

**TAKE HOME FINAL EXAM FOR MATH517: PDE      SUBMIT SOLUTIONS TO FIVE PROBLEMS BY NOON, DEC. 19**

- (1) Recall that if  $c_n$  are chosen to be the Fourier coefficients of  $g(x)$ , where  $g$  is a  $2\pi$ -periodic function defined on the circle  $\mathbb{S}^1$ , then

$$u(t, x) = \sum_{n=-\infty}^{\infty} c_n e^{-n^2 t} e^{inx}$$

is the solution to

$$\begin{cases} u_t(t, x) - u_{xx}(t, x) = 0, & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{S}^1, \\ u(0, x) = g(x), & \text{for } x \in \mathbb{S}^1, \end{cases}$$

where the initial data was proved to be taken in the sense of  $L^2(\mathbb{S}^1)$  for  $g \in L^2(\mathbb{S}^1)$ .

- (a) Prove that for  $g \in C^1(\mathbb{S}^1)$ , the  $u$  defined above is in  $C^0([0, \infty) \times \mathbb{S}^1)$  and takes on the initial data in pointwise sense.  
 (b) Prove that for  $g \in C^0(\mathbb{S}^1)$ , the  $u$  defined above is in  $C^0([0, \infty) \times \mathbb{S}^1)$  and takes on the initial data in pointwise sense. (Hint: think of maximum principle).  
 (c) Define

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx} \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{S}^1,$$

then the  $u$  above can be thought of as  $H_t * g$ . Prove that

$$H_t(x) \geq 0 \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{S}^1.$$

In fact, the stronger statement  $H_t(x) > 0$  holds for  $(t, x) \in \mathbb{R}^+ \times \mathbb{S}^1$ .

- (2) Consider the parabolic operator  $L[u] = u_t - \sum_{i,j=1}^n a_{ij}(x, t) \partial_{x_i x_j}^2 u(x, t)$  in  $Q_1 := \{(x, t) : |x| < 1, 0 < t < 1\}$ , where we assume that for some  $0 < \lambda \leq \Lambda$ ,  $\lambda I \leq (a_{ij}(x, t)) \leq \Lambda I$  for all  $(x, t) \in Q_1$ . Assume, in addition, that  $a_{ij}(x, t) \in C^1(Q_1)$ , and there exists  $M > 0$  such that

$$\frac{|\partial_x a_{ij}(x, t)|}{\lambda} \leq M$$

for all  $(x, t) \in Q_1$ . Suppose that  $u(x, t)$  is a solution to  $L[u] = 0$  in  $Q_1$  and  $\partial_x^3 u, \partial_{xt}^2 u \in C(Q_1)$ . Modify Bernstein's method to prove that there exists some  $A > 0$  depending on  $M$  and  $\Lambda/\lambda$ , such that

$$\max\{|\partial_x u(x, t)| : |x| \leq 1/2, \frac{3}{4} \leq t \leq 1\} \leq A \max_{Q_1} |u(x, t)|.$$

- (3) (a) Let  $\Delta_n$  denote the Laplace operator in  $\mathbb{R}^n$ . Suppose  $u \in C^2(\mathbb{R} \times \mathbb{R}^n)$  is a solution of

$$u_{tt}(t, x) - \Delta_n u(t, x) + k^2 u(t, x) = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}^n.$$

Then  $v(t, x, x_{n+1}) = [A \cos(kx_{n+1}) + B \sin(kx_{n+1})] u(t, x)$  solves

$$v_{tt}(t, x, x_{n+1}) - \Delta_{n+1} v(t, x, x_{n+1}) = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}^{n+1}.$$

(b) Use the method of descent to solve the Cauchy problem

$$\begin{cases} u_{tt}(t, x) - \Delta_2 u(t, x) \pm k^2 u(t, x) = 0, & \text{in } \mathbb{R} \times \mathbb{R}^2, \\ u(0, x) = g(x), \\ u_t(0, x) = h(x). \end{cases}$$

(4) Consider the Cauchy problem for the wave equation in  $\mathbb{R} \times \mathbb{R}^n$  with initial data given on the hyperplane  $H = \{(t, x) : \nu_0 t + \sum_{i=1}^n \nu_i x_i = 0\}$ :

$$\begin{cases} u_{tt}(t, x) - \Delta_n u(t, x) = f(t, x), & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(t, x) = u_0(t, x), & \text{for } (t, x) \in H, \\ \nu_0 u_t(t, x) + \sum_{i=1}^n \nu_i u_{x_i}(t, x) = u_1(t, x), & \text{for } (t, x) \in H. \end{cases} \quad (\text{h})$$

(a) Make the change of variables  $(t, x) \mapsto (\tau = \nu_0 t + \sum_{i=1}^n \nu_i x_i, x)$  and  $v(\tau, x) = u(t, x)$ . Derive an IVP for  $v$  and prove that  $H$  is characteristic iff  $\nu_0^2 - \sum_{i=1}^n \nu_i^2 = 0$ .

(b) Prove that, for  $n = 1$ , IVP (h) is well posed iff  $H$  is non-characteristic.

(c) Prove that, for  $n \geq 2$ , IVP (h) is well posed iff  $H$  is space-like, i.e.,  $\nu_0^2 - \sum_{i=1}^n \nu_i^2 > 0$ .

(5) Use the Hadamard-Petrowsky criterion to discuss the wellposedness to the Cauchy problem (with  $\{t = 0\}$  as initial surface) for the following operators

(a)  $\partial_t \pm \Delta^2$ .

(b)  $\partial_t + i\Delta + c$ , where  $c$  is a constant complex parameter, possibly non-real.

(6) Solve  $u_y + uu_x = 2$  with initial data  $u(x, 0) = -x$  and discuss the maximal domain for this solution to be  $C^1$ .

(7) Solve  $u_y = 4u_x^2$  with initial data  $u(x, 0) = x^2$  and discuss the maximal domain for this solution to be  $C^1$ .

(8) Solve the following initial value problem (locally, if necessary):  $u_y = u_x^3$ ,  $u(x, 0) = 2x^{3/2}$ . Also prove that if  $u(x, y) \in C^1(\mathbb{R}^2)$  is a global solution to  $u_y = u_x^3$ , then  $u$  is a linear function of  $x$  and  $y$ .