

Blow-up Analysis for Solutions of $-\Delta u = V e^u$ in Dimension Two

YANYAN LI & ITAI SHAFRIR

1. Introduction. In this article we are interested in the asymptotic behavior of sequences of solutions of

$$(1) \quad -\Delta u_n = V_n(x)e^{u_n}$$

on a bounded domain $\Omega \subset \mathbf{R}^2$ with V_n a non-negative continuous function. A solution of (1) will always refer to a continuous function satisfying (1) in the distributional sense. The standard elliptic regularity theory will give $C^{1,\alpha}$ regularity. Equations like (1) appear, for instance, in problems of prescribing Gauss curvature on \mathbf{S}^2 , see Chang-Yang ([C-Y]) and the references therein. The difference with respect to our situation is that in [C-Y] solutions on all of \mathbf{S}^2 are studied, while we are interested in solutions on bounded domains. In the case $V_n(x) \equiv \lambda_n$, with u_n satisfying a boundary condition, related problems were studied by Nagasaki and Suzuki ([N-S]). Here we are concerned with the case where no boundary condition is imposed on u_n , following the approach of Brezis and Merle ([B-M]). A special case of their main result is the following:

Theorem 0 ([B-M]). *Assume $\{u_n\}$ is a sequence of solutions of (1) satisfying, for some positive constants C_0, C_1 , that*

$$(2) \quad 0 \leq V_n(x) \leq C_0, \quad \forall x \in \Omega,$$

and

$$(3) \quad \int_{\Omega} e^{u_n} \leq C_1.$$

Then there exists a subsequence $\{u_{n_k}\}$ satisfying one of the following alternatives:

- (i) $\{u_{n_k}\}$ is bounded in $L_{loc}^{\infty}(\Omega)$,
- (ii) $u_{n_k} \rightarrow -\infty$ uniformly on compact subsets of Ω ,
- (iii) there exists a finite blow-up set $S = \{a_1, \dots, a_m\} \subset \Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_{n_k}\} \subset \Omega$, $x_{n_k} \rightarrow a_i$, $u_{n_k}(x_{n_k}) \rightarrow \infty$, and $u_{n_k}(x) \rightarrow -\infty$ uniformly on compact subsets of $\Omega \setminus S$. Moreover, $V_{n_k} e^{u_{n_k}} \rightharpoonup \sum_{i=1}^m \alpha_i \delta_{a_i}$ in the sense of measure with $\alpha_i \geq 4\pi$, $\forall i$.

It was conjectured by Brezis and Merle (Open Problem 4 in [B-M]) that, under some additional regularity assumptions on $\{V_n\}$ (i.e., $V_n \rightarrow V$ in $C(\bar{\Omega})$), the mass α_i appearing in alternative (iii) should be of the form $\alpha_i = 8\pi m_i$ with m_i being some positive integer. Here we give an affirmative answer to the conjecture.

Theorem. *Suppose $V_n \in C^0(\bar{\Omega})$, $V_n \geq 0$ in Ω , $V_n \rightarrow V$ in $C^0(\bar{\Omega})$. Let $\{u_n\}$ be a sequence of solutions of (1) with $\|e^{u_n}\|_{L^1(\Omega)} \leq C_1$ for some positive constant C_1 . Assume alternative (iii) in Theorem 0 holds. Then for each i , $\alpha_i = 8\pi m_i$ with m_i being some positive integer.*

Remark. An interesting question is whether $\alpha_i > 8\pi$ may actually occur. We tend to believe that this is not possible, at least under suitable uniform regularity assumption on $\{V_n\}$. We do not know the answer even for $V_n \equiv 1$. Under more restrictive assumptions, namely, $\sup_{\partial\Omega} u_n - \inf_{\partial\Omega} u_n \leq C_2$ and $\|\nabla V_n\|_{L^\infty} \leq C_1$, Wolansky ([W]) has proved that $\alpha_i = 8\pi$ for each i (note that we are trying to avoid any boundary assumption on u_n).

Our proof consists of two main ingredients. First, by a blow-up around a local maximum of u_n , we are led to a solution of $-\Delta u = ce^u$ in \mathbf{R}^2 for some $c > 0$ with $\int_{\mathbf{R}^2} e^u < \infty$. A result of Chen-Li ([C-L], see also [C-W]) characterizes all these solutions, in particular, $\int_{\mathbf{R}^2} ce^u = 8\pi$. Applying this blow-up procedure around other local maxima, we obtain each time a contribution of 8π to the limiting mass. Since $\bar{\lim} \int_{\Omega} V_n e^{u_n} \leq C$, we must stop after a finite number of steps, say m . The second ingredient consists of showing that the limiting mass equals $8\pi m$. For that matter we need to show that there is no contribution of mass outside the m disjoint balls (whose radii going to zero) which contain a contribution of $8\pi m$ mass. Here we use a “sup+inf” type inequality proved in [S]: for a solution of $-\Delta u = Ve^u$ in Ω , with $0 < a \leq V \leq b$, we have

$$(4) \quad \sup_K u + C_1 \inf_{\Omega} u \leq C_2,$$

with positive constants $C_1 \geq 1$ and C_2 depending only on a, b , and the compact set $K \subset \Omega$. See also [B-L-S] and [T] for other “sup+inf” type inequalities.

It is evident that in order to prove our theorem, we only need to establish the following.

Proposition 1. *For $R > 0$, let $\{V_n\}$ be a sequence of functions satisfying*

$$(5) \quad 0 \leq V_n \rightarrow V \quad \text{in } C^0(\bar{B}_R),$$

where $B_R = \{x \in \mathbf{R}^2 \mid |x| < R\}$. Let $\{u_n\}$ be a sequence of solutions of (1) in $\Omega = B_R$ with the following properties:

$$(6) \quad \max_{\bar{B}_R} u_n \rightarrow \infty,$$

$$(7) \quad \max_{\bar{B}_R \setminus B_r} u_n \rightarrow -\infty, \quad \forall r \in (0, R).$$

$$\lim_{n \rightarrow \infty} \int_{B_R} V_n e^{u_n} = \alpha,$$

and for some positive constant C_0 ,

$$(8) \quad \int_{B_R} e^{u_n} \leq C_0, \quad \forall n.$$

Then $\alpha = 8\pi m$ for some positive integer m .

Proposition 2. *In addition to the hypotheses in Proposition 1, we assume for some positive constant C_2 that*

$$(9) \quad u_n(x) + 2\log|x| \leq C_2 \quad \forall x \in B_R, \forall n.$$

Then there exists some constant $C_3 > 0$ depending only on C_2 , $V(0)$ and the modulo of continuity of V at 0 such that

$$(10) \quad u_n(0) \geq \max_{\bar{B}_R} u_n - C_3, \quad \text{for all large } n,$$

$$(11) \quad \alpha = 8\pi,$$

and for any $k_n \rightarrow \infty$,

$$(12) \quad \lim_{n \rightarrow \infty} \int_{B_{R/2} \setminus B_{k_n \bar{\delta}_n}} V_n e^{u_n} = 0,$$

where $\bar{\delta}_n = e^{-u_n(0)/2}$.

2. Proofs. The proofs of the propositions above are divided into several lemmas.

Lemma 1. *Under the hypotheses of the Proposition 1, we have $V(0) > 0$ and $\alpha \geq 8\pi$.*

Proof. Let $x_n \in B_R$, $u_n(x_n) = \max_{x \in B_R} u_n(x)$. It follows from (6) and (7) that

$$(13) \quad x_n \rightarrow 0, \quad u_n(x_n) \rightarrow \infty.$$

Let $\delta_n = e^{-u_n(x_n)/2}$. It follows from (13) that $\delta_n \rightarrow 0$. For $|x| \leq (2\delta_n)^{-1}R$, we consider the sequence of functions $\tilde{u}_n(x) = u_n(\delta_n x + x_n) + 2 \log \delta_n$ which satisfy

$$\begin{aligned} -\Delta \tilde{u}_n(x) &= V_n(\delta_n x + x_n) e^{\tilde{u}_n(x)} && \forall x \in B_{R/(2\delta_n)}, \\ \tilde{u}_n(0) &= 0, \\ \tilde{u}_n(x) &\leq 0 && \forall x \in B_{R/(2\delta_n)}, \\ \int_{B_{R/(2\delta_n)}} e^{\tilde{u}_n} &\leq C_0 && \forall n. \end{aligned}$$

For each $r > 0$ the sequence $\{\tilde{u}_n\}$ is well defined in B_r for n large enough. It follows from Theorem 0 that only alternative (i) may occur, hence $\{\tilde{u}_n\}$ is bounded in $L^\infty_{\text{loc}}(B_r)$ and, by standard elliptic estimates, also in $C^{1,\alpha}_{\text{loc}}(B_r)$ ($0 < \alpha < 1$). We may do the same for a sequence $r_k \rightarrow \infty$, and pass to a diagonal subsequence (which we will still denote as $\{u_n\}$) converging in $C^{1,\alpha}_{\text{loc}}(\mathbf{R}^2)$ to \tilde{u} which satisfies

$$\begin{aligned} -\Delta \tilde{u} &= V(0) e^{\tilde{u}} && \text{in } \mathbf{R}^2, \\ \tilde{u}(0) &= 0, \\ \tilde{u} &\leq 0 && \text{in } \mathbf{R}^2, \\ \int_{\mathbf{R}^2} e^{\tilde{u}} &\leq C_0. \end{aligned}$$

Since $V(0) = 0$ would imply that \tilde{u} is harmonic in \mathbf{R}^2 with $\int_{\mathbf{R}^2} e^{\tilde{u}} < \infty$ which is impossible, we have $V(0) > 0$. It follows from a result of Chen and Li ([C-L]), that $\tilde{u}(x) = \log \{1/(1 + \gamma^2|x|^2)^2\}$ with $\gamma = (V(0)/8)^{1/2}$. In particular, $V(0) \int_{\mathbf{R}^2} e^{\tilde{u}} = 8\pi$. It follows that for every $r > 0$,

$$\alpha = \lim_{n \rightarrow \infty} \int_{B_R} V_n e^{u_n} \geq \lim_{n \rightarrow \infty} \int_{B_{r\delta_n}} V_n e^{u_n} = V(0) \int_{B_r} e^{\tilde{u}}.$$

Sending r to ∞ , we have $\alpha \geq 8\pi$. The proof of Lemma 1 is completed. □

The following lemma is a consequence of the Harnack inequality.

Lemma 2. For $R > 0$, $0 < R_0 \leq R/4$, we set $\tilde{\Omega} = \{x \in \mathbf{R}^2 \mid R_0 < |x| < R\}$. Let u be a solution of $-\Delta u = V(x)e^u$ in $\tilde{\Omega}$ with $\|V\|_{L^\infty(\tilde{\Omega})} \leq C_1$ and $u(x) + 2\log|x| \leq C_2$, $\forall x \in \tilde{\Omega}$ for some positive constants C_1 and C_2 . Then there exist constants $\beta \in (0, 1)$ and C_3 depending only on C_1 and C_2 such that

$$\sup_{\partial B_r} u \leq C_3 + \beta \inf_{\partial B_r} u + 2(\beta - 1) \log r, \quad \forall 2R_0 \leq r \leq \frac{R}{2}.$$

We emphasize that the constants β and C_3 are independent of R_0 and R .

Proof. See Appendix.

Next we give a direct consequence of the ‘‘sup + inf’’ type inequality in [S].

Lemma 3. Let $V \in L^\infty(B_R)$ satisfy $a \leq V(x) \leq b$, $\forall x \in B_R$, where a, b are positive constants. Suppose that u is a solution of $-\Delta u = V(x)e^u$ in B_R . Then

$$(14) \quad u(0) + C_1 \inf_{B_R} u + 2(C_1 + 1) \log R \leq C_2,$$

where $C_1 \geq 1$ and C_2 are constants depending only on a and b .

Proof. See Appendix.

Proof of Proposition 2. It is clear that we can pass to subsequences freely in the process of the proof. Due to (7) and Lemma 1, we may pass to a smaller disc and assume that for some positive constants a and b we have

$$(15) \quad a \leq V_n(x) \leq b \quad \forall x \in B_R, \forall n.$$

Let $x_n \in B_R$, $u_n(x_n) = \max_{x \in \bar{B}_R} u_n(x)$. Set $\tilde{u}_n(x) = u_n(\delta_n x + x_n) + 2\log \delta_n$ with $\delta_n = e^{-u_n(x_n)/2}$. It follows from (6) and (7) that (13) holds. It follows from (9) that

$$(16) \quad |x_n| \leq e^{C_2/2} \delta_n.$$

Arguing as in the proof of Lemma 1 we conclude, after passing to a subsequence, that \tilde{u}_n converges in $C_{loc}^{1,\alpha}(\mathbf{R}^2)$ to $\tilde{u}(x) = \log\{1/(1 + \gamma^2|x|^2)^2\}$ with $\gamma = (V(0)/8)^{1/2}$. As mentioned previously, \tilde{u} satisfies $-\Delta \tilde{u} = V(0)e^{\tilde{u}}$, $\tilde{u}(0) = 0$ and $V(0) \int_{\mathbf{R}^2} e^{\tilde{u}} = 8\pi$.

Noticing that $\tilde{u}_n(-(\delta_n)^{-1}x_n) = u_n(0) + 2\log \delta_n$, we deduce from (16) and the explicit formula of \tilde{u} that

$$\liminf_{n \rightarrow \infty} \tilde{u}_n(-(\delta_n)^{-1}x_n) \geq \tilde{u}(e^{C_2/2}).$$

Hence (10) follows for every $C_3 > -\tilde{u}(e^{C_2/2})$. We fix any such C_3 , that we may take bigger than C_2 .

For any given $\varepsilon > 0$, we may choose $k_0 = k_0(\varepsilon) \gg 1$ with the property that, for all $k \geq k_0$, there exists $n_0 = n_0(k, \varepsilon)$ such that for all $n \geq n_0(k, \varepsilon)$ we have

$$(17) \quad \begin{aligned} 8\pi - \frac{\varepsilon}{2} &\leq \int_{B_{k/2}} V_n(\delta_n x + x_n) e^{\tilde{u}_n(x)} dx \\ &\leq \int_{B_{2e^{C_3/2}k}} V_n(\delta_n x + x_n) e^{\tilde{u}_n(x)} dx \leq 8\pi + \frac{\varepsilon}{2}. \end{aligned}$$

It follows from (10) that

$$(18) \quad \delta_n \leq \bar{\delta}_n \leq e^{C_3/2} \delta_n.$$

Using (16), (17), (18) and a change of variable, we obtain for all $k \geq \max\{k_0, 2e^{C_2/2}\}$ that

$$(19) \quad \limsup_{n \rightarrow \infty} \left| \int_{B_{k\bar{\delta}_n}} V_n e^{u_n} - 8\pi \right| \leq \frac{\varepsilon}{2}.$$

It follows from Lemma 3 that for each $r \in (0, R]$ we have (note that u_n is superharmonic and, therefore, $\inf_{\partial B_r} u_n = \inf_{B_r} u_n$):

$$(20) \quad \inf_{\partial B_r} u_n \leq C - \frac{1}{C_1} u_n(0) - 2 \left(1 + \frac{1}{C_1} \right) \log r.$$

Here and in the following C denotes various constants depending only on the parameters.

Next we apply Lemma 2 to obtain for some $\beta \in (0, 1)$ and $C > 0$ that

$$(21) \quad \sup_{\partial B_r} u_n \leq C + \beta \inf_{\partial B_r} u_n + 2(\beta - 1) \log r, \quad \forall r \in \left(0, \frac{R}{2} \right].$$

Combining (20) with (21) we conclude that

$$\sup_{\partial B_r} u_n \leq C - \frac{\beta}{C_1} u_n(0) - 2 \left(\frac{\beta}{C_1} + 1 \right) \log r, \quad \forall r \in \left(0, \frac{R}{2} \right],$$

namely,

$$e^{u_n(x)} \leq C(\bar{\delta}_n)^{2\beta/C_1} |x|^{-2(\beta/C_1+1)}, \quad \forall 0 < |x| \leq \frac{R}{2}.$$

Finally, for any $k > 1$ we have

$$(22) \quad \int_{B_{R/2} \setminus B_{k\bar{\delta}_n}} V_n e^{u_n} \leq C(\bar{\delta}_n)^{2\beta/C_1} \int_{k\bar{\delta}_n}^\infty r^{-2(\beta/C_1+1)} r dr = Ck^{-2\beta/C_1}.$$

By elementary considerations, (12) follows from (22) and (11) follows from (19) and (22). Proposition 2 is established. \square

Lemma 4. For $R > 0$, let $\{V_n\}$ be a sequence of functions satisfying (5). Let $\{u_n\}$ be a sequence of solutions of (1) in $\Omega = B_R$ satisfying (6), (7) and (8). Then, after passing to a subsequence, there exist m ($1 \leq m \leq V(0)C_0/(8\pi)$) sequences of points $\{x_n^{(j)}\}_{j=0}^{m-1}$ in B_R and m sequences of positive numbers $\{k_n^{(j)}\}_{j=0}^{m-1}$ with $\lim_{n \rightarrow \infty} x_n^{(j)} = 0$ and $\lim_{n \rightarrow \infty} k_n^{(j)} = \infty$ ($0 \leq j \leq m-1$) such that

$$(23) \quad u_n(x_n^{(j)}) = \max_{|x-x_n^{(j)}| \leq k_n^{(j)} \delta_n^{(j)}} u_n(x) \rightarrow \infty, \quad \forall 0 \leq j \leq m-1,$$

where $\delta_n^{(j)} = e^{-u_n(x_n^{(j)})/2}$,

$$(24) \quad B_{2k_n^{(i)} \delta_n^{(i)}}(x_n^{(i)}) \cap B_{2k_n^{(j)} \delta_n^{(j)}}(x_n^{(j)}) = \emptyset, \\ \forall i \neq j, 0 \leq i, j \leq m-1,$$

$$(25) \quad \frac{\partial}{\partial t} u_n(ty + x_n^{(j)})|_{t=1} < 0, \\ \forall \delta_n^{(j)} \leq |y| \leq 2k_n^{(j)} \delta_n^{(j)}, 0 \leq j \leq m-1.$$

$$(26) \quad \lim_{n \rightarrow \infty} \int_{B_{2k_n^{(j)} \delta_n^{(j)}}(x_n^{(j)})} V_n e^{u_n} = \lim_{n \rightarrow \infty} \int_{B_{k_n^{(j)} \delta_n^{(j)}}(x_n^{(j)})} V_n e^{u_n} = 8\pi, \\ \forall 0 \leq j \leq m-1,$$

$$(27) \quad \max_{x \in \bar{B}_R} \{u_n(x) + 2 \log \min_{0 \leq j \leq m-1} |x - x_n^{(j)}|\} \leq C, \quad \forall n.$$

Proof. Let $x_n^{(0)} = x_n$ with $u_n(x_n) = \max_{\bar{B}_R} u_n$. Set $\tilde{u}_n^{(0)}(x) = u_n(\delta_n^{(0)}x + x_n^{(0)}) + 2 \log \delta_n^{(0)}$ with $\delta_n^{(0)} = e^{-u_n(x_n^{(0)})/2}$. It is easy to see that $\tilde{u}_n^{(0)}$ satisfies

$$-\Delta \tilde{u}_n^{(0)}(x) = V_n(\delta_n^{(0)}x + x_n^{(0)})e^{\tilde{u}_n^{(0)}(x)} \quad \forall x \in B_{R/(2\delta_n^{(0)})}, \\ \tilde{u}_n^{(0)}(0) = 0, \\ \tilde{u}_n^{(0)}(x) \leq 0 \quad \forall x \in B_{R/(2\delta_n^{(0)})}, \\ \int_{B_{R/(2\delta_n^{(0)})}} e^{\tilde{u}_n^{(0)}} \leq C_0 \quad \forall n.$$

As in the proof of Lemma 1, after passing to a diagonal subsequence we may select $k_n^{(0)} \rightarrow \infty$ such that

$$\|\tilde{u}_n^{(0)} - \tilde{u}\|_{C^{1,\alpha}(B_{2k_n^{(0)}})} \rightarrow 0, \quad 0 < \alpha < 1,$$

$$\int_{B_{2k_n^{(0)}} \delta_n^{(0)}(x_n^{(0)})} V_n e^{u_n} = \int_{B_{2k_n^{(0)}}} V_n (\delta_n^{(0)} \cdot + x_n^{(0)}) e^{\tilde{u}_n^{(0)}} \rightarrow 8\pi,$$

$$\int_{B_{k_n^{(0)}} \delta_n^{(0)}(x_n^{(0)})} V_n e^{u_n} = \int_{B_{k_n^{(0)}}} V_n (\delta_n^{(0)} \cdot + x_n^{(0)}) e^{\tilde{u}_n^{(0)}} \rightarrow 8\pi,$$

$$\frac{\partial}{\partial t} u_n(ty + x_n^{(0)})|_{t=1} < 0, \quad \forall \delta_n^{(0)} \leq |y| \leq 2k_n^{(0)} \delta_n^{(0)}.$$

Recall that \tilde{u} is the function defined in the proof of Lemma 1.

Clearly $x_n^{(0)}$ and $k_n^{(0)}$ satisfy (23), (26) and (25) with $m = 1$. Next we suppose that the ℓ sequences $\{x_n^{(j)}\}_{j=0}^{\ell-1}$, $\{k_n^{(j)}\}_{j=0}^{\ell-1}$ ($\ell \geq 1$) have already been chosen satisfying (23), (24), (25) and (26) with $m = \ell$. If $\max_{x \in B_R} [u_n(x) + 2 \log \min_{0 \leq j \leq \ell-1} |x - x_n^{(j)}|] \leq C$, $\forall n$, we stop and define $m = \ell$. Otherwise, we define $\bar{x}_n^{(\ell)}$ as a point where $M_n = \max_{x \in B_R} [u_n(x) + 2 \log \min_{0 \leq j \leq \ell-1} |x - x_n^{(j)}|]$ is attained, so $M_n \rightarrow \infty$ (after passing to a subsequence) and, in particular, $u_n(\bar{x}_n^{(\ell)}) \rightarrow \infty$. Denoting $\bar{\delta}_n^{(\ell)} = e^{-u_n(\bar{x}_n^{(\ell)})/2}$, $M_n \rightarrow \infty$ reads as $\min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}|/\bar{\delta}_n^{(\ell)} \rightarrow \infty$. First we observe that for all $|x| \leq \frac{1}{2} \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}|/\bar{\delta}_n^{(\ell)}$, we have

$$(28) \quad \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} + \bar{\delta}_n^{(\ell)} x - x_n^{(j)}| \geq \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}| - \bar{\delta}_n^{(\ell)} |x|$$

$$\geq \frac{1}{2} \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}|.$$

Set $\tilde{u}_n(x) = u_n(\bar{x}_n^{(\ell)} + \bar{\delta}_n^{(\ell)} x) + 2 \log \bar{\delta}_n^{(\ell)}$. It follows (using the definition of $\bar{\delta}_n^{(\ell)}$, the maximality of $x_n^{(\ell)}$ and (28)) that \tilde{u}_n satisfies

$$-\Delta \tilde{u}_n(x) = V_n (\bar{\delta}_n^{(\ell)} x + x_n^{(\ell)}) e^{\tilde{u}_n(x)}, \quad |x| \leq \frac{1}{2} \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}|/\bar{\delta}_n^{(\ell)},$$

$$\tilde{u}_n(0) = 0,$$

$$\tilde{u}_n(x) \leq 2 \log 2, \quad |x| \leq \frac{1}{2} \min_{0 \leq j \leq \ell-1} |\bar{x}_n^{(\ell)} - x_n^{(j)}|/\bar{\delta}_n^{(\ell)}.$$

We conclude, after passing to a subsequence, that \tilde{u}_n converges in $C_{loc}^{1,\alpha}(\mathbf{R}^2)$ ($0 < \alpha < 1$) to $\bar{u} = \log(\bar{a}^2/(1 + \gamma^2 \bar{a}^2 |x - \bar{x}|^2)^2)$ with $\gamma = (V(0)/8)^{1/2}$, $\bar{a} > 0$ and $\bar{x} \in \mathbf{R}^2$.

Since \bar{u} should satisfy

$$\bar{u}(0) = 0, \quad \bar{u}(x) \leq 2 \log 2, \quad \forall x \in \mathbf{R}^2,$$

we deduce that

$$(29) \quad 1 \leq \bar{a} \leq 2, \quad |\bar{x}| \leq \frac{1}{2\gamma}.$$

Set

$$C = \max_{|x| \leq 1/\gamma} \bar{u}(x) - \min_{|x| \leq 1/\gamma} \bar{u}(x).$$

Passing to a diagonal subsequence we may choose $k_n^{(\ell)} \rightarrow \infty$ such that

$$(30) \quad \|\tilde{u}_n - \bar{u}\|_{C^{1,\alpha}(B_{4k_n^{(\ell)}})} \rightarrow 0,$$

$$\int_{B_{1/2e - C/2k_n^{(\ell)}}} V_n(\bar{\delta}_n^{(\ell)} \cdot + \bar{x}_n^{(\ell)}) e^{\tilde{u}_n} \rightarrow 8\pi,$$

$$\int_{B_{4k_n^{(\ell)}}} V_n(\bar{\delta}_n^{(\ell)} \cdot + \bar{x}_n^{(\ell)}) e^{\tilde{u}_n} \rightarrow 8\pi,$$

$$\frac{\partial}{\partial t} \tilde{u}_n(ty + \bar{x}) \Big|_{t=1} < 0, \quad \forall 1 \leq |y| \leq 4k_n^{(\ell)}.$$

Let $y_n^{(\ell)} \in B_{3k_n^{(\ell)}}$ satisfy

$$(31) \quad \tilde{u}_n(\bar{x} + y_n^{(\ell)}) = \max_{y \in \bar{B}_{4k_n^{(\ell)}}} \tilde{u}_n(\bar{x} + y),$$

and set $x_n^{(\ell)} = \bar{x}_n^{(\ell)} + \bar{\delta}_n^{(\ell)}(\bar{x} + y_n^{(\ell)})$. Using (29), (30) and the explicit expression of \bar{u} , it is not difficult to see that $y_n^{(\ell)} \rightarrow 0$ and

$$(32) \quad u_n(\bar{x}_n^{(\ell)}) \leq u_n(x_n^{(\ell)}) \leq u_n(\bar{x}_n^{(\ell)}) + C.$$

Introducing $\tilde{u}_n^{(\ell)} = u_n(\delta_n^{(\ell)}x + x_n^{(\ell)}) + 2\log \delta_n^{(\ell)}$ with $\delta_n^{(\ell)} = e^{-u_n(x_n^{(\ell)})/2}$. It follows from (32) that

$$(33) \quad \delta_n^{(\ell)} \leq \bar{\delta}_n^{(\ell)} \leq e^{C/2} \delta_n^{(\ell)},$$

and

$$u_n(x_n^{(\ell)}) = \max_{|x - x_n^{(\ell)}| \leq k_n^{(\ell)} \delta_n^{(\ell)}} u_n(x) \rightarrow \infty.$$

We conclude that

$$\begin{aligned} \|\tilde{u}_n^{(\ell)} - \tilde{u}\|_{C^{1,\alpha}(B_{k_n^{(\ell)}})} &\rightarrow 0, \\ \int_{B_{2k_n^{(\ell)}\delta_n^{(\ell)}}(x_n^{(\ell)})} V_n e^{u_n} &= \int_{B_{2k_n^{(\ell)}}} V_n(\delta_n^{(\ell)} \cdot + x_n^{(\ell)}) e^{\tilde{u}_n^{(\ell)}} \rightarrow 8\pi, \\ \int_{B_{k_n^{(\ell)}\delta_n^{(\ell)}}(x_n^{(\ell)})} V_n e^{u_n} &= \int_{B_{k_n^{(\ell)}}} V_n(\delta_n^{(\ell)} \cdot + x_n^{(\ell)}) e^{\tilde{u}_n^{(\ell)}} \rightarrow 8\pi, \\ \frac{\partial}{\partial t} u_n(ty + x_n^{(\ell)})|_{t=1} &< 0, \quad \forall \delta_n^{(\ell)} \leq |y| \leq 2k_n^{(\ell)}\delta_n^{(0)}. \end{aligned}$$

It is not difficult to see that $\{x_n^{(j)}\}_{j=0}^\ell$ and $\{k_n^{(j)}\}_{j=0}^\ell$ satisfy (23), (24), (25) and (26) with $m = \ell + 1$.

We continue in this manner until (27) holds. We must stop after a finite number of steps since each time we find a mass of 8π near $x_n^{(j)}$. It follows that $m \leq C_0 V(0)/(8\pi)$. Lemma 4 is established. \square

In order to conclude the proof of Proposition 1, we only need to show that the mass contribution outside the chosen neighborhoods of the m centers of mass $x_n^{(0)}, \dots, x_n^{(m-1)}$ tends to zero. Namely,

$$\limsup_{n \rightarrow \infty} \int_{B_R \setminus \bigcup_{j=0}^{m-1} B_{k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)})} V_n e^{u_n} = 0.$$

We will use the following lemma which deals with a slightly more general situation than ours, but is useful for an induction proof.

Lemma 5. *For $R > 0$, let $\{V_n\}$ be a sequence of functions satisfying (5). We assume that $\{u_n\}$ is a sequence of solutions of (1) satisfying (7) and (8) on B_R for some positive constant C_0 , $\{x_n^{(j)}\}_{j=0}^{m-1}$ are m ($m \geq 1$) sequences of points in B_R and $\{r_n^{(j)}\}_{j=0}^{m-1}$ are m sequences of positive numbers which satisfy*

$$(34) \quad u_n(x_n^{(j)}) \rightarrow \infty, \quad \forall 0 \leq j \leq m-1,$$

$$(35) \quad \lim_{n \rightarrow \infty} \frac{r_n^{(j)}}{\delta_n^{(j)}} = \infty, \quad 0 \leq j \leq m-1,$$

where $\delta_n^{(j)} = e^{-u_n(x_n^{(j)})/2}$,

$$(36) \quad B_{r_n^{(i)}}(x_n^{(i)}) \cap B_{r_n^{(j)}}(x_n^{(j)}) = \emptyset, \quad \forall i \neq j, \quad 0 \leq i, j \leq m-1,$$

$$(37) \quad \max_{x \in \bar{B}_R \setminus \cup_{j=0}^{m-1} B_{r_n^{(j)}}(x_n^{(j)})} \{u_n(x) + 2 \log \min_{0 \leq j \leq m-1} |x - x_n^{(j)}|\} \leq C, \quad \forall n,$$

and

$$(38) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_{B_{2r_n^{(j)}}(x_n^{(j)})} V_n e^{u_n} \\ &= \lim_{n \rightarrow \infty} \int_{B_{r_n^{(j)}}(x_n^{(j)})} V_n e^{u_n} = \beta_j, \quad 0 \leq j \leq m-1. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} \int_{B_R} V_n e^{u_n} = \sum_{j=0}^{m-1} \beta_j.$$

Proof. By rescaling the variable and applying Theorem 0 we see that $\beta_j > 0$ ($0 \leq j \leq m-1$). Because of (7) and Lemma 1 we may also assume (15) for some positive constants a and b . It follows from (34) and (7) that $\lim_{n \rightarrow \infty} x_n^{(j)} = 0$, $\forall 0 \leq j \leq m-1$. We prove the lemma by induction on m . First we prove the lemma for $m = 1$. Without loss of generality we assume $x_n^{(0)} = 0 \forall n$. We can also assume that $\lim_{n \rightarrow \infty} r_n^{(0)} = 0$, since otherwise the lemma obviously holds due to (7).

Applying Lemma 2 we obtain, by using (37), that

$$\sup_{\partial B_r} u_n \leq C + \beta \inf_{\partial B_r} u_n + 2(\beta - 1) \log r, \quad \forall 2r_n^{(0)} \leq r \leq \frac{R}{2}.$$

Applying Lemma 3 we obtain, since $\inf_{\partial B_r} u_n = \inf_{B_r} u_n$, that

$$\inf_{\partial B_r} u_n \leq C - \frac{1}{C_1} u_n(0) - 2 \left(1 + \frac{1}{C_1}\right) \log r, \quad \forall 0 < r < R.$$

Combining the two inequalities above, we obtain

$$\sup_{\partial B_r} u_n \leq C - \frac{\beta}{C_1} u_n(0) - 2 \left(\frac{\beta}{C_1} + 1\right) \log r, \quad \forall 2r_n^{(0)} \leq r \leq \frac{R}{2},$$

namely,

$$e^{u_n(x)} \leq C(\delta_n^{(0)})^{2\beta/C_1} |x|^{-2(\beta/C_1+1)}, \quad \forall 2r_n^{(0)} \leq |x| \leq \frac{R}{2}.$$

It follows, by using (35), that

$$\begin{aligned} \int_{B_{R/2} \setminus B_{2r_n^{(0)}}} V_n e^{u_n} &\leq C(\delta_n^{(0)})^{2\beta/C_1} \int_{2r_n^{(0)}}^\infty r^{-2(\beta/C_1+1)} r \, dr \\ &= C \left(\frac{\delta_n^{(0)}}{2r_n^{(0)}} \right)^{2\beta/C_1} \rightarrow 0. \end{aligned}$$

Using (38), (7) and the above, we obtain that

$$\int_{B_R} V_n e^{u_n} \rightarrow \beta_0.$$

This establishes Lemma 5 for $m = 1$.

In the following we assume that Lemma 5 holds for $1, \dots, m - 1$ ($m \geq 2$) and we prove that it also holds for m .

By relabelling the indices we may assume that

$$d_n = |x_n^{(0)} - x_n^{(1)}| = \min\{|x_n^{(i)} - x_n^{(j)}| \mid 0 \leq i, j \leq m - 1, i \neq j\}.$$

Without loss of generality we may assume that $x_n^{(0)} = 0$. There are two cases.

Case 1.

There exists some constant A such that

$$|x_n^{(i)} - x_n^{(j)}| \leq Ad_n, \quad \forall 0 \leq i, j \leq m - 1, i \neq j.$$

In this case we will establish

$$(39) \quad \lim_{n \rightarrow \infty} \int_{B_{4Ad_n}} V_n e^{u_n} = \lim_{n \rightarrow \infty} \int_{B_{2Ad_n}} V_n e^{u_n} = \sum_{j=0}^{m-1} \beta_j.$$

Once (39) is established, we introduce $r_n^{\prime(0)} = 2Ad_n$, $\beta'_0 = \sum_{j=0}^{m-1} \beta_j$ and $x_n^{\prime(0)} = 0$. We then apply Lemma 5 for $m = 1$ to conclude

$$\int_{B_R} V_n e^{u_n} = \beta'_0 = \sum_{j=0}^{m-1} \beta_j.$$

Thus, it suffices to prove (39). Consider $\tilde{u}_n(x) = u_n(d_n x) + 2 \log d_n$ for $|x| \leq R/d_n$. Let us denote

$$\begin{aligned} \tilde{V}_n(x) &= V_n(d_n x), & |x| &\leq \frac{R}{d_n}, \\ \tilde{x}_n^{(j)} &= x_n^{(j)} / d_n, & 0 \leq j &\leq m - 1, \\ \tilde{\delta}_n^{(j)} &= e^{-\tilde{u}_n(\tilde{x}_n^{(j)})/2} = \delta_n^{(j)} / d_n, & 0 \leq j &\leq m - 1, \\ \tilde{r}_n^{(j)} &= r_n^{(j)} / d_n, & 0 \leq j &\leq m - 1. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} \tilde{x}_n^{(0)} &= 0, \\ \tilde{u}_n(\tilde{x}_n^{(j)}) &\rightarrow \infty, & 0 \leq j \leq m-1, \\ \lim_{n \rightarrow \infty} \tilde{r}_n^{(j)} / \tilde{\delta}_n^{(j)} &= \infty, & 0 \leq j \leq m-1, \\ B_{\tilde{r}_n^{(i)}}(\tilde{x}_n^{(i)}) \cap B_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)}) &= \emptyset, \quad \forall i \neq j, \quad 0 \leq i, j \leq m-1, \\ \max_{x \in \bar{B}_{R/d_n} \setminus \cup_{j=0}^{m-1} B_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \{ \tilde{u}_n(x) + 2 \log \min_{0 \leq j \leq m-1} |x - \tilde{x}_n^{(j)}| \} &\leq C, \quad \forall n, \end{aligned}$$

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{B_{2\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{V}_n e^{\tilde{u}_n} \\ &= \lim_{n \rightarrow \infty} \int_{B_{\tilde{r}_n^{(j)}}(\tilde{x}_n^{(j)})} \tilde{V}_n e^{\tilde{u}_n} = \beta_j, \quad 0 \leq j \leq m-1. \end{aligned}$$

Since $|\tilde{x}_n^{(j)}| \leq A, \forall n, j$, we may assume that

$$\tilde{x}_n^{(j)} \rightarrow \tilde{x}^{(j)}, \quad 0 \leq j \leq m-1.$$

It follows from Theorem 0 that

$$(40) \quad \tilde{u}_n \rightarrow -\infty \text{ uniformly on any compact set of } \mathbf{R}^2 \setminus \bigcup_{j=0}^{m-1} \{ \tilde{x}^{(j)} \}.$$

It is easy to see that

$$(41) \quad 1 \leq |\tilde{x}^{(i)} - \tilde{x}^{(j)}| \leq A, \quad \forall 0 \leq i, j \leq m-1, i \neq j.$$

For each $0 \leq j \leq m-1$, if (along a subsequence) $\lim_{n \rightarrow \infty} \tilde{r}_n^{(j)} > 0$, we use (40) to obtain

$$(42) \quad \int_{B_{1/2}(\tilde{x}^{(j)})} \tilde{V}_n e^{\tilde{u}_n} \rightarrow \beta_j.$$

If (along a subsequence) $\lim_{n \rightarrow \infty} \tilde{r}_n^{(j)} = 0$, we may apply the case $m = 1$ of Lemma 5 to conclude (42). It follows (using (42) and (40)) that

$$\lim_{n \rightarrow \infty} \int_{B_{4A}} \tilde{V}_n e^{\tilde{u}_n} = \lim_{n \rightarrow \infty} \int_{B_{2A}} \tilde{V}_n e^{\tilde{u}_n} = \sum_{j=0}^{m-1} \beta_j.$$

By a change of variable, (39) follows and the proof of the lemma in Case 1 is completed.

The remaining case is the following.

Case 2.

There is a proper subset J of $\{0, 1, \dots, m - 1\}$ containing $\{0, 1\}$ and some constant A such that

$$\begin{aligned} |x_n^{(j)}| &\leq Ad_n, & \forall j \in J, \\ \lim_{n \rightarrow \infty} |x_n^{(j)}|/d_n &= \infty, & \forall j \notin J. \end{aligned}$$

Without loss of generality we may assume $J = \{0, 1, \dots, k - 1\}$, $2 \leq k \leq m - 1$.

In this case we consider $\tilde{u}_n(x) = u_n(d_n x) + 2 \log d_n$ for $|x| \leq 4A$. We may apply the same argument as in Case 1 to obtain

$$\lim_{n \rightarrow \infty} \int_{B_{2Ad_n}} V_n e^{u_n} = \lim_{n \rightarrow \infty} \int_{B_{Ad_n}} V_n e^{u_n} = \sum_{j=0}^{k-1} \beta_j.$$

Set $r'_n{}^{(0)} = Ad_n$, $\beta'_0 = \sum_{j=0}^{k-1} \beta_j$ and $x'^{(0)} = 0$. We may apply Lemma 5 for $m - k + 1$ (induction hypothesis) to the sequences $x'_n{}^{(0)}$, $\{x_n^{(j)}\}_{j=k}^{m-1}$ with the radii $r'_n{}^{(0)}$, $\{r_n^{(j)}\}_{j=k}^{m-1}$ and the masses β'_0 , $\{\beta_j\}_{j=k}^{m-1}$ to obtain

$$\int_{B_R} V_n e^{u_n} \rightarrow \beta'_0 + \sum_{j=k}^{m-1} \beta_j = \sum_{j=0}^{m-1} \beta_j.$$

For this last step we need to verify the hypotheses (34)–(38) for applying the case $m - k + 1$ of Lemma 5. Let us only indicate how to show (36) since the others are easier. We only need to show that

$$B_{Ad_n}(0) \cap B_{r_n^{(j)}}(x_n^{(j)}) = \emptyset, \quad \forall k \leq j \leq m - 1.$$

Suppose the above does not hold for some $k \leq j \leq m - 1$. Then we have $Ad_n + r_n^{(j)} \geq |x_n^{(j)}|$. It follows that

$$\frac{r_n^{(j)}}{d_n} \rightarrow \infty, \quad \frac{r_n^{(j)}}{|x_n^{(j)}|} \rightarrow 1.$$

Therefore, we have $B_{Ad_n}(0) \subset B_{2r_n^{(j)}}(x_n^{(j)})$ which, in view of (36) and the inequality $\beta_0 > 0$, contradicts (38). Lemma 5 is established. □

Proof of the Proposition 1 completed. It suffices to apply Lemma 5 to the sequences $\{x_n^{(j)}\}$ found in Lemma 4, with $r_n^{(j)} = k_n^{(j)} \delta_n^{(j)}$ and $\beta_j = 8\pi$.

Example. In the following we construct for each $m \geq 1$, a sequence of solutions $\{u_n\}$ to $-\Delta u = e^u$ in D , the unit disc of \mathbf{C} (identified as \mathbf{R}^2), which blows up at exactly m points a_1, \dots, a_m in D . Here the mass is $\alpha_i = 8\pi, \forall i$ (see Remark). We use Liouville's representation for solutions of $-\Delta u = e^u$ ([L]): $u(z) = \log\{8|f'(z)|^2/(1+|f(z)|^2)^2\}$ with f an holomorphic function such that $f'(z) \neq 0$. For some fixed distinct points b_1, \dots, b_m in \mathbf{C} we consider the functions $\{f_n\}_{n=1}^\infty, f_n(z) = np(z), p(z) = (z - b_1) \cdots (z - b_m)$, and the corresponding $\{v_n\}$ given by Liouville's formula. The sequence $\{v_n\}$ does blow up at b_1, \dots, b_m , but it is not defined at the points c_1, \dots, c_{m-1} , the zeroes of the polynomial $p'(z)$. So we choose a smooth bounded and simply connected domain Ω containing b_1, \dots, b_m but avoiding c_1, \dots, c_{m-1} . Let ψ be the Riemann mapping of D onto Ω . We define on D the sequence $u_n(z) = v_n(\psi(z)) + 2\log(|\psi'(z)|)$. It is easy to see that $\{u_n\}$ is a sequence of solutions on D which blows up at $\{a_1, \dots, a_m\} = \{\psi^{-1}(b_1), \dots, \psi^{-1}(b_m)\}$.

Appendix. In this Appendix we provide the proofs of Lemmas 2 and 3.

Proof of Lemma 2. In the sequel we will denote by C various positive constants depending only on C_1 and C_2 . For $r \in [2R_0, R/2]$, we consider $\tilde{u}(x) = u(rx) + 2\log r$ which satisfies

$$-\Delta \tilde{u}(x) = V(rx)e^{\tilde{u}(x)} \quad \text{in } B_2 \setminus \bar{B}_{1/2}.$$

We have $\tilde{u}(x) = u(rx) + 2\log(r|x|) - 2\log|x| \leq C_2 + 2\log 2$ for all $\frac{1}{2} \leq |x| \leq 2$. It follows that $|V(rx)e^{\tilde{u}(x)}| \leq C$ in $B_2 \setminus \bar{B}_{1/2}$. Let w be the solution of

$$\begin{cases} -\Delta w = V(rx)e^{\tilde{u}(x)} & \text{in } B_2 \setminus B_{1/2}, \\ w = 0 & \text{on } \partial(B_2 \setminus B_{1/2}). \end{cases}$$

It follows from the standard elliptic theory that $\|w\|_{L^\infty} \leq C$. We conclude that the harmonic function $h = w - \tilde{u}$ is bounded below on $B_2 \setminus B_{1/2}$ by some constant $-C$. It follows from the Harnack inequality that there exists some universal constant $\beta \in (0, 1)$ such that $\sup_{\partial B_1}(h + C) \leq \beta^{-1} \inf_{\partial B_1}(h + C)$. Returning to the original u we find the desired estimate.

Proof of Lemma 3. We consider $\tilde{u}(x) = u(Rx) + 2\log R$ which satisfies $-\Delta \tilde{u}(x) = V(Rx)e^{\tilde{u}(x)}$ in B_1 . It is enough to apply (4) for \tilde{u} with $\Omega = B_1$ and $K = \{0\}$ to conclude (14).

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YANYAN LI
Department of Mathematics
Rutgers University
New Brunswick, New Jersey 08903

ITAI SHAFRIR
Department of Mathematics
Universite de Metz
Ile du Saulcy
F-57045 Metz Cedex 01 France

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