} \int_{Q_R(z_0)} h^q dz &\leq b \left(\frac{1}{|Q_{4R}(z_0)|} \int_{Q_{4R}(z_0)} h dz \right)^q \\ &\quad + \frac{1}{|Q_{4R}(z_0)|} \int_{Q_{4R}(z_0)} H^q dz + \theta \frac{1}{|Q_{4R}(z_0)|} \int_{Q_{4R}(z_0)} h^q dz, \end{aligned}$$

for every $Q_{4R}(z_0) \subset Q$. Then there exist constants $\epsilon > 0$ and $C > 0$, depending only on b, q, θ, n and $|Q|$, such that for all $p \in [q, q + \epsilon)$ and all $Q_{4R}(z_0) \subset Q$,

$$\left(\frac{1}{|Q_R(z_0)|} \int_{Q_R(z_0)} h^p dz \right)^{\frac{1}{p}} \leq C \left(\left(\frac{1}{|Q_{4R}(z_0)|} \int_{Q_{4R}(z_0)} h^q dz \right)^{\frac{1}{q}} + \left(\frac{1}{|Q_{4R}(z_0)|} \int_{Q_{4R}(z_0)} H^p dz \right)^{\frac{1}{p}} \right).$$

This result was used in [16] to derive partial regularity of solutions of some nonlinear parabolic systems satisfying strongly elliptic condition (A.2). We first establish interior estimates in Theorem A.1 when $f \equiv 0$.

Proposition A.4. *Under the hypotheses of Theorem A.1 with $f \equiv 0$, there exists $2 < p_0 < 2^*$, depending only on n, N, λ and Λ , such that for any $Q_{5R} \subset \omega \times (0, T)$, and $2 \leq p < p_0$, we have*

$$\frac{1}{|Q_R|} \int_{Q_R} |\nabla u|^p dz \leq C \left(\frac{1}{|Q_{4R}|} \int_{Q_{4R}} |\nabla u|^2 \right)^{\frac{p}{2}} + \frac{1}{|Q_{4R}|} \int_{Q_{4R}} |g|^p dz, \quad (\text{A.3})$$

where C depends only on n, N, λ , and Λ .

Let $\chi(x)$ be a function in $C_c^\infty(B_2(x_0))$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $B_1(x_0)$ and $|\nabla \chi| \leq 2$. We denote

$$\chi_{2R}(x) = \chi \left(x_0 + \frac{x - x_0}{R} \right),$$

and let $\tau_{2R} \in C^\infty([t_0 - (2R)^2, t_0])$,

$$0 \leq \tau_{2R} \leq 1, \quad \tau_{2R} = 1 \quad \text{on} \quad [t_0 - R^2, t_0], \quad \tau_{2R} = 0 \quad \text{on} \quad \left[t_0 - (2R)^2, t_0 - \left(\frac{3}{2}R \right)^2 \right],$$

and they satisfy

$$|\nabla \chi_{2R}| \leq \frac{C(n)}{R}, \quad |\nabla \tau_{2R}| \leq \frac{C(n)}{R^2}.$$

We note that

$$\int_{B_{2R}} \chi_{2R}^4 = \frac{1}{R^n} \int_{B_2} \chi^4, \quad \int_{B_{2R}} \chi_{2R}^2 = \frac{1}{R^n} \int_{B_2} \chi^2. \quad (\text{A.4})$$

Define the weighted means of $u(x, t)$ in $B_{2R}(x_0)$ as

$$\bar{u}(t) := \bar{u}_{x_0, 2R}(t) = \frac{\int_{B_{2R}(x_0)} u(x, t) \chi_{2R}^2 dx}{\int_{B_{2R}(x_0)} \chi_{2R}^2(x) dx}.$$

Lemma A.5. Let $A \in \mathcal{A}(\lambda, \Lambda)$ and let $u \in \mathring{V}(\omega \times (0, T), \mathbb{R}^N)$ be a weak solution of (A.1) with $f \equiv 0$. Then for all $Q_{2R} \subset Q$, we have

$$\sup_{t_0 - R^2 \leq t \leq t_0} \int_{B_R} |u - \bar{u}_{2R}|^2 dx + \int_{Q_R} |\nabla u|^2 dz \leq \frac{C}{R^2} \int_{Q_{2R}} |u - \bar{u}_{2R}|^2 dz + C \int_{Q_{2R}} |g|^2 dz, \quad (\text{A.5})$$

where C depends only on n, N, λ and Λ .

Proof. Let $z_0 = (x_0, t_0) \in Q$, and $Q_{2R}(z_0) \subset Q$. By the definition of weak solutions, we test (A.1) with

$$\varphi = (u_h - \bar{u}_h) \eta^2,$$

where $\eta = \eta_{2R} = \chi_{2R} \tau_{2R}$,

$$u_h(x, t) := \frac{1}{2h} \int_{t-h}^{t+h} \tilde{u}(x, s) ds, \quad \text{for } 0 < h < R^2 \quad (\text{A.6})$$

and

$$\tilde{u}(x, t) = \begin{cases} u(x, \tau), & t \geq \tau, \\ u(x, t), & t_0 - (2R)^2 < t < \tau, \\ u(x, t_0 - (2R)^2), & t \leq t_0 - (2R)^2. \end{cases}$$

Then we have, for $\tau \in [t_0 - R^2, t_0]$,

$$\begin{aligned} & \int_{B_{2R}} (u\varphi)(\cdot, t_0) dx - \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} u \varphi_t dx dt + \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i dx dt \\ &= \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} g^\alpha D_\alpha \varphi dx dt. \end{aligned} \quad (\text{A.7})$$

By our choice of the test function, we know that the term

$$\int_{t_0 - (2R)^2}^{\tau} \left(\int_{B_{2R}} (u_h - \bar{u}_h) \chi^2 dx \right) (\partial_t \bar{u}) \tau_{2R}^2 dt = 0.$$

It follows that

$$\int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} \bar{u} \partial_t ((u_h - \bar{u}_h) \eta^2) dx dt = \int_{B_{2R}} (\bar{u} (u_h - \bar{u}_h) \eta^2)(x, \tau) dx.$$

Then we have

$$\begin{aligned} & \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} u \varphi_t dx dt \\ &= \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} u \partial_t ((u_h - \bar{u}_h) \eta^2) dx dt \\ &= \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u}) \partial_t ((u_h - \bar{u}_h) \eta^2) dx dt + \int_{B_{2R}} (\bar{u} (u_h - \bar{u}_h) \eta^2)(x, \tau) dx \\ &= \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u}) (u_h - \bar{u}_h)_t \eta^2 dx dt \\ &\quad + \int_{t_0 - (2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u}) (u_h - \bar{u}_h) (\partial_t (\eta^2)) dx dt + \int_{B_{2R}} (\bar{u} (u_h - \bar{u}_h) \eta^2)(x, \tau) dx. \end{aligned} \quad (\text{A.8})$$

First, we will show that

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})(u_h - \bar{u}_h)_t \eta^2 dx dt \\ &= \frac{1}{2} \int_{B_{2R}} ((u - \bar{u})^2 \eta^2)(x, \tau) dx - \frac{1}{2} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} ((u - \bar{u})^2 \partial_t(\eta^2))(x, \tau) dx. \end{aligned} \quad (\text{A.9})$$

By the definition of u_h , (A.6), we have, for $0 < h < \frac{R^2}{2}$,

$$\begin{aligned} & \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})(u_h - \bar{u}_h)_t \eta^2 dx dt \\ &= \frac{1}{2h} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})(x, t) \{(\tilde{u} - \bar{\tilde{u}})(x, t+h) - (\tilde{u} - \bar{\tilde{u}})(x, t-h)\} \eta^2(x, t) dx dt \\ &= \frac{1}{2h} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} \{ (u - \bar{u})(x, t) (\tilde{u} - \bar{\tilde{u}})(x, t+h) \eta^2(x, t+h) \\ &\quad - (u - \bar{u})(x, t) (\tilde{u} - \bar{\tilde{u}})(x, t-h) \eta^2(x, t) \} dx dt \\ &\quad + \frac{1}{2h} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})(x, t) (u - \bar{u})(x, t+h) \{ \eta^2(x, t) - \eta^2(x, t+h) \} dx dt \\ &= \frac{1}{2h} \int_{\tau}^{\tau+h} \int_{B_{2R}} (u - \bar{u})(x, t-h) (u - \bar{u})(x, \tau) \eta^2(x, t) dx dt \\ &\quad + \frac{1}{2h} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})(x, t) (u - \bar{u})(x, t+h) \{ \eta^2(x, t) - \eta^2(x, t+h) \} dx dt \\ &=: \text{I}_h + \text{II}_h. \end{aligned}$$

Clearly,

$$\lim_{h \rightarrow 0} \text{II}_h = -\frac{1}{2} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})^2 \partial_t(\eta^2) dx dt$$

and

$$\begin{aligned} & \left| 2\text{I}_h - \int_{B_{2R}} ((u - \bar{u})^2 \eta^2)(x, \tau) dx \right| \\ &\leq \frac{1}{h} \int_{\tau}^{\tau+h} \int_{B_{2R}} |(u - \bar{u})(x, t-h) - (u - \bar{u})(x, \tau)| |u - \bar{u}|(x, \tau) \eta^2 dx dt \\ &\leq \frac{C(R)}{h} \int_{\tau}^{\tau+h} \|u(\cdot, t-h) - u(\cdot, \tau)\|_{L^2(\omega)} \cdot \|(u - \bar{u})(x, \tau) \eta^2(x, \tau)\|_{L^2(\omega)} dt \\ &\rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned}$$

For the last step above, we have used the fact that $u \in V^{1,0}(\omega \times (0, T))$ (see Lemma 2.2, i.e., the conclusion and its proof are valid under our hypotheses). Thus we obtain (A.9). Again, using $u \in V^{1,0}(\omega \times (0, T))$, we have

$$\lim_{h \rightarrow 0} \int_{B_{2R}} |u - u_h|^2(x, \tau) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |u - u_h|^2 dx dt = 0.$$

It follows that

$$\lim_{h \rightarrow 0} \int_{B_{2R}} |(u - \bar{u}) - (u_h - \bar{u}_h)|^2(x, \tau) = 0,$$

and

$$\lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |(u - \bar{u}) - (u_h - \bar{u}_h)|^2 dx dt = 0.$$

Then we have

$$\lim_{h \rightarrow 0} \int_{B_{2R}} \bar{u}((u - \bar{u}) - (u_h - \bar{u}_h)) \eta^2(x, \tau) = 0,$$

$$\lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u}) \{ (u - \bar{u}) - (u_h - \bar{u}_h) \} \partial_t (\eta^2) dx dt = 0,$$

i.e.,

$$\lim_{h \rightarrow 0} \int_{B_{2R}} \bar{u} (u_h - \bar{u}_h) \eta^2(x, \tau) dx = \int_{B_{2R}} \bar{u} (u - \bar{u}) \eta^2(x, \tau) dx \quad (\text{A.10})$$

and

$$\lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u}) (u_h - \bar{u}_h) \partial_t (\eta^2) dx dt = \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})^2 \partial_t (\eta^2) dx dt. \quad (\text{A.11})$$

By (A.9)–(A.11), we have from (A.8),

$$\begin{aligned} \lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} u \varphi_t dx dt &= \int_{B_{2R}} \bar{u} (u - \bar{u})(x, \tau) \eta^2(x, \tau) dx + \frac{1}{2} \int_{B_{2R}} (u - \bar{u})^2(x, \tau) \eta^2(x, \tau) dx \\ &\quad + \frac{1}{2} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})^2 \partial_t (\eta^2) dx dt. \end{aligned} \quad (\text{A.12})$$

Since $\|u_h - u\|_{V(Q_{2R})} \rightarrow 0$ as $h \rightarrow 0$, it follows that

$$\lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta} u^j D_{\alpha} \varphi^i dx dt = \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta} (u - \bar{u})^j D_{\alpha} [(u - \bar{u}) \eta^2]^i dx dt \quad (\text{A.13})$$

and

$$\lim_{h \rightarrow 0} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} g^{\alpha} D_{\alpha} \varphi = \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} g^{\alpha} D_{\alpha} [(u - \bar{u}) \eta^2] dx dt. \quad (\text{A.14})$$

Then sending $h \rightarrow 0$ in (A.7), from (A.12)–(A.14), we have

$$\begin{aligned} &\frac{1}{2} \int_{B_{2R}} (u - \bar{u})^2(x, \tau) \eta^2(x, \tau) dx + \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta} (u - \bar{u})^j D_{\alpha} [(u - \bar{u}) \eta^2]^i dx dt \\ &= \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (g^{\alpha} D_{\alpha} [(u - \bar{u}) \eta^2]) dx dt + \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})^2 \partial_t (\eta^2) dx dt. \end{aligned}$$

By Cauchy's inequality, for any $\epsilon > 0$,

$$\begin{aligned} &\int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (g^{\alpha} D_{\alpha} [(u - \bar{u}) \eta^2]) dx dt \\ &\leq \frac{\epsilon}{4} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |\nabla [(u - \bar{u}) \eta]|^2 dx dt + C(\epsilon) \int_{Q_{2R}(z_0)} |g|^2 dz. \end{aligned}$$

By a simple computation, we have

$$\int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} (u - \bar{u})^2 \partial_t (\eta^2) dx dt \leq \frac{C}{R^2} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |u - \bar{u}|^2 dx dt$$

and

$$\begin{aligned} &\int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta} (u - \bar{u})^j D_{\alpha} [(u - \bar{u}) \eta^2]^i dx dt \\ &= \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta} [(u - \bar{u})^j \eta] D_{\alpha} [(u - \bar{u})^i \eta] dx dt \\ &\quad - \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} [(D_{\beta} \eta) (u - \bar{u})^j] D_{\alpha} [(u - \bar{u})^i \eta] dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta}[(u-\bar{u})^j \eta][(D_{\alpha}\eta)(u-\bar{u})^i] dx dt \\
& - \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} [(D_{\beta}\eta)(u-\bar{u})^j][(D_{\alpha}\eta)(u-\bar{u})^i] dx dt.
\end{aligned}$$

Then by Cauchy's inequality again and (1.2), we have

$$\begin{aligned}
& \frac{1}{2} \int_{B_{2R}} (u-\bar{u})^2(x, \tau) \eta^2(x, \tau) dx + \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} A_{ij}^{\alpha\beta} D_{\beta}[(u-\bar{u})^j \eta] D_{\alpha}[(u-\bar{u})^i \eta] dx dt \\
& \leq \epsilon \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |\nabla[(u-\bar{u})\eta]|^2 dx dt + \frac{C(\epsilon)}{R^2} \int_{t_0-(2R)^2}^{\tau} \int_{B_{2R}} |u-\bar{u}|^2 dx dt + C(\epsilon) \int_{Q_{2R}(z_0)} |g|^2 dz.
\end{aligned}$$

Therefore, by the weak parabolic condition (1.3), taking $\epsilon = \frac{\lambda}{2}$, we have

$$\int_{B_R} (u-\bar{u})^2(x, \tau) dx + \int_{Q_R} |\nabla(u-\bar{u})|^2 dz \leq \frac{C}{R^2} \int_{Q_{2R}(z_0)} |u-\bar{u}|^2 dz + C \int_{Q_{2R}(z_0)} |g|^2 dz,$$

where C depends only on n, N, λ and Λ . The proof of Lemma A.5 is completed. $\square$

Proof of Proposition A.4. Using (A.4), we have

$$\int_{B_{2R}} |u-\bar{u}_{2R}|^2 \leq C \int_{B_{2R}} |u-\bar{u}_{4R}|^2,$$

where C depends only on n . From Lemma A.5 with R replaced by $2R$, for $t_0 - (2R)^2 \leq t \leq t_0$,

$$\int_{B_{2R}} |u-\bar{u}_{4R}|^2 dx \leq \frac{C}{R^2} \int_{Q_{4R}} |u-\bar{u}_{4R}|^2 dz + \int_{Q_{4R}} |g|^2 dz. \quad (\text{A.15})$$

By Hölder's inequality, Poincaré's inequality and (A.15), we have

$$\begin{aligned}
\int_{B_{2R}} |u-\bar{u}_{2R}|^2 dx &= \left(\int_{B_{2R}} |u-\bar{u}_{2R}|^2 dx \right)^{\frac{n}{n+2}} \left(\int_{B_{2R}} |u-\bar{u}_{2R}|^2 dx \right)^{\frac{2}{n+2}} \\
&\leq C \left(\int_{B_{2R}} |\nabla u|^{\frac{2n}{n+2}} dx \right) \left(\frac{1}{R^2} \int_{Q_{4R}} |u-\bar{u}_{4R}|^2 dz + \int_{Q_{4R}} |g|^2 dz \right)^{\frac{2}{n+2}} \\
&\leq C \left(\int_{B_{2R}} |\nabla u|^{\frac{2n}{n+2}} dx \right) \left(\int_{Q_{4R}} |\nabla u|^2 dz + \int_{Q_{4R}} |g|^2 dz \right)^{\frac{2}{n+2}}. \quad (\text{A.16})
\end{aligned}$$

Integrating over t leads to, for every $\epsilon > 0$,

$$\begin{aligned}
\iint_{Q_{2R}} |u-\bar{u}_{2R}|^2 dz &\leq C \left(\int_{Q_{2R}} |\nabla u|^{\frac{2n}{n+2}} dz \right) \left(\int_{Q_{4R}} |\nabla u|^2 dz + \int_{Q_{4R}} |g|^2 dz \right)^{\frac{2}{n+2}} \\
&\leq \epsilon \left(\int_{Q_{4R}} (|\nabla u|^2 + |g|^2) dz \right) + \frac{C}{\epsilon^{\frac{2}{n}}} \left(\int_{Q_{2R}} |\nabla u|^{\frac{2n}{n+2}} dz \right)^{\frac{n+2}{n}}.
\end{aligned}$$

Using (A.5), we have

$$\begin{aligned}
\frac{1}{|Q_{2R}|} \int_{Q_{2R}} |\nabla u|^2 dz &\leq \frac{C}{R^2} \frac{1}{|Q_{2R}|} \int_{Q_{2R}} |u-\bar{u}_{2R}|^2 dz + C \frac{1}{|Q_{2R}|} \int_{Q_{2R}} |g|^2 dz \\
&\leq \frac{C\epsilon}{R^2} \left(\frac{1}{|Q_{4R}|} \int_{Q_{4R}} (|\nabla u|^2 + |g|^2) dz \right) \\
&\quad + \frac{C}{\epsilon^{\frac{2}{n}} R^{n+4}} \left(\int_{Q_{4R}} |\nabla u|^{\frac{2n}{n+2}} dz \right)^{\frac{n+2}{n}} + C \frac{1}{|Q_{2R}|} \int_{Q_{2R}} |g|^2 dz.
\end{aligned}$$

Taking $\epsilon > 0$ such that $\frac{C\epsilon}{R^2} = \frac{1}{2}$, we have

$$\epsilon^{\frac{2}{n}} R^{n+4} = C R^{\frac{4}{n} + n + 4} = C R^{\frac{4 + (n+4)n}{n}} = C R^{\frac{(n+2)^2}{n}}.$$

So

$$\frac{1}{|Q_R|} \int_{Q_R} |\nabla u|^2 dz \leq \frac{1}{2} \frac{1}{|Q_{4R}|} \int_{Q_{4R}} |\nabla u|^2 dz + C \left(\frac{1}{|Q_{4R}|} \int_{Q_{4R}} |\nabla u|^{\frac{2n}{n+2}} dz \right)^{\frac{n+2}{n}} + C \frac{1}{|Q_{4R}|} \int_{Q_{4R}} |g|^2 dz.$$

Then taking $h = |\nabla u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$ and $H = |g|^{\frac{2n}{n+2}}$, we obtain, in view of Proposition A.3, (A.3) and have proved Proposition A.4. $\square$

Given the interior estimates Proposition A.4, we now only need to establish boundary estimates analogous to (A.3).

The completion of the proof of Theorem A.1 when $f \equiv 0$. Since $\partial\omega$ is Lipschitz continuous, there exists $\bar{R} > 0$ such that for all $\bar{x} \in \partial\omega$, $\partial\omega \cap B_{\bar{R}}(\bar{x})$ is the graph of a Lipschitz function with controlled Lipschitz constant. In view of Proposition A.4, we only need to establish (A.3) for all $Q_{4R}(z_0)$ with $t_0 \leq T$, $0 < R < \frac{1}{8}\bar{R}$. Note that we allow $\Omega_T \setminus Q_{4R} \neq \emptyset$, and where u and g have been extended as zero outside $\Omega_T := \omega \times (0, T)$.

There are three cases: Case 1, where $B_{\frac{3}{2}R}(x_0) \cap \omega^c = \emptyset$, can be seen as the interior case, and has been settled; Case 2, where $B_{\frac{3}{2}R}(x_0) \subset \omega^c$, is trivial; we only need to consider Case 3, where $B_{\frac{3}{2}R}(x_0) \cap \partial\omega^c \neq \emptyset$.

For Case 3, in order to prove (A.3) in $Q_{4R}^+ := Q_{4R}(z_0) \cap \Omega_T$, we need only to replace $\bar{u}_{x_0, 2R}(t)$ by

$$\bar{u}_{x_0, 2R}^+(t) = \frac{\int_{B_{2R}(x_0) \cap \omega} u(x, t) \chi_{2R}^2 dx}{\int_{B_{2R}(x_0) \cap \omega} \chi_{2R}^2(x) dx},$$

and let $\chi(x)$ be a function in $C_c^\infty(B_2(x_0))$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $B_{\frac{3}{2}}(x_0)$ and $|\nabla \chi| \leq 4$, take

$$\chi_{2R}(x) = \chi\left(x_0 + \frac{x - x_0}{R}\right),$$

satisfying $|\nabla \chi_{2R}| \leq \frac{C(n)}{R}$. Then by the same way, we could obtain (A.3). The choice of R and ball $B_{2R}(x_0)$ guarantees the validity of the Sobolev inequality used in (A.3).

It follows that for some $p > 2$, the L^p norm of $|\nabla u|$ is controlled by the L^2 norm of $|\nabla u|$ and the L^p norm of g . On the other hand, we know that the L^2 norm of ∇u is controlled by the L^2 norm of g . Therefore we have shown that, for some $p > 2$,

$$\int_{\Omega_T} |\nabla u|^p dz \leq C \int_{\Omega_T} |g|^p dz. \quad (\text{A.17})$$

Appendix A.2 Completion of the proof of Theorem A.1

In order to complete the proof of Theorem A.1, we need the following lemma.

Lemma A.6. *Under the condition of Theorem A.1 with $g \equiv 0$, and let p_0 be as in Proposition A.4. Then for all $2 \leq p < p_0$, $f \in L^p(0, T, L^2(\omega, \mathbb{R}^N))$, we have $u \in L^p(\Omega_T)$ and*

$$\int_{\Omega_T} |\nabla u|^p dx dt \leq C \int_0^T \left(\int_{\omega} |f|^2 dx \right)^{p/2},$$

where C depends only on n, N, λ, Λ and ω .

Proof. Let U be the solution of

$$U \in H_0^1(\omega, \mathbb{R}^N) \cap H^2(\omega, \mathbb{R}^N), \quad -\Delta U^i = f^i.$$

It is known that

$$\|U\|_{W^{2,2}(\omega)} \leq C \|f\|_{L^2(\omega, \mathbb{R}^N)},$$

where C depends only on n, N and ω , and by the imbedding theorems with respect to x ,

$$\int_{\omega} |DU|^p dx \leq C \|U\|_{W^{2,2}(\omega)}^p \leq C \|f\|_{L^2(\omega, \mathbb{R}^N)}^p. \quad (\text{A.18})$$

Then the weak solution of (A.1) with $g \equiv 0$, u , satisfies, for a.e. $\tau \in (0, T)$ and for all $\varphi \in \overset{\circ}{W}_2^{1,1}(\Omega_T; \mathbb{R}^N)$,

$$\int_{\omega} (u\varphi)(\cdot, \tau) dx - \int_{\Omega_T} (A_{ij}^{\alpha\beta} \partial_{\beta} u^j \partial_{\alpha} \varphi^i - u \varphi_t) dx dt = \int_{\Omega_T} \partial_{\alpha} U^i \partial_{\alpha} \varphi^i dx dt.$$

By (A.17) and (A.18), we conclude that

$$\int_{\Omega_T} |\nabla u|^p dx dt \leq C \int_{\Omega_T} |DU|^p dx dt \leq C \int_0^T \left(\int_{\omega} |f|^2 dx \right)^{p/2} dt,$$

and this is the required assertion. $\square$

Proof of Theorem A.1. Combining the proof of Theorem A.1 with $f \equiv 0$ and Lemma A.6, the proof is completed. $\square$