# Existence of Multiple Solutions of Semilinear Elliptic Equations in $R^N$

YANYAN LI

## Abstract

We study the existence of multiple solutions of semilinear elliptic equations in $R^N$ with growth of nonlinearities below the critical Sobolev exponent.

## Acknowledgement

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## Introduction

In this paper we present some results on the existence of multiple solutions of the following equations:

$$-\triangle u + u - g(x,u) = 0, \quad x \in R^N \tag{1}$$

In Section 1 we discuss it only for a model case:

$$\begin{cases} -\triangle u + u - q(x)|u|^{\sigma}u = 0 & \text{in } R^N \\ 0 < \sigma < \frac{4}{N-2} & \text{if } N \geq 3 \\ 0 < \sigma < +\infty & \text{if } N = 1,2 \\ q \in C(R^N) & \\ q \in L^{\infty}(R^N) + L^{\beta}(R^N) & \\ q(x) > 0 & \text{for some } x \in R^N \end{cases} \tag{2}$$

Where

$$\beta = \frac{2N}{2N - (N-2)(\sigma+2)} \quad \text{if } N \geq 3$$

$$\beta > 1 \quad \text{if } N = 2$$

$$\beta = 1 \quad \text{if } N = 1$$

In Section 2 we use the same idea to deal with the more general case (1).

For similar problems on a bounded domain of $R^N$ ,there have been many results. (see [18] and the references there.) We usually use variational method to deal with semilinear elliptic equations and apply Ljusternik-Schnirelmann type theory to explore the invariance of functionals under group action. Since we are working in infinite dimensional space we need some compactness hypothesis on the functional. One of such hypothesis is Palais-Smale condition. If we study semilinear elliptic equations with growth of nonlinearities below critical exponent on a bounded domain of $R^N$ ,usually the associated functional satisfies the Palais-Smale condition. But this is not true any more for equations on $R^N$ . The main difference here is that $H^1(R^N)$ does not imbed to $L^p(R^N)$ compactly for $2 \leq p < \frac{2N}{N-2}$ , $N \geq 3$; $2 \leq p < +\infty, N = 1,2$. Therefore we need to analyze on which interval does the functional satisfies the Palais-Smale condition. This has been studied by P. L. Lions (see [15]

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,[16] and the references there). Here we give a somewhat different proof for this. In order to obtain multiple solutions we make use of the properties of the nonlinearity to prove that certain minmax values we have constructed really lie in the interval where the associated functional satisfies Palais-Smale condition. The existence of one positive solution of (1) has been studied by P. L. Lions,M. Esteban, Weiyue Ding,Weiming Ni ,Dong Zhang and some others (see [9],[16] ,[15],[22] ,[10] and the references there). The existence of at least two solutions has been studied by Xiping Zhu (see [23]). For the existence of infinitely many solutions there have been work by Berestycki and Taubes (see [5]). Our reselt is different from theirs in the way that we make use of the properties of the nonlinearity in bounded region while their results concern hypotheses at infinity.

In a separate paper (see [14]) we apply this method to study the existence of multiple solutions of the standing waves of Schrödinger equations. In this case we need to analyze where the constrained functional satisfies Palais-Smale condition and construct apropriate minmax values in the interval where the constrained functional satisfies Palais-Smale condition.

# 1 Existence of Multiple Solutions of (2)

We study the existence of multiple solutions of (2) and therefore (2) is assumed throught this section. First we point out where the difficulty lies. Consider the corresponding problem on a bounded domain:

$$\begin{cases} -\triangle u + u - q(x)|u|^\sigma u = 0 & \text{in } \Omega \subset R^N \\ 0 < \sigma < \frac{4}{N-2} & \text{if } N \geq 3 \\ 0 < \sigma < +\infty & \text{if } N = 1,2 \\ q(x) \in C(\Omega) \cap L^\beta(\Omega) & \\ q(x) > 0 & \text{for some } x \in \Omega \end{cases} \qquad (3)$$

We can apply Ljusternik-Schnirelmann type theory to prove that (3) has infinitely many solutions in $H_0^1(\Omega)$ ( See [18] ). The difference between (2) and (3) lies in the fact that the embedding from $H^1(R^N)$ to $L^{\sigma+2}(R^N)$ is no more compact. This is easily seen by taking any nontrivial function $u \in H^1(R^N)$ and letting $u_n(x) := u(x+n)$,clearly $u_n$ converges weakly in $H^1(R^N)$ but not strongly in $L^{\sigma+2}(R^N)$ as $n$ goes to $+\infty$ .

If $q(x)$ in (2) satisfies some further hypotheses,the kind of loss of compactness as above will not occur,therefore the standard Ljusternik-Schnirelmann theory will apply. More precisely we have the following two propositions:

**Propsition 1.1**: In addition to (2) if we assume that $\lim_{|x|\to\infty} q(x) = 0$ , then (2) has infinitly many solutions in $H^1(R^N)$ .

The above result is essentially proved in [20]. There is only a slight difference, namely,we assume here that $q(x)$ is positive somewhere not that,as in [20], positive everywhere.But only a minor modification is needed.

**Proposition 1.2**: In addition to (2) if we assume that $q(x)$ is a radially symmetric function, namely, $q(x) = q(y)$ for $|x| = |y|$, then (2) has infinitely many solutions in $H^1(R^N)$.

The proof of Proposition 1.2 rests on the fact that the embedding from $H^1(R^N)$ to $L^p(R^N)$ $(2 < p < \frac{2N}{N-2}$ if $N \geq 3$, $2 < p < +\infty$ if $N = 1,2)$

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is actually compact among radially symmetric functions (See [21] ). For a proof see [9] and the references there. In fact stronger results are proved in [9] in the radially symmetric case.

If we do not assume further hypotheses on $q(x)$, (2) may not have any nontrivial solutions in $H^1(R^N)$ . We give such an example due to M. J. Esteban and P. L. Lions ( See [10] ):

**Example:** Let $q(x)$ be a smooth bounded positive function defined on $R^N$ with bounded derivatives and $q_{x_1}$ be nonnegative but not identically equal to zero,where $q_{x_1}$ denotes the partial derivative of $q$ in $x_1$ direction. Then any solution of

$$-\triangle u + u - q(x)|u|^{\sigma}u = 0$$

which belongs to $L^p(R^N)$ for some $p > \sigma + 1$ has to be identically equal to zero. Where $0 < \sigma < \frac{4}{N-2}$ if $N \geq 3$ , $0 < \sigma < +\infty$ if N=1,2.

**Proof:** Let $u \in L^p(R^N)$ $(p > \sigma + 1)$ be a solution. It follows from the standard bootstrap method that $u \in H^1(R^N)$. Multiply the equation by $u_{x_1}$ and integrate by parts,we obtain :

$$\int q_{x_1}|u|^{\sigma+2} = 0$$

which implies that $u$ has to vanish on an open set. By some well known unique continuation results, u is identically equal to zero.

The example above shows that the loss of compactness we encounter is genuine, not only because of the method we use.

The existence of one positive solution of (2) under some further hypotheses on $q(x)$ has been discussed by M. J. Esteban, P.L.Lions,Weiyue Ding andWeiming Ni and some others. (See [9],[10],[16] and the references there )

According to their results, either of the following additional hypotheses on $q(x)$ will give rise to at least one positive solution of (2):

$$\begin{cases} \lim\limits_{|x|\to\infty} q(x) = q_{\infty} > 0 & \\ q(x) > q_{\infty} & x \in R^N \end{cases} \qquad (4)$$

There exist positive constants $R, \alpha > 0$ ,such that,

$$\begin{cases} \lim\limits_{|x|\to\infty} q(x) = q_\infty > 0 \\ q(x) > q_\infty + |x|^{-\alpha} \qquad |x| > R, \quad x \in R^N \end{cases} \tag{5}$$

There are other known hypotheses on $q(x)$ which will ensure the existence of at least one positive solution. Since they are basically of the same nature we are not going to list them all here. See the references above for more information. One can also find a survey on the subject in [17],[16].

The existence of at least two pairs of nontrivial solutions of (2) has been studied by Xiping Zhu [23] by using a technique in [6] and the concentration compactness principle introduced by P. L. Lions in [15].

In the following we are going to apply minmax principle to study the existence of multiple solutions of (2) under some further hypotheses on $q(x)$.

**Theorem1.1**: In addition to (2),let $q_\infty$ be a positive constant, $n$ be a positive integer. Suppose that:

$$\limsup_{|x|\to\infty} q(x) \le q_\infty \tag{6}$$

and there exist $n$ functions $u_j \in H^1$ with disjoint supports, $j = 1, \ldots, n$,such that,

$$\int q(x)|u_j|^{\sigma+2} > 0 \qquad j = 1, \ldots, n \tag{7}$$

$$\sum_{j=1}^{n} \frac{(\int |\nabla u_j|^2 + u_j^2)^{\frac{\sigma+2}{\sigma}}}{(\int q(x)|u_j|^{\sigma+2})^{\frac{2}{\sigma}}} < S_\infty^{\frac{\sigma+2}{\sigma}} \tag{8}$$

where

$$S_\infty := \inf_{u \in H^1 \setminus \{0\}} \frac{\int |\nabla u|^2 + u^2}{(\int q_\infty |u|^{\sigma+2})^{\frac{2}{\sigma+2}}}$$

Then (2) has at least $n$ pairs of nontrivial solutions in $H^1(R^N)$ and one of them is positive.

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**Remark 1.1:** Qualitatively Theorem1.1 implies that the larger $q(x)$ is in the bounded region comparing to $\lim\limits_{|x|\to\infty} \sup q(x)$ ,the more solutions we are able to obtain.

**Remark 1.2:** The existence of one positive solution is classical. See [9],[15],[16] and [17].

Theorem 1.1 has the following corollary:

**Corollary 1.1:** In addition to (2) if $\lim\limits_{|x|\to\infty} \sup\ q(x) \leq 0$,then (2) has infinitely many solutions.

**Proof of the corollary 1.1:**

Since $q(x)$ is continuous and positive somewhere,there exists an open set $O \subset R^N$ ,such that, $q(x) \geq c > 0\, \forall x \in O$ ,where $c$ is some constant.

For any integer $n$ , take $n$ disjoint balls $B_1, \ldots, B_n$, $\bigcup_{j=1}^n B_j \subset O$ and any functions $u_j \in H_0^1(B_j) \setminus \{0\}$ $(j = 1, \ldots, n)$ .

$$C_j := \frac{(\int |\nabla u_j|^2 + u_j^2)^{\frac{\sigma+2}{\sigma}}}{(\int q(x)|u_j|^{\sigma+2})^{\frac{2}{\sigma}}} > 0 \qquad j = 1, \ldots, n$$

Choose $q_\infty > 0$ so small that

$$\sum_{j=1}^n C_j < S_\infty^{\frac{\sigma+2}{\sigma}}, \quad j = 1, \ldots, n.$$

According to Theorem 1.1 (2) has at least n pairs of nontrivial solutions. The conclusion of the corollary follows from the fact that we can choose $n$ to be arbitrarily large.

**Corollary 1.2:** Suppose that $q(x) \in L^\infty(R^N) \cap C^\infty(R^N), \psi(x) \in C_0^\infty(R^N) \setminus \{0\}$ and $\psi(x) \geq 0$ . Consider:

$$\begin{cases} -\triangle u + u - q_\lambda(x)|u|^\sigma u = 0 & \text{in } R^N \\ 0 < \sigma < \frac{4}{N-2} & \text{if } N \geq 3 \\ 0 < \sigma < +\infty & \text{if } N = 1, 2 \end{cases}$$

Where $q_\lambda(x) = q(x) + \lambda\psi(x)$ .

The number of solutions of the above equation tends to $+\infty$ as $\lambda$ tends to $+\infty$.

In order to prove Theorem 1.1 we first introduce some notations:

$$\begin{aligned}
\|u\| &:= \int_{R^N} |\bigtriangledown u|^2 + u^2 \\
J(u) &:= \frac{1}{2}\int_{R^N} |\bigtriangledown u|^2 + u^2 - \frac{1}{\sigma+2}\int_{R^N} q(x)|u|^{\sigma+2} \\
J_\infty(u) &:= \frac{1}{2}\int_{R^N} |\bigtriangledown u|^2 + u^2 - \frac{1}{\sigma+2}\int_{R^N} q_\infty|u|^{\sigma+2} \\
V_\infty &:= \{u \in H^1(R^N) \setminus \{0\} : 0 =< J_\infty'(u), u >\} \\
m_\infty &:= \inf\nolimits_{u \in V_\infty} J_\infty(u)
\end{aligned}$$

where $< \cdot, \cdot >$ denotes the paring between $H^1(R^N)$.

**Remark 1.3:** By simple calculation we have

$$< J_\infty'(u), u >= \int |\bigtriangledown u|^2 + u^2 - \int q_\infty |u|^{\sigma+2}$$

**Remark 1.4:** $m_\infty$ and $S_\infty$ are actually attained by some positive radially symmetric functions. Therefore $m_\infty$ and $S_\infty$ are positive numbers.

**Lemma 1.1:**

$$S_\infty = \{(\frac{1}{2} - \frac{1}{\sigma+2})m_\infty\}^{\frac{\sigma}{\sigma+2}}$$

**Proof:** $\forall \epsilon > 0$, there exists $u \in H^1(R^N) \setminus \{0\}$,such that,

$$\frac{\int |\bigtriangledown u|^2 + u^2}{(\int q_\infty |u|^{\sigma+2})^{\frac{2}{\sigma+2}}} \leq S_\infty + \epsilon \tag{9}$$

Let $\lambda > 0$ be defined by

$$\int |\bigtriangledown u|^2 + u^2 = \lambda^\sigma \int q_\infty |u|^{\sigma+2}$$

above equation tends to $+\infty$ as $\lambda$

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$\frac{\cdot u^2}{)^{\frac{2}{\sigma+2}}} \le S_\infty + \epsilon$ (9)

$= \lambda^\sigma \int q_\infty|u|^{\sigma+2}$

Then we have

$$\lambda^\sigma(\int q_\infty|u|^{\sigma+2})^{\frac{\sigma}{\sigma+2}} \le S_\infty + \epsilon \tag{10}$$

Let

$$w := \lambda u$$

clearly

$$w \in V_\infty$$

Therefore according to the definition of $m_\infty$ and (10) we have

$$\begin{aligned} m_\infty &\le J_\infty(w) \\ &= (\frac{1}{2} - \frac{1}{\sigma + 2})\int q_\infty|w|^{\sigma+2} \\ &= (\frac{1}{2} - \frac{1}{\sigma + 2})\{\lambda^\sigma(\int q_\infty|u|^{\sigma+2})^{\frac{\sigma}{\sigma+2}}\}^{\frac{\sigma+2}{\sigma}} \\ &\le (\frac{1}{2} - \frac{1}{\sigma + 2})(S_\infty + \epsilon)^{\frac{\sigma+2}{\sigma}} \end{aligned}$$

Let $\epsilon \to 0$,we have

$$S_\infty \ge \{(\frac{1}{2} - \frac{1}{\sigma + 2})^{-1}m_\infty\}^{\frac{\sigma}{\sigma+2}} \tag{11}$$

On the other hand, $\forall\epsilon > 0$, there exists some $w \in V_\infty$ with $J_\infty(w) \le m_\infty + \epsilon$,namely,

$$\begin{aligned} \int|\nabla w|^2 + w^2 &= \int q_\infty|w|^{\sigma+2} \\ \epsilon + m_\infty &\ge (\frac{1}{2} - \frac{1}{\sigma + 2})\int|\nabla w|^2 + w^2 \end{aligned}$$

Therefore

$$\begin{aligned} S_\infty &\le \frac{\int|\nabla w|^2 + w^2}{(\int q_\infty|w|^{\sigma+2})^{\frac{2}{\sigma+2}}} \\ &= (\int|\nabla w|^2 + w^2)^{\frac{\sigma}{\sigma+2}} \\ &\le \{(\frac{1}{2} - \frac{1}{\sigma + 2})^{-1}m_\infty\}^{\frac{\sigma}{\sigma+2}} \end{aligned}$$

Let $\epsilon \to 0$,we have

$$S_\infty \ge \{(\frac{1}{2} - \frac{1}{\sigma + 2})^{-1}m_\infty\}^{\frac{\sigma}{\sigma+2}} \tag{12}$$

The conclusion of Lemma 1.1 follows from (11) and (12).

To look for $H^1(R^N)$ solutions of (2) we use a variational approach, namely, to look for critical points of $J(u)$. It is well known that $J \in C^1(H^1(R^n\ ), R)$ under the assumption of Theorem 1.1. Due to the fact that the embedding from $H^1(R^N)$ to $L^p(R^N)$ $\ (2 \le p < \frac{2N}{N-2}$ if $N \ge 3, \quad 2 \le p$ if $N = 1,2)$ is not compact,J does not, in general,satisfies $(PS)_c$ for all real values $c$. But we can prove that $J$ satisfies $(PS)_c$ for suitable values $c$.

**Definition 1.1**: We say that $J$ satisfies $(PS)_c$ for some real value $c$,if for any sequence $\{u_n\} \subset H^1(R^N)$ satisfying

$$J(u_n) \to c$$

$$J^{'}(u_n) \to 0 \quad \text{strongly in} \quad H^{-1}(R^N)$$

there exists a subsequence of $\{u_n\}$ which converges strongely in $H^1(R^N)$.

**Lemma 1.2** : Under the assumption of Theorem 1.1, $J$ satisfies $(PS)_c$ for any value $c$ lying in the open interval $(-\infty, m_\infty)$.

**Remark 1.2**: Lemma 1.2 is sharp in the sense that if $q(x) \to q_\infty$ as $|x| \to \infty$, $J$ does not satify $(PS)_{m_\infty}$.

Lemma 1.2 has been proved by P. L. Lions (see [16] ). For completeness we give a somewhat different proof.

**Proof of Lemma 1.2:**

Let $\{u_n\} \subset H^1(R^N)$ be a sequence satifying:

$$J(u_n) \to C^{'} < m_\infty$$

$$J^{'}(u_n) \to 0 \quad \text{strongly in } H^{-1}(R^N)$$

namely

$$\frac{1}{2}\int |\nabla u_n|^2 + u_n^2 - \frac{1}{\sigma+2}\int q(x)|u_n|^{\sigma+2} \to C^{'} \tag{13}$$

follows from (11) and (12).

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$$\int q(x)|u_n|^{\sigma+2} \to C' \qquad (13)$$

$$| < J'(u_n), v > | \leq o(1)\|v\| \qquad \forall v \in H^1(R^N) \qquad (14)$$

Substitute $u_n$ for $v$ in (14) we have

$$| < J'(u_n), u_n > | \leq o(1)\|u_n\|$$

namely

$$-o(1)\|u_n\| \leq \int |\nabla u_n|^2 + u_n^2 - \int q(x)|u_n|^{\sigma+2} \leq o(1)\|u_n\| \qquad (15)$$

Multiply (15) by $-(\sigma+2)^{-1}$ and add it to (13) we have

$$(\frac{1}{2} - \frac{1}{\sigma+2})\|u_n\|^2 \leq C + o(1)\|u_n\|$$

Clearly $\{\|u_n\|\}$ is bounded.

According to some well known facts in functional analysis,there exists $u_\infty \in H^1(R^N)$, such that,along a subsequence of $\{u_n\}$(still being denoted as $\{u_n\}$),

$$u_n \rightharpoonup u_\infty \qquad \text{weakly in} \qquad H^1(R^N)$$

$$u_n \to u_\infty \text{ a.e. in } R^N$$

We claim that $u_\infty$ satisfies:

$$< J'(u_\infty), u_\infty >= 0 \qquad (16)$$

In fact,we have

$$\begin{aligned} o(1) &= \int \nabla u_n \nabla u_\infty + u_n u_\infty - \int q(x)|u_n|^\sigma u_n u_\infty \\ &= o(1) + \|u_\infty\|^2 - \int q(x)|u_n|^\sigma u_n u_\infty \\ &= o(1) + \|u_\infty\|^2 - \int q(x)|u_\infty|^{\sigma+2} \end{aligned}$$

The last equality follows from some elementary real analysis argument. Next we want to prove the strong convergence of $\{u_n\}$ in $H^1(R^N)$. There are only two posibilities for $\{u_n\}$ :

**(1)**: $\forall \delta > 0$, there exists $\tilde{R} > 0$,such that, $\forall n > \tilde{R}$,

$$\int_{|x| \geq \tilde{R}} |\nabla u_n^2| + u_n^2 < \delta$$

(2): There exists $\delta_0 > 0$,such that, $\forall \tilde{R} > 0$, there exists $n = n(\tilde{R}) \geq \tilde{R}$ with

$$\int_{|x|\geq\tilde{R}} |\nabla u_n|^2 + u_n^2 \geq \delta_0$$

Case (1) corresponds to the case that there is no fixed amount of positive mass of $\{u_n\}$ slipping away to infinity,which is easier to handle. Let us deal with it first,more precisely,we prove that case (1) leads to the strong convergence of $\{u_n\}$ in $H^1(R^N)$.

$\forall \delta > 0$,there exists $\tilde{R} > 0$ , such that , $\forall n > \tilde{R}$ we have

$$\int_{|x|\geq\tilde{R}} |\nabla u_n|^2 + u_n^2 < \delta \tag{17}$$

By the weak lower semicontinuity of the above integral and the fact that $u_n \rightharpoonup u_\infty$ weakly in $H^1(R^N)$,we have

$$\int_{|x|\geq\tilde{R}} |\nabla u_\infty|^2 + |u_\infty|^2 \leq \delta \tag{18}$$

Since $J'(u_n) \to 0$ strongly in $H^{-1}(R^N)$, we have:

$$|\int \nabla u_n \nabla v + u_n v - \int q(x)|u_n|^\sigma u_n v| \leq o(1)\|v\| \qquad \forall v \in H^1(R^N) \tag{19}$$

Use Hölder inequality,Sobolev inequality,(17),(18) and the property of $q(x)$ we obtain $\forall v \in H^1(R^N)$ that

$$\begin{cases} |\int_{|x|\geq\tilde{R}} q(x)|u_n|^\sigma u_n v| + |\int_{|x|\geq\tilde{R}} q(x)|u_\infty|^\sigma u_\infty v| & \leq \quad c\delta^{\frac{\sigma+1}{2}}\|v\| \\ |\int_{|x|\leq\tilde{R}} q(x)|u_n|^\sigma u_n v - \int_{|x|\leq\tilde{R}} q(x)|u_\infty|^\sigma u_\infty v| & \leq \quad o(1)\|v\| \end{cases} \tag{20}$$

Combine (19) with (20) we obtain

$$|\int \nabla u_n \nabla v + u_n v - \int q(x)|u_\infty|^\sigma u_\infty v| \leq C\delta^{\frac{\sigma+1}{2}}\|v\| + o(1)\|v\| \tag{21}$$

Where $C$ is some constant independent of $n$ and $\delta$.

Since $\delta > 0$ can be chosen arbitrarily small,(21) implies the strong convergence of $\{u_n\}$ in $H^1(R^N)$.

t, $\forall \tilde{R} > 0$, there exists $n = n(\tilde{R}) \geq \tilde{R}$

$|^2 + u_n^2 \geq \delta_0$

at there is no fixed amount of positive ty,which is easier to handle. Let us ove that case (1) leads to the strong

at , $\forall n > \tilde{R}$ we have

$$|^2 + u_n^2 < \delta \tag{17}$$

f the above integral and the fact that

$$+ |u_\infty|^2 \leq \delta \tag{18}$$

$^N$), we have:

$$\leq o(1)\|v\| \qquad \forall v \in H^1(R^N) \tag{19}$$

ality,(17),(18) and the property of

$$\begin{aligned} )|u_\infty|^\sigma u_\infty v| &\leq c\delta^{\frac{\sigma+1}{2}}\|v\| \\ )|u_\infty|^\sigma u_\infty v| &\leq o(1)\|v\| \end{aligned} \tag{20}$$

$$v| \leq C\delta^{\frac{\sigma+1}{2}}\|v\| + o(1)\|v\| \tag{21}$$

of $n$ and $\delta$.
small,(21) implies the strong con-

In order to prove Lemma 1.2 we only need to rule out case (2). we argue by contradiction. If case (2) occurs, for that $\delta_0$,take $\tilde{R} = l = 1, 2, 3, \ldots$ ,there exists $n_l \geq l$,such that,

$$\int_{|x|\geq l} |\nabla u_{n_l}|^2 + u_{n_l}^2 \geq \delta_0 \tag{22}$$

Take $\epsilon > 0$ so small that

$$C\epsilon + C' < m_\infty$$

Here and later,$C > 0$ denotes some constant which will be determined in the following calculation, C is independent of $\epsilon$.

Since $u_\infty \in H^1(R^N)$ and $\limsup\limits_{|x|\to R_0} q(x) \leq q_\infty$, there exists $R_0 > 0$, such that,

$$\int_{|x|\geq R_0} |\nabla u_\infty|^2 + u_\infty^2 + |q(x)|u_\infty|^{\sigma+2} < \epsilon \tag{23}$$

$$q(x) \leq q_\infty + \epsilon \qquad |x| \geq R_0 \tag{24}$$

Because of the boundedness of $\{\|u_{n_l}\|\}$ , there exists some integer $j_\epsilon$,such that,

$$j_\epsilon \cdot \epsilon > \|u_{n_l}\|^2 \qquad l = 1, 2, 3, \ldots \tag{25}$$

Consider the annuli $\{I_k\}_1^{j_\epsilon}$,where

$$I_k = \{x \in R^N : R_0 + k - 1 \leq |x| \leq R_0 + k\} \quad k = 1, \ldots, j_\epsilon$$

Because of (25), for any $l$ , there exists some $I \subset \{I_1, \ldots, I_{j_\epsilon}\}$, such that,

$$\int_I |\nabla u_{n_l}|^2 + u_{n_l}^2 < \epsilon \tag{26}$$

There are only finitely many annuli but infinitely many $\{u_{n_l}\}$ , there must exists at least one annulus $I \subset \{I_1, \ldots, I_{j_\epsilon}\}$ for which (26) holds for infinitely many $l$. Take such a subsequence and , for simplicity,still denote it as $\{u_{n_l}\}$. Let us denote the above annulus as

$$I = \{x \in R^N : R \leq |x| \leq R + 1\}, \qquad R \geq R_0$$

Construct a function $\rho_1 \in C^\infty(R^N)$ satisfying:

$$\rho_1(x) = \begin{cases} 1 & |x| \le R \\ 0 & |x| \ge R+1 \\ \text{between 0 and 1} & R \le |x| \le R+1 \end{cases}$$

$$|\bigtriangledown \rho_1(x)| \le 2 \qquad \forall x \in R^N$$

Let
$$\rho_2(x) = 1 - \rho_1(x) \qquad \forall x \in R^N$$

For $l = 1, 2, 3, \ldots,$ let
$$v_l = \rho_1(x) u_{n_l}$$
$$w_l = \rho_2(x) u_{n_l}$$

We have
$$u_{n_l} = v_l + w_l$$
$$\operatorname{supp} v_l \subset B_{R+1} \qquad \operatorname{supp} w_l \subset R^N \setminus B_R$$

With (26) and the properties of $\rho_1$, it is easy to see that

$$\int |q(x)||v_l|^{\sigma+1}|w_l| + \int |q(x)||v_l||w_l|^{\sigma+1} + \int |v_l||w_l| + \int |\bigtriangledown v_l||\bigtriangledown w_l| \le C\epsilon$$

Therefore we have
$$o(1) = < J'(u_{n_l}), v_l > = < J'(v_l), v_l > + O(\epsilon)$$

Here and later ,$O(\epsilon)$ denotes some quantity bounded by $C\epsilon$

Similarly we have
$$o(1) = < J'(w_l), w_l > + O(\epsilon)$$

namely
$$\int |\bigtriangledown v_l|^2 + v_l^2 = \int q(x)|v_l|^{\sigma+2} + O(\epsilon) \qquad (27)$$
$$\int |\bigtriangledown w_l|^2 + w_l^2 = \int q(x)|w_l|^{\sigma+2} + O(\epsilon) \qquad (28)$$

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$x) \qquad \forall x \in R^N$

$\rho_1(x)u_{n_l}$

$\rho_2(x)u_{n_l}$

$= v_l + w_l$

$\operatorname{supp} w_l \subset R^N \setminus B_R$

$\rho_1$ , it is easy to see that

$^{\sigma+1} + \int |v_l||w_l| + \int |\nabla v_l||\nabla w_l| \leq C\epsilon$

$>=< J^{'}(v_l), v_l > +O(\epsilon)$

uantity bounded by $C\epsilon$

$w_l), w_l > +O(\epsilon)$

$$\int q(x)|v_l|^{\sigma+2} + O(\epsilon) \tag{27}$$

$$\int q(x)|w_l|^{\sigma+2} + O(\epsilon) \tag{28}$$

On the other hand

$$C^{'} + o(1) = J(u_{n_l}) = J(v_l) + J(w_l) + O(\epsilon)$$

Use (27),we have

$$J(v_l) = (\frac{1}{2} - \frac{1}{\sigma+2})\|v_l\|^2 + O(\epsilon) \geq O(\epsilon)$$

Hence we have

$$C^{'} \geq \frac{1}{2}\int |\nabla w_l|^2 + w_l^2 - \frac{1}{\sigma+2}\int q(x)|w_l|^{\sigma+2} + O(\epsilon) \tag{29}$$

We are going to use (28) and (29) to get a contradiction , which means that case (2) never occurs.

The following formulas hold only for large $l$ ,at least $l > R+1$.It follows from (22),(28) and the choice of $\epsilon$ that

$$\int |\nabla w_l|^2 + w_l^2 \geq \delta_0$$

$$\int q_\infty |w_l|^{\sigma+2} \geq \frac{\delta_0}{2} \tag{30}$$

Let

$$\lambda := \lambda_{l,\epsilon} := \{\frac{\int |\nabla w_l|^2 + w_l^2}{\int q_\infty |w_l|^{\sigma+2}}\}^{\frac{1}{\sigma}}$$

Use (28) and (30),we have

$$\lambda \leq 1 + C\epsilon \tag{31}$$

Let $\xi_l = \lambda w_l$, then $\xi_l \in V_\infty$. Use (28),(31) we have that

$$m_\infty \leq (\frac{1}{2} - \frac{1}{\sigma+2})\int q(x)|w_l|^{\sigma+2} + C\epsilon \tag{32}$$

Combine (28) and (29) we have

$$C^{'} \geq (\frac{1}{2} - \frac{1}{\sigma+2})\int q(x)|w_l^{\sigma+2} - C\epsilon \tag{33}$$

It follows from (32) and (33) that $m_\infty \leq C^{'} + C\epsilon$ ,which contradicts to the choice of $\epsilon$. This concludes the proof.

**Proof of Theorem 1.1:**

Let

$$X_n = \{t_1u_1 + \ldots + t_nu_n : (t_1, \ldots, t_n) \in R^N\}$$

where $u_1, \ldots, u_n$ are those functions in the hypotheses. Clearly $X_n \cap \{u \in H^1(R^N) : J(u) \geq 0\}$ is bounded. Because of Lemma 1.2 and Theorem A in the Appendix ,to conclude the proof we only need to get the following estimates:

$$\sup_{u\in X_n} J(u) < m_\infty \tag{34}$$

According to Lemma 1.1 and (5),there exists $\tilde{S}_\infty < S_\infty$,such that

$$\sum_{j=1}^{n} \frac{(\int |\bigtriangledown u_j|^2 + u_j^2)^{\frac{\sigma+2}{\sigma}}}{(\int q(x)|u_j|^{\sigma+2})^{\frac{2}{\sigma}}} < \tilde{S}_\infty^{\frac{\sigma+2}{\sigma}} \tag{35}$$

Take any $u \in X_n$, $u = t_1u_1 + \ldots + t_nu_n$, $(t_1, \ldots, t_n) \in R^N$.We have

$$\begin{cases} \int q(x)|u|^{\sigma+2} = \sum_{j=1}^{n} |t_j|^{\sigma+2} B_j \\ \int |\bigtriangledown u|^2 + u^2 = \sum_{j=1}^{n} t_j^2 A_j \end{cases} \tag{36}$$

where

$$A_j = \int |\bigtriangledown u_j|^2 + u_j^2, \quad B_j = \int q(x)|u_j|^{\sigma+2}, \quad j = 1, \ldots, n$$

By Hölder's inequality we have

$$\begin{aligned} \sum t_j^2 A_j &\leq \{\sum(\frac{A_j}{B_j^{\frac{2}{\sigma+2}}})^{\frac{\sigma+2}{\sigma}}\}^{\frac{\sigma}{\sigma+2}} \{\sum |t_j|^{\sigma+2} B_j\}^{\frac{2}{\sigma+2}} \\ &< \tilde{S}_\infty \{\sum t_j^2 A_j\}^{\frac{2}{\sigma+2}} \end{aligned}$$

Hence

$$\sup_{u\in X_n} J(u) < m_\infty \tag{37}$$

Theorem 1.1 follows from Lemma 1.1 and Theorem A in the Appendix .

$$+ t_n u_n : (t_1, \dots, t_n) \in R^N\}$$

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$$x_n J(u) < m_\infty \tag{34}$$

5),there exists $\tilde{S}_\infty < S_\infty$,such that

$$\frac{{}_j|^2 + u_j^2)^{\frac{\sigma+2}{\sigma}}}{)|u_j|^{\sigma+2})^{\frac{2}{\sigma}}} < \tilde{S}_\infty^{\frac{\sigma+2}{\sigma}} \tag{35}$$

$\cdot\cdot + t_n u_n, \quad (t_1, \dots, t_n) \in R^N$.We have

$$\begin{aligned} {}^{+2} &= \sum_{j=1}^{n} |t_j|^{\sigma+2} B_j \\ u^2 &= \sum_{j=1}^{n} t_j^2 A_j \end{aligned} \tag{36}$$

$$= \int q(x)|u_j|^{\sigma+2}, \ \ j = 1, \dots, n$$

$$)^{\frac{\sigma+2}{\sigma}}\}^{\frac{\sigma}{\sigma+2}} \{\sum |t_j|^{\sigma+2} B_j\}^{\frac{2}{\sigma+2}}$$

$$4_j\}^{\frac{2}{\sigma+2}}$$

$$J(u) < m_\infty \tag{37}$$

1.1 and Theorem A in the Appendix .

## 2 Existence of multiple solutions of more general equations

In this section we study the equation (1). Suppose that $g(x,u)$ satisfies the following hypotheses:

$$g(x,u) \ \text{ is continuous and odd in } u \tag{38}$$

$$|g(x,u)| \le b_1(x)|u| + b_2(x)|u|^{\sigma+1} \tag{39}$$

where

$$\begin{array}{ll} (x,u) \in R^N \times R & \\ 0 < \sigma < \frac{4}{N-2} & \text{if } N \ge 3 \\ 0 < \sigma < +\infty & \text{if} N = 1,2 \\ b_1 \in L^\alpha(R^N) + L^\infty(R^N) & \\ b_2 \in L^\beta(R^N) + L^\infty(R^N) & \\ \alpha = \frac{N}{2} & \text{if } N \ge 3 \\ \alpha > 1 & \text{if } N = 2 \\ \alpha = 1 & \text{if } N = 1 \\ \beta = \frac{2N}{2N-(N-2)(\sigma+2)} & \text{if } N \ge 3 \\ \beta > 1 & \text{if } N = 2 \\ \beta = 1 & \text{if } N = 1 \end{array}$$

There exists $\theta \in (0, \frac{1}{2})$,such that,

$$G(x,u) \le \theta u g(x,u) \qquad \forall (x,u) \in R^N \times R \tag{40}$$

where $G(x,u) := \int_0^u g(x,s)ds$

$$g(x,u)u^{-1} \text{ is nondecreasing in } u \qquad \forall u \ge 0 \quad x \in R^N \tag{41}$$

**Remark 2.1**: (40) and (41) follow from the following stronger assumption:

$$g \in C^1 \text{ and there exists } \bar{\theta} \in (0,1), \text{such that,}$$

$$g(x,u) \le \bar{\theta} u g_u(x,u) \qquad \forall (x,u) \in R^N \times R^+$$

Suppose that $h : R^N \times R \to R$ satisfies the following properties: There exists $R_1 \gg 1$, such that

$$g(x,u)u \le h(x,u)u \qquad |x| > R_1, u \in R \tag{42}$$

$$\left\{ \begin{array}{l} h \in C(R^N \times R, R), \quad u(\cdot) \mapsto h(\cdot, u(\cdot))u(\cdot) \\ \text{is a continuous map from } H^1(R^N) \text{ into } L^1(R^N). \end{array} \right. \tag{43}$$

There exists $\epsilon_1 > 0$, such that

$$h(x,tu)tu \ge t^{2+\epsilon_1} h(x,u)u \qquad t \ge 1, x \in R^N, u \in R. \tag{44}$$

For all $w \in H_0^1(R^N) \cap L^\infty(R^N)$, $\operatorname{supp} w \subset R^N \setminus B_{R_2}$:

$$\lim_{t \to 0^+} \sup \frac{1}{t^2} \int h(x,tw)tw = 0 \tag{45}$$

Where $R_2 \gg 1$ is some positive constant.

Let

$$\begin{aligned} H(x,u) &:= \int_0^u h(x,s)ds && (x,u) \in R^N \times R \\ J_g(u) &:= \frac{1}{2} \int |\nabla u|^2 + u^2 - \int G(x,u)dx && u \in H^1(R^N) \\ J_h(u) &:= \frac{1}{2} \int |\nabla u|^2 + u^2 - \int H(x,u)dx && u \in H^1(R^N) \end{aligned}$$

According to (42),for any $u \in H^1(R^N)$ with $\operatorname{supp} u \subset R^N \setminus B_{R_2}$,we have

$$\int G(x,u)dx \le \int H(x,u)dx \tag{46}$$

Let

$$m_h := \inf_{u \in V_h} J_h(u)$$

where

$$V_h := \{u \in H^1(R^N) \setminus \{0\} : \int |\nabla u|^2 + u^2 - \int h(x,u)u = 0\}$$

It is well known that $J_g, J_h \in C^1(H^1(R^N), R)$ under (38),(39),(42).

satisfies the following properties: There

$$,u)u \qquad |x| > R_1, u \in R \tag{42}$$

$$), \quad u(\cdot) \mapsto h(\cdot, u(\cdot))u(\cdot)$$
ap from $H^1(R^N)$ into $L^1(R^N)$. (43)

$$t)u \qquad t \geq 1, x \in R^N, u \in R. \tag{44}$$

upp $w \subset R^N \setminus B_{R_2}$:

$$\frac{1}{2} \int h(x,tw)tw = 0 \tag{45}$$

onstant.

$$; \qquad (x,u) \in R^N \times R$$

$$+ u^2 - \int G(x,u)dx \quad u \in H^1(R^N)$$

$$+ u^2 - \int H(x,u)dx \quad u \in H^1(R^N)$$

$H^1(R^N)$ with supp $u \subset R^N \setminus B_{R_2}$,we have

$$)dx \leq \int H(x,u)dx \tag{46}$$

$$= \inf_{u \in V_h} J_h(u)$$

$$: \int |\nabla u|^2 + u^2 - \int h(x,u)u = 0\}$$

$C^1(H^1(R^N), R)$ under (38),(39),(42).

**Theorem 2.1**: In addition to (38) through (45), there exist $n$ functions with disjoint supports $u_j \in H_0^1(R^N) \setminus \{0\}$, $j = 1, \ldots, n$, such that,

$$\sup_{u \in X_n} J_g(u) < m_h$$

where $n$ is a positive integer and

$$X_n := \{t_1 u_1 + \ldots + t_n u_n : (t_1, \ldots, t_n) \in R^n\} \subset H^1(R^N)$$

is a $n$ dimensional space.

Then (1) has at least n pairs of nontrivial solutions in $H^1(R^n)$ and one of them is positive.

In the following we denote $J_g$ as $J$.

**Remark 2.1** The existence of one positive solution under the hypotheses is classical,see [16] ,[9] and the references there.

**Lemma 2.1**: Under the hypotheses of Theorem 2.1, $J$ satisfies $(PS)_c$ for $c \in (-\infty, m_h)$.

**Proof:** Let $\{u_n\} \subset H^1(R^N)$ be a sequence satifying:

$$J(u_n) \to C' < m_h$$

$$J'(u_n) \to 0 \quad \text{strongly in } H^{-1}(R^N)$$

By a similar argument as in the proof of Lemma 1.2 we obtain the boundedness of $\{\|u_n\|\}$ and the existence of $u_\infty \in H^1(R^N)$ ,such that,along a sequence of $\{u_n\}$ (still being denoted as $\{u_n\}$)

$$\begin{cases} u_n \rightharpoonup u_\infty & \text{weakly in } H^1(R^N) \\ u_n \to u_\infty & \text{a.e. in } R^N \end{cases} \tag{47}$$

$$< J'(u_\infty), u_\infty >= 0 \tag{48}$$

There are only two possibilities for $\{u_n\}$.

**Case (1):** $\forall 0 < \delta < 1$, there exists $\tilde{R} > 0$, such that, $\forall n \in \tilde{R}$,

$$\int_{|x| \geq \tilde{R}} |\nabla u_n|^2 + u_n^2 < \delta$$

Case (2): There exists $\delta_0 > 0$, such that, $\forall \tilde{R} > 0$, there exists $n = n(\tilde{R}) \geq \tilde{R}$,such that

$$\int_{|x|\geq \tilde{R}} |\nabla u_n|^2 + u_n^2 < \delta_0$$

As in the proof of Lemma 1.2 we will prove that Case (1) leads to the strong convergence of $\{u_n\}$ in $H^1(R^N)$. In fact, $\forall \delta > 0$,there exists $\tilde{R} > 0$,such that, $\forall n > \tilde{R}$ we have

$$\int_{|x|\geq \tilde{R}} |\nabla u_n|^2 + u_n^2 < \delta \tag{49}$$

By the lower semicontinuity of the above integral and (47) we have

$$\int_{|x|\geq \tilde{R}} |\nabla u_\infty|^2 + u_\infty^2 \leq \delta \tag{50}$$

Since $J'(u_n) \to 0$ strongly in $H^1(R^N)$ we have

$$|\int \nabla u_n \nabla v + u_n v - \int g(x, u_n) v| \leq o(1)\|v\| \quad v \in H^1(R^n) \tag{51}$$

Use Hölder inequality,Sobolev inequality,Rellich lemma on bounded domains and (39)we have ,for any $v \in H^1(R^N)$

$$\begin{cases} |\int_{|x|\geq \tilde{R}} g(x,u_n)v| + |\int_{|x|\geq \tilde{R}} g(x,u_\infty)v| & \leq \quad C\delta^{\frac{1}{2}}\|v\| \\ |\int_{|x|\leq \tilde{R}} g(x,u_n)v - \int_{|x|\leq \tilde{R}} g(x,u_\infty)v| & \leq \quad o(1)\|v\| \end{cases} \tag{52}$$

Where $C$ is some constant independent of $\delta$ and $n$. Combine (51) and (52) we have

$$|\int \nabla u_n \nabla v + u_n v - \int g(x, u_\infty) v| \leq C\delta^{\frac{1}{2}}\|v\| + o(1)\|v\| \quad v \in H^1(R^N) \tag{53}$$

Since $\delta > 0$ can be chosen arbitrarily small,(53) implies the strong convergence of $\{u_n\}$ in $H^1(R^N)$.

To conclude the proof we only need to rule out Case (2) . As before we argue by contradiction .If Case (2) occurs, for that $\delta_0$,take $\tilde{R} = l = 1, 2, 3, \ldots$,there exists $n_l \geq l$,such that

$$\int_{|x|\geq l} |\nabla u_{n_l}|^2 + u_{n_l}^2 \geq \delta_0 \tag{54}$$

$> 0$, such that, $\forall \bar{R} > 0$, there exists $n =$

$$|\nabla u_n|^2 + u_n^2 < \delta_0$$

.2 we will prove that Case (1) leads to the $H^1(R^N)$. In fact, $\forall \delta > 0$,there exists $\tilde{R} >$

$$|\nabla u_n|^2 + u_n^2 < \delta \tag{49}$$

of the above integral and (47) we have

$$\nabla u_\infty|^2 + u_\infty^2 \le \delta \tag{50}$$

$H^1(R^N)$ we have

$$(x, u_n)v| \le o(1)\|v\| \quad v \in H^1(R^n) \tag{51}$$

inequality,Rellich lemma on bounded do- $v \in H^1(R^N)$

$$\begin{aligned} |\textstyle\int_{|x|\ge\tilde{R}} g(x,u_\infty)v| &\le C\delta^{\frac{1}{2}}\|v\| \\ \textstyle\int_{|x|\le\tilde{R}} g(x,u_\infty)v| &\le o(1)\|v\| \end{aligned} \tag{52}$$

ndent of $\delta$ and $n$. Combine (51) and (52)

$$v| \le C\delta^{\frac{1}{2}}\|v\| + o(1)\|v\| \quad v \in H^1(R^N) \tag{53}$$

itrarily small,(53) implies the strong con-

y need to rule out Case (2) . As before se (2) occurs, for that $\delta_0$,take $\tilde{R} = l =$ hat

$$u_{n_l}|^2 + u_{n_l}^2 \ge \delta_0 \tag{54}$$

Take $\epsilon > 0$ so small that

$$C\epsilon + C' < m_h$$

Here and later,$C > 0$ denotes some constant being determined in the following calculation,which is independend of $\epsilon$.

According to (43) and the fact that $u_\infty \in H^1(R^N)$ , there exists $R_0 > \max\{R_1, R_2\}$,such that

$$\int_{|x|\ge R_0} |\nabla u_\infty|^2 + u_\infty^2 + |q(x)||u_\infty|^{\sigma+2} < \epsilon \tag{55}$$

$$g(x,u)u \le h(x,u)u \qquad \forall |x| > R_2, u \in R \tag{56}$$

As in the proof of Theorem 1.1 we can find $R \ge R_0, I = \{x \in R^N : R \le |x| \le R+1\}$ a subsequence of $\{u_{n_l}\}$, which will still be denoted it as $\{u_{n_l}\}$,such that,

$$\int_I |\nabla u_{n_l}|^2 + u_{n_l}^2 < \epsilon \tag{57}$$

Let

$$v_l = \rho_1(x)u_{n_l}, \qquad w_l = \rho_2(x)u_{n_l} \quad l = 1,2,3,\ldots$$

we have

$$u_{n_l} = v_l + w_l, \quad \operatorname{supp} v_l \subset B_{R+1}, \quad \operatorname{supp} w_l \subset R^N \setminus B_R, \quad l = 1,2,3,\ldots$$

where $\rho_1, \rho_2$ are the same functions as in the proof of Theorem 1.1.

With (57) and the properties of $\rho_1, \rho_2$ and $g$ it is not difficult to see ,by using Sobolev embedding theorem,that

$$< J'(v_l), v_l > \; = \; o(1) + O(\epsilon) \tag{58}$$

$$< J'(w_l), w_l > \; = \; o(1) + O(\epsilon) \tag{59}$$

$$J(v_l) + J(w_l) \; = \; C' + o(1) + O(\epsilon) \tag{60}$$

Notice that according to (40),(48),(58) and Sobolev embedding theorem we have

$$J(v_l) = \frac{1}{2}\int_{B_{R+1}} \{g(x,v_l)v_l - G(x,v_l)\} + o(1) + O(\epsilon)$$

$$\int_{B_{R+1}} g(x,u_\infty)u_\infty \ge o(1) + O(\epsilon)$$

$$J(v_l) \ge o(1) + O(\epsilon)$$

It then follows that
$$J(w_l) \le C' + o(1) + O(\epsilon) \tag{61}$$

Here and later the formulas hold only for large $l$ .

Writing (59) and (61) more explicitly we have
$$\int |\nabla w_l|^2 + w_l^2 = \int g(x, w_l) w_l + O(\epsilon) \tag{62}$$
$$\frac{1}{2}\int |\nabla w_l|^2 + w_l^2 - \int G(x, w_l) \le C' + O(\epsilon) \tag{63}$$

According to (54) ,for $l > R + 1$,
$$\int |\nabla w_l|^2 + w_l^2 \ge \delta_0.$$

For $\epsilon > 0$ small,use (62) we have
$$\int g(x, w_l) \ge \frac{1}{2}\delta_0. \tag{64}$$

Consider
$$\xi(t) = \frac{1}{t^2}\int h(x, tw_l) tw_l \quad t \in R.$$

For $t = 1 + C\epsilon > 1$,where $C$ is large but independent of $\epsilon$, we deduce from (43),(44) and (64) that
$$\xi(t) \ge t^{\epsilon_1}\int g(x, w_l) w_l \ge \int g(x, w_l) w_l + O(\epsilon) = \int |\nabla w_l|^2 + w_l^2.$$

On the other hand, $\lim_{t\to 0^+}\xi(t) = 0$ according to (45). By the continuity of $\xi$,there exists some $t = t(l, \epsilon) \le 1 + C\epsilon$,such that,
$$\xi(t) = \int |\nabla w_l|^2 + w_l^2$$

Let $\eta_{l,\epsilon} := tw_l$,then $\eta_{l,\epsilon} \in V_h$, hence
$$\begin{aligned} m_h &\le J_{h,\epsilon}(\eta_l) \\ &= \frac{1}{2}t^2\int |\nabla w_l|^2 + w_l^2 - \int H(x, tw_l) \\ &\le \frac{t^2}{2}\int g(x, w_l) w_l + O(\epsilon) - \int G(x, tw_l) \end{aligned}$$

$$C' + o(1) + O(\epsilon) \tag{61}$$

only for large $l$ .
xplicitly we have

$$w_l^2 = \int g(x, w_l)w_l + O(\epsilon) \tag{62}$$

$$-\int G(x, w_l) \leq C' + O(\epsilon) \tag{63}$$

$$w_l|^2 + w_l^2 \geq \delta_0.$$

$$(x, w_l) \geq \frac{1}{2}\delta_0. \tag{64}$$

$$h(x, tw_l)tw_l \quad t \in R.$$

arge but independent of $\epsilon$, we deduce from

$$g(x, w_l)w_l + O(\epsilon) = \int |\nabla w_l|^2 + w_l^2.$$

$= 0$ according to (45). By the continuity of
$+ C\epsilon$,such that,

$$\int |\nabla w_l|^2 + w_l^2$$

:e

$$w_l|^2 + w_l^2 - \int H(x, tw_l)$$

$$w_l)w_l + O(\epsilon) - \int G(x, tw_l)$$

It follows immediately from (41) that

$$\frac{t^2}{2}\int g(x, w)w - \int G(x, tw) \text{ is nondecreasing for } 0 \leq t \leq 1.$$

Therefore if $t \leq 1$,

$$m_h \leq J(w_l) + O(\epsilon) \leq C' + O(\epsilon)$$

If $t \geq 1$, remember that $t \leq 1 + C\epsilon$,by the mean value theorem we have

$$m_h \leq C' + O(\epsilon)$$

This is a contradiction to the choice of $\epsilon$.

Proof of Theorem 2.1 follows immediatly from the application of Theorem A of the Appendix by letting $E = H^1(R^N), J = J_g$.

## 3 Appendix : A Minmax Lemma

Let $E$ be a real Banach space and $\Sigma(E) \equiv \Sigma$ the collection of $A \subset E \setminus \{0\}$ with $A$ closed in $E$ and symmetric with respect to the origin,namely, $-x \in A$ whenever $x \in A$. The set $A \subset E$ is said to have genus $n$ (denoted by $\gamma(A) = n$) if there exists an odd map $\varphi \in C(A, R^n \setminus \{0\})$ and $n$ is the smallest integer having this property. If $A = \phi$, we write $\gamma(A) = 0$ and if there is no such $n$, we set $\gamma(A) = +\infty$. For the properties of the genus,see [18].

Suppose that $E$ is an infinite dimensional Banach space, $J \in C^1(E, R)$, satisfying:
For some positive integer $n$, positive constant $C'$,

$$J \text{ satisfies } (PS)_c \quad \forall c \in (0, C') \tag{65}$$

$$J \text{ is even,namely } J(-u) = J(u) \tag{66}$$

There exists $\rho, \alpha > 0$, such that

$$\begin{cases} J > 0 & \text{in } B_\rho \setminus \{0\} \\ J \geq \alpha & \text{on } \partial B_\rho \end{cases} \tag{67}$$

There exists a $n-$dimensional subspace $X_n$ of $E$ ,such that

$$\begin{cases} X_n \cap \widehat{A}_0 \text{ is bounded} \\ \sup_{u \in X_n} J(u) < C' \end{cases} \tag{68}$$

where

$$\widehat{A}_0 := \{u \in E : J(u) \geq 0\}$$

Let

$$\Gamma^* := \{h : \ h \text{ is a homeomorphism of } E \text{ onto } E, \quad h(0) = 0, h(B_1) \subset \widehat{A}_0, h \text{ is odd } \}$$

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Mathematics Department
Rutgers University
New Brunswick, NJ 08903