Name

QUIZ 11 — Math 250:09

1. (4 points) Answer True or False to each of the following questions. If your answer to a question is False, you NEED to provide a counter-example.

a) If the different columns of an $n \times n$ matrix are orthogonal to each other, then this matrix is an orthogonal matrix. F

$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, Columns are orthogonal to each other, but A is not orthogonal

b) If the determinant of an $n \times n$ matrix is ± 1, then this matrix is an orthogonal matrix.

F

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\det A = 1$, but A is not an orthogonal matrix

c) If A is an orthgonal matrix, then so is A^{-1}. T

d) If A is an $n \times n$ matrix and there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A, the A is symmetric. T

2. (16 ponits) Given the matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 3 \end{bmatrix}$.

a) Find the characteristic polynomial for A and verify that $\lambda = -1$ and $\lambda = 4$ are the eigenvalues of A.

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & 3-\lambda \end{bmatrix} = (-1-\lambda) \det \begin{bmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$= (-1-\lambda)[-\lambda(3-\lambda) - 4] = (-1-\lambda)(\lambda^2 - 3\lambda - 4)$$

$$= (-1-\lambda)(\lambda - 4)(\lambda + 1).$$

So $\lambda = -1$, $\lambda = 4$ are the eigenvalues of A.

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b) Find an orthonormal basis for each eigenspace of A.

For $\lambda = -1$, solve $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -2z \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{-1}. Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$, a unit vector.

For $\lambda = 4$, solve $\begin{bmatrix} -5 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 2y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. Then $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$, and are also eigenvectors of A.

c) Find an orthogonal matrix P such that $P^t AP$ is diagonal.

From (b), if we use $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$, then

P is an orthogonal matrix, and

$AP = P\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow P^t A P = P^{-1} A P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$