An extension to a classical theorem of Liouville and applications

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Dedicated to Louis Nirenberg and Peter Lax with admiration and friendship

A classical theorem of Liouville says that a positive harmonic function in $\mathbb{R}^n$ must be a constant. The Laplace operator Δ is invariant under rigid motion: For any rigid motion $T : \mathbb{R}^n \to \mathbb{R}^n$, and for any function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta(u \circ T) = (\Delta u) \circ T.$$

Recall that T is called a rigid motion if $Tx \equiv Ox + b$ for some orthogonal $n \times n$ matrix O and some vector $b \in \mathbb{R}^n$.

Instead of rigid motions, let us look at Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$, and we look at operators invariant under Möbius transformations. A map $\varphi : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ is called a Möbius transformation, if it is a composition of a finitely many of the following three types of transformations:

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$$\text{A translation}: \quad x \to x + \bar{x}, \ \text{where } \bar{x} \ \text{is a given point in } \mathbb{R}^n,$$
$$\text{A dilation}: \quad x \to ax, \ \text{where } a \text{ is a positive number},$$
$$\text{A Kelvin transformation}: \quad x \to \tfrac{x}{|x|^2}.$$

For a function u on $\mathbb{R}^n$, let

$$u_\varphi := |J_\varphi|^{\frac{n-2}{2n}}(u \circ \varphi)$$

where J_φ denotes the Jacobian of φ.

Let $H(s, p, M)$ be a smooth function in its variables, where $s > 0$, $p \in \mathbb{R}^n$ and $M \in \mathcal{S}^{n\times n}$, the set of $n \times n$ real symmetric matrices. We say that the second order fully nonlinear operator $H(u, \nabla u, \nabla^2 u)$ is conformally invariant if

$$H(u_\varphi, \nabla u_\varphi, \nabla^2 u_\varphi) \equiv H(u, \nabla u, \nabla^2 u) \circ \varphi$$

holds for all positive smooth functions u and all Möbius transformations φ.

It was proved in [11] that a conformally invariant operator $H(u, \nabla u, \nabla^2 u)$ must be of the form

$$H(u, \nabla u, \nabla^2 u) \equiv f(\lambda(A^u))$$

where

$$A^u := w\nabla^2 w - \frac{|\nabla w|^2}{2} I, \qquad \text{with } w = u^{-\frac{2}{n-2}},$$

$\lambda(A^u)$ denotes the eigenvalues of A^u, I is the $n \times n$ identity matrix, $f(\lambda)$ is some symmetric function in $\lambda = (\lambda_1, \cdots, \lambda_n)$.

Let φ be a Möbius transformation, then for some $n \times n$ orthogonal matrix functions $O(x)$ (i.e. $O(x)O(x)^t = I$), depending on φ,

$$A^{u_\varphi}(x) \equiv O(x)A^u(\varphi(x))O^t(x).$$

Thus $f(\lambda(A^u))$ is a conformally invariant operator for all symmetric functions f.

Taking $f(\lambda) = \sigma_1(\lambda) := \lambda_1 + \cdots + \lambda_n$, we have a simple expression:

$$\sigma_1(\lambda(A^u)) \equiv -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u.$$

In general, $f(\lambda(A^u))$ is a fully nonlinear operator, and is rather complex even for $f(\lambda) = \sigma_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$, the $k-$th elementary symmetric function. The expression for σ_2 is still quite pleasant:

$$\sigma_2(\lambda(A^u)) \equiv \frac{1}{2}\left(\sigma_1(A^u)^2 - (A^u)^t A^u\right),$$

where $(A^u)^t$ denotes the transpose matrix of A^u.

For a Riemannian metric g,

$$A_g := \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right)$$

is called Schouten tensor. Let $g_{Eucl.} = \delta_{ij}dx^i dx^j$ be the Euclidean metric, and let, for a positive function u, $g := u^{\frac{4}{n-2}} g_{Eucl}$ be conformal to the Euclidean metric, then A^u is associated with the Schouten tensor A_g as follows:

$$A_g = u^{\frac{4}{n-2}} A^u_{ij} dx^i dx^j.$$

The so called σ_k-Yamabe problem was studied in [19] and [20], which involves the Schouten tensor A_g. Equations involving σ_2 and A_g, with applications in topology and geometry, were studied in [2].

Let

$$\Gamma \subset \mathbb{R}^n \text{ be open convex symmetric cone with vertex at the origin} \tag{1}$$

satisfying

$$\Gamma_n := \{\lambda \mid \lambda_i > 0, \forall\, i\} \subset \Gamma \subset \{\lambda \mid \sum_{i=1}^n \lambda_i > 0\} =: \Gamma_1. \tag{2}$$

Naturally, Γ being symmetric means that $(\lambda_1, \cdots, \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \cdots, \lambda_{i_n}) \in \Gamma$ for any permutation $(\lambda_{i_1}, \cdots, \lambda_{i_n})$ of $(\lambda_1, \cdots, \lambda_n)$.

Theorem 1 *([15], [16]) Let u be a positive function in $C^{0,1}_{loc}(\mathbb{R}^n)$ satisfying*

$$\lambda(A^u) \in \partial\Gamma \quad \text{in } \mathbb{R}^n \text{ in the viscosity sense.} \tag{3}$$

Then $u \equiv$ Constant.

A positive continuous function u is said to satisfy (3) when the following holds: if $x_0 \in \mathbb{R}^n$, $\psi \in C^2$, $(u-\psi)(x_0) = 0$, $u - \psi \le 0$ near x_0, then $\lambda(A^\psi(x_0)) \in \mathbb{R}^n \setminus \Gamma$; if $\psi \in C^2$, $(u-\psi)(x_0) = 0$, $u - \psi \ge 0$ near x_0, then $\lambda(A^\psi(x_0)) \in \overline{\Gamma}$.

It is not difficult to see that if u is a positive $C^{1,1}$ function in $\mathbb{R}^n$ satisfying $\lambda(A^u) \in \partial\Gamma$ a.e. in $\mathbb{R}^n$, then u satisfies (3).

Fully nonlinear second order elliptic equations with $\lambda(\nabla^2 u)$ in such Γ was first studied by Caffarelli, Nirenberg and Spruck in [1]. Equation (3) is a fully nonlinear degenerate elliptic equation. The proof of Theorem 1 makes use of the method of moving spheres, a variant of the method of moving planes as in Gidas, Ni and Nirenberg [5].

For $u \in C^2$, $\lambda(A^u) \in \partial\Gamma_1$ means $\Delta u = 0$, and Theorem 1 says that a positive harmonic function in $\mathbb{R}^n$ is constant — a classical theorem of Liouville.

Let

$$\Gamma_k := \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\},$$

then $\Gamma = \Gamma_k$ satisfies (1) and (2)..

Such Liouville theorem was proved, and used in deriving global apriori C^0 and C^1 estimates for solutions, in [3] for $u \in C^{1,1}_{loc}$, $\Gamma = \Gamma_2$ and $n = 4$; in [10] for $u \in C^{1,1}_{loc}$, $\Gamma = \Gamma_2$ and $n = 3$; in ([14, v1]) for $u \in C^{1,1}_{loc}$, Γ satisfying (1), (2) and $n \geq 3$. Our proofs of the Liouville theorems in [14] and [15] are entirely different from arguments in [3] and [10]. In particular, the proof of Theorem 1 in [15] uses only the following properties of the set of $C^{0,1}_{loc}(\mathbb{R}^n)$ viscosity solutions of $\lambda(A^u) \in \partial\Gamma$: It is a subset of the set of positive continuous superharmonic functions in $\mathbb{R}^n$, it is conformally invariant, it is invariant under positive constant scalar multiplication, and some weak comparison principle holds for functions in it.

The motivation of our study of such Liouville properties of entire solutions of $\lambda(A^u) \in \partial\Gamma$ is to answer the following questions concerning local gradient estimates of solutions to general second order conformally invariant fully nonlinear elliptic equations.

Assume

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is a symmetric function,} \tag{4}$$

$$f \text{ is homogeneous of degree 1,} \tag{5}$$

$$f > 0, f_{\lambda_i} := \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \ f|_{\partial\Gamma} = 0, \tag{6}$$

$$\sum_{i=1}^n f_{\lambda_i} \geq \delta, \quad \text{in } \Gamma \text{ for some } \delta > 0. \tag{7}$$

Note that $(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$ satisfies (4)-(7).

Question 1 *Let $n \geq 3$, (f, Γ) satisfy (1), (2), (4)-(7). For constants $0 < a < \infty$ and $0 < h \leq \bar{h} \leq 1$, let u be a C^3 function in B_2, a ball of radius 2 in $\mathbb{R}^n$, satisfying*

$$f(\lambda(A^u)) = h, \ 0 < u \leq a, \ \lambda(A^u) \in \Gamma, \quad \textit{in } B_2. \tag{8}$$

Is it true that

$$|\nabla \log u| \leq C \quad \textit{in } B_1 \tag{9}$$

for some constant C depending only on a, $\bar{h}$ and (f, Γ)?

A more general question on Riemannian manifolds is

Question 2 *Let g be a smooth Riemannian metric on $B_2 \subset \mathbb{R}^n$, $n \geq 3$, (f, Γ) satisfy (1), (2), (4)-(7). For a positive number a and a positive function $h \in C^1(B_2)$, let $u \in C^3(B_2)$ satisfy, with $\tilde{g} := u^{\frac{4}{n-2}} g$,*

$$f(\lambda(A_{\tilde{g}})) = h, \ 0 < u \leq a, \quad \lambda(A_{\tilde{g}}) \in \Gamma, \quad \text{in } B_2,$$

where $\lambda(A_{\tilde{g}})$ denotes the eigenvalues of $A_{\tilde{g}}$ with respect to $\tilde{g}$. Is it true that

$$\|\nabla \log u\|_g \leq C \quad \text{in } B_1$$

for some constant C depending only on a, g, $\|h\|_{C^1(B_2)}$ and (f, Γ)?

Theorem 1 is a crucial ingredient in our proof of the following optimal local gradient estimate.

Theorem 2 *([15], [16]) The answer to Question 2 is "Yes".*

Such local gradient estimates have been studied by a number of authors. See [7] for $(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$ as well as the efforts of achieving further generality in [11], [8], [6] and [17]. If f is assumed in addition to be concave in Γ which includes $(\sigma_k^{\frac{1}{k}}, \Gamma_k)$, such local gradient estimate, depending on $\|h\|_{C^2(B_2)}$ instead of $\|h\|_{C^1(B_2)}$, is a consequence of the Liouville theorem for $C^{1,1}_{loc}$ solutions in ([14, v1]) and the estimates in [11], see [15], [16] and [18]; a completely different proof of this is given in [4]. The concavity of f is not assumed in Theorem 2.

In the rest of this note, we assume Theorem 1 and outline the arguments, in three steps, which give an affirmative answer to Question 1. For details, as well as for the proofs of Theorem 1 and Theorem 2, see [15].

A subtlety of (9) is that the constant C does not depend on the lower bound of u. Indeed we have

Step 1. ([12]) If we further assume in (8) that $u \geq b$ in B_2 for some positive constant b, then (9) holds for some constant C depending only on a, b, $\bar{h}$ and (f, Γ).

This result was extended to manifolds with boundary under prescribed mean curvature boundary conditions in [9], see theorem 1.3 there.

Next, we first estimate semi-Hölder norm of $\log u$ instead of the gradient of $\log u$. Namely, we have

Step 2. For $0 < \alpha < 1$, $[\log u]_{\alpha, B_1} \leq C(\alpha)$.

For $|x| < 3/2$ and $0 < \delta < 1/8$, let

$$[v]_{\alpha,\delta}(x) := \sup_{|y-x|<\delta} \frac{|v(y) - v(x)|}{|y - x|^\alpha},$$

$$\delta(v, x; \alpha) := \begin{cases} \infty & \text{if } [v]_{\alpha,1}(x) < 1, \\ \mu \text{ where } \mu \in (0,1],\ \mu^\alpha [v]_{\alpha,\mu}(x) = 1 & \text{if } [v]_{\alpha,1}(x) \geq 1. \end{cases}$$

We establish Step 2 by contradiction. Suppose the contrary, then, for some positive constants a and $h_i \leq 1$, there exists a sequence of solutions $\{u_i\}$ to $f(\lambda(A^{u_i})) = h_i$, $\lambda(A^{u_i}) \in \Gamma$ in B_2 such that $\inf_{x \in B_1} \delta(\log u_i, x; \alpha) \to 0$. For simplicity, we assume

$$\delta(\log u_i, 0) := \delta(\log u_i, 0; \alpha) = \inf_{x \in B_1} \delta(\log u_i, x; \alpha) \to 0.$$

Rescale u_i:

$$v_i(y) := \frac{1}{u_i(0)} u_i(\epsilon_i y), \qquad |y| \leq \frac{1}{\epsilon_i},$$

where

$$\epsilon_i := \delta(\log u_i, 0).$$

Then,

$$[\log v_i]_{\alpha,1}(0) = 1, \tag{10}$$

and, for all $1 < \beta < \infty$ there exists some positive constant $C(\beta)$ such that

$$[\log v_i]_{\alpha,1}(x) \leq C(\beta) \qquad \forall\ |x| \leq \beta.$$

Since $\log v_i(0) = 0$, we have, for some positive constant $C(\beta)$,

$$\frac{1}{C(\beta)} \leq v_i(x) \leq C(\beta) \qquad \forall\ |x| \leq \beta. \tag{11}$$

Since u_i satisfies

$$f(\lambda(A^{u_i})) = h_i,\ 0 < u_i < a,\ \lambda(A^{u_i}) \in \Gamma,$$

v_i satisfies

$$f(\lambda(A^{v_i})) = \epsilon_i^2 u_i(0)^{\frac{4}{n-2}} h_i,\ \lambda(A^{v_i}) \in \Gamma. \tag{12}$$

Thus, in view of (11) and Step 1,

$$|\nabla v_i(x)| \leq C(\beta) \qquad \forall\ |x| \leq \beta.$$

Passing to a subsequence, we have, for any $\alpha < \gamma < 1$,

$$v_i \to v \qquad \text{in } C^{\gamma}_{loc}(\mathbb{R}^n) \tag{13}$$

for some positive function v in $C^{0,1}_{loc}(\mathbb{R}^n)$.

It follows from (10) and (13) that

$$[\log v]_{\alpha,1}(0) = 1.$$

In particular, v is not a constant.

On the other hand, it is easy to see from (12) that v is a $C^{0,1}_{loc}(R^n)$ viscosity solution of $\lambda(A^v) \in \partial\Gamma$. By Theorem 1, v is a constant. A contradiction. Step 2 is established.

Step 3. $|\nabla \log u| \le C$.

Step 2 implies the Harnack inequality:

$$\sup_{B_{\frac{3}{2}}} u \le C \inf_{B_{\frac{3}{2}}} u.$$

Let

$$w := \frac{u}{u(0)}.$$

Then w satisfies

$$f(\lambda(A^w)) = u(0)^{\frac{4}{n-2}} h, \quad \frac{1}{C} \le w \le C, \qquad \text{in } B_{\frac{3}{2}}.$$

By Step 1,

$$|\nabla \log u| = |\nabla \log w| \le C \qquad \text{in } B_1.$$

Step 3 is established.

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