

QUIZ 5 — Math 250:C2

1. (4 points) Answer true or false to each of the following questions.

- a) For any two matrices A and B , if $AB = I_n$ for some positive integer n , then A is invertible. F.
- b) If A is an $n \times n$ matrix such that $Ax = 0$ has only the solution $x = 0$, then A is invertible. T.
- c) If A is an $n \times n$ matrix such that $Ax = b$ is consistent for every $b \in \mathbb{R}^n$, then A is invertible. T.
- d) An $n \times n$ matrix A is invertible if and only if its columns are linearly independent. T.

2. (8 points) Given the matrix

$$A = \begin{bmatrix} 1 & -3 & 0 & -2 \\ -1 & 5 & 1 & 0 \\ 2 & -8 & 2 & 4 \end{bmatrix}.$$

Find the LU decomposition of A .

$$A \xrightarrow{r_1+r_2} \begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & 2 & 1 & -2 \\ 2 & -8 & 2 & 4 \end{bmatrix} \xrightarrow{(-2)r_1+r_3} \begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & 2 & 1 & -2 \\ 0 & -2 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow{r_2+r_3} \begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & 6 \end{bmatrix} := U.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

turn over to continue

3. (8 points) Given the LU decomposition of A as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & -5 & -2 \end{bmatrix}.$$

Use the LU decomposition of A to solve

$$Ax = \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}.$$

Set $U\vec{x} = \vec{y}$, then $L\vec{y} = \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}$, i.e. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}$

$$\Rightarrow \begin{cases} y_1 = 3 \\ -y_1 + y_2 = 10 \\ 3y_1 + 2y_2 + y_3 = -5 \end{cases} \Rightarrow \begin{cases} y_1 = 3 \\ y_2 = 13 \\ y_3 = -5 - 3y_1 - 2y_2 = -5 - 3(3) - 2(13) \\ = -40 \end{cases}$$

Next, we solve $U\vec{x} = \begin{pmatrix} 3 \\ 13 \\ -40 \end{pmatrix}$, i.e.

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ -40 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 + 2x_4 = 3 \\ 2x_2 + 4x_3 + 3x_4 = 13 \\ -5x_3 - 2x_4 = -40 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 - x_3 - 2x_4 + 3 \\ x_2 = -2x_3 - \frac{3}{2}x_4 + \frac{13}{2} \\ x_3 = -\frac{2}{5}x_4 + 8 \end{cases}$$

$$x_2 = -2\left(-\frac{2}{5}x_4 + 8\right) - \frac{3}{2}x_4 + \frac{13}{2} = -\frac{19}{2} - \frac{7}{10}x_4$$

$$x_1 = 3 - \frac{19}{2} - \frac{7}{10}x_4 + \frac{2}{5}x_4 - 8 - 2x_4 = -\frac{29}{2} - \frac{23}{10}x_4$$

So the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{29}{2} - \frac{23}{10}x_4 \\ -\frac{19}{2} - \frac{7}{10}x_4 \\ -8 - \frac{2}{5}x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{29}{2} \\ -\frac{19}{2} \\ 8 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{23}{10} \\ -\frac{7}{10} \\ -\frac{2}{5} \\ 1 \end{pmatrix}$$

QUIZ 6 — Math 250:C2

1. (4 points) Answer True or False to each of the following questions.

a) For any $n \times n$ matrices A and B , $\det(A+B) = \det A + \det B$. Fb) The determinant of any square matrix equals the product of the diagonal entries of its reduced row echelon form. Fc) If A^{-1} exists, then $\det A \neq 0$. Td) If A is any square matrix and c is a scalar, then $\det(cA) = c \det A$. F

2. (8 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$A \xrightarrow{-r_1+r_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 2 & -2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{\begin{matrix} -r_2+r_3 \\ r_2+r_4 \end{matrix}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{(-2)r_3+r_4} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since all the steps preserve determinants, $\det A = 0$.

3. (8 points) Determine the value(s) of c for which the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -2 & -c \\ 0 & c & -10 \end{bmatrix}$$

is not invertible.

Expanding in 3rd row,

$$\det \mathbf{A} = c \cdot (-1)^{3+2} \det \begin{bmatrix} -1 & 1 \\ 3 & -c \end{bmatrix} + (-10) \cdot (-1)^{3+3} \det \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$= -c(c-3) - 10 \cdot (2-3)$$

$$= -c^2 + 3c + 10 = -(c-5)(c+2) = 0$$

$$\text{N} \mid c=5 \text{ or } c=-2.$$

QUIZ 7 — Math 250:C2

1. (5 points) Answer True or False to each of the following questions.

- a) A basis for the row space of a matrix is also a basis for the row space of its reduced row echelon form. T
- b) The null space of any matrix equals the null space of its reduced row echelon form. T
- c) The dimension of column space of a matrix equals the dimension of the column space of its reduced row echelon form. T
- d) The columns of a 3×4 matrix are linearly dependent. T
- e) The dimension of $\text{span}\{u_1, \dots, u_k\}$ is equal to k . F

2. (15 points) Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

(a). Find a smallest spanning set for the row space of A .

$$A \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} := R$$

Since $\text{Row}(A) = \text{Row}(R)$, and $\text{Row}(R) = \text{span}\{[1 \ 2 \ 0 \ 1], [0 \ -1 \ 1 \ -1]\}$, we find that $\{[1 \ 2 \ 0 \ 1], [0 \ -1 \ 1 \ -1]\}$ is a smallest spanning set of $\text{Row}(A)$.

turn over to continue

(b). Find the dimension and a basis for the nullspace of A .

From computations in (a),
$$1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1 + 2x_2 + x_4 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ x_3, x_4 \text{ free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2(x_3 - x_4) - x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Null}(A)$, and its dimension is 2.

(c). Find the dimension and a basis for the column space of A .

From the echelon form R in (a), $\text{Col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Since these two vectors are l.i., they form a basis for

$\text{Col}(A)$, and \dim of $\text{col}(A)$ is 2.

QUIZ 8 — Math 250:K2

1. (4 points) Answer True or False to each of the following questions.

- a) If two $n \times n$ matrices have the same characteristic polynomials, then they have the same eigenvectors. F
- b) The eigenvalues of a matrix are equal to those of its reduced row echelon form. F
- c) The dimension of eigenspace of a matrix corresponding to an eigenvalue equals its multiplicity. F
- d) If A is a diagonalizable matrix, then there is a unique diagonal matrix D such that $A = PDP^{-1}$ for some invertible P . F

2. (16 points) Given the matrix

$$A = \begin{bmatrix} -7 & 6 & 6 \\ 0 & -1 & 0 \\ -12 & 12 & 11 \end{bmatrix}.$$

(a). Verify that the characteristic polynomial of A is $-(\lambda + 1)^2(\lambda - 5)$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -7-\lambda & 6 & 6 \\ 0 & -1-\lambda & 0 \\ -12 & 12 & 11-\lambda \end{bmatrix} = (-1-\lambda) \det \begin{bmatrix} -7-\lambda & 6 \\ -12 & 11-\lambda \end{bmatrix} \quad (\text{cofactor exp in 2nd row})$$

$$= (-1-\lambda) [(-7-\lambda)(11-\lambda) - (-12)6] = (-1-\lambda) [\lambda^2 - 4\lambda - 5]$$

$$= -(\lambda+1)^2(\lambda-5).$$

(b). Find the eigenvalues of A .

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2 The eigenvalues of A are the roots of $-(\lambda+1)^2(\lambda-5) = 0$.
So $\lambda = -1$ is an eigenvalue of A with multiplicity 2;
 $\lambda = 5$ is an eigenvalue of A with multiplicity 1.

turn over to continue

(c). Find a basis for each eigenspace of A .

For $\lambda = -1$, Solve $\begin{bmatrix} -7-(-1) & 6 & 6 \\ 0 & -(-1) & 0 \\ -12 & 12 & 11-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e.

4 $\begin{cases} -x_1 + x_2 + x_3 = 0 \\ x_2, x_3 \text{ free} \end{cases}$. So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A with $\lambda = -1$.

For $\lambda = 5$, Solve $\begin{bmatrix} -7-5 & 6 & 6 \\ 0 & -1-5 & 0 \\ -12 & 12 & 11-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

4 $\begin{bmatrix} -12 & 6 & 6 \\ 0 & -6 & 0 \\ -12 & 12 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} 2x_1 - x_2 - x_3 = 0 \\ x_2 = 0 \\ x_3 = \text{free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$. $\left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A with $\lambda = 5$.

(d). Determine whether A is diagonalizable. If it is, find a diagonal matrix D and invertible matrix Q such that $A = QDQ^{-1}$.

2 From (c), A has 3 linearly independent eigenvectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let $Q = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then

$$A = QDQ^{-1}$$