

Overview. *In the next couple lectures, we aim to develop methods to solve boundary value problem of the kind*

$$\begin{cases} \Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \quad (1)$$

In the beginning lectures, we discussed the separation of variables method to solve (1) in the case U is a round disk in \mathbb{R}^2 and $f \equiv 0$ in U . The same method can also be made to work on higher dimensional round balls, and the eigenfunction expansion method can be used to solve (1) on such domains when f is not identically 0. However these methods leave us no clue as for how to approach (1) on general domains. The first property of solutions to (1) that we will develop is the maximum principle, which states that for a bounded domain U , there exists a constant $C > 0$ depending only on the domain U such that

$$|u|_{C^0(\bar{U})} \leq |u|_{C^0(\partial U)} + C|\Delta u|_{C^0(\bar{U})}. \quad (2)$$

One immediate consequence of the maximum principle is the uniqueness of solution to (1) in the class $C^2(U) \cap C^0(\bar{U})$: suppose $u, v \in C^2(U) \cap C^0(\bar{U})$ are solutions to (1), then $|u - v|_{C^0(\partial U)} = 0$, therefore

$$|u - v|_{C^0(\bar{U})} \leq |u - v|_{C^0(\partial U)} + C|\Delta(u - v)|_{C^0(\bar{U})} = 0,$$

thus $u \equiv v$ in U . Another consequence of the maximum principle is the following convergence property: suppose that $u_k \in C^2(U) \cap C^0(\bar{U})$ is the unique solution to (1) with f_k, g_k replacing f, g , respectively, and suppose that there exists $f \in C^0(\bar{U})$ and $g \in C^0(\partial U)$ such that $f_k \rightarrow f$ in $C^0(\bar{U})$ and $g_k \rightarrow g$ in $C^0(\partial U)$, then we see through (2) applied to $u_k - u_l$ that $\{u_k\}$ is Cauchy in $C^0(\bar{U})$, therefore, there exists a limit $u \in C^0(\bar{U})$ with $u = g$ on ∂U . In order for u to solve (1) in the classical sense, we need to find conditions which guarantee that $u_k \rightarrow u$ not only in $C^0(\bar{U})$, but also in $C_{local}^2(U)$, at least after extracting a subsequence. This can be achieved via Arzela-Ascoli theorem if we can prove that derivatives of u_k up to second order are equicontinuous on any compact subset of U . This result can be proved relatively easily for solutions to (1) when $f \equiv 0$ in U , namely, for harmonic functions. See Theorem 4 below. The precise statements on the convergence of solutions to (1) are given in Theorems 6 and 7 below. The equicontinuity of derivatives of solutions of (1) for general f requires some control on the modulus of continuity of f , and can be developed using potential representation. There is a theory, called the Schauder theory, that generalizes such estimates to solutions of elliptic equations with Hölder continuous variable coefficients. We will not have time to develop this theory in this course. Our main focus will be to find conditions on U and g such that we have a reasonably complete result on the solvability of (1) for the case $f \equiv 0$. Although the convergence result in Theorem 7 looks like a plausible approach, a complete result would require the solvability of (1) for a dense set of data. This can be provided by Poincaré's method of balayage. An alternative approach is to use Perron's method which involves the concept of subharmonic functions. In order to develop the derivatives estimates for harmonic functions in Theorem 4, we first discuss Green's representation and some of

its consequences, including the regularity properties of harmonic functions, the concept of Green's functions, and the construction of Green's function on round balls and half spaces in any dimension.

First we state the Green's identity.

Proposition 1. *Suppose that U is a bounded domain with piecewise C^1 boundary, and $u, v \in C^2(\bar{U})$. Then*

$$\int_U [u(x)\Delta v(x) - v(x)\Delta u(x)] dx = \int_{\partial U} \left[u(x)\frac{\partial v(x)}{\partial n(x)} - v(x)\frac{\partial u(x)}{\partial n(x)} \right] d\sigma(x). \quad (3)$$

Remark 1. (3) continues to hold when u and v satisfy the weaker condition that $u, v \in C^1(\bar{U}) \cap C^2(U)$ as long as Δu and Δv are in $L^1(U)$.

Suppose that $u \in C^2(U)$ is harmonic in U , and $B_R(x_0) \subset U$, then applying (3) to u and $v \equiv 1$ on $B_r(x_0)$ for any $0 < r < R$, we obtain

$$\begin{aligned} 0 &= \int_{\partial B_r(x_0)} \frac{\partial u(x)}{\partial n(x)} d\sigma(x) \\ &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u(x_0 + r\omega)}{\partial r} d\omega \\ &= r^{n-1} \frac{\partial}{\partial r} \left(\int_{\partial B_1(0)} u(x_0 + r\omega) d\omega \right), \end{aligned}$$

from which it follows that $\int_{\partial B_1(0)} u(x_0 + r\omega) d\omega$ is a constant independent of r for $0 < r < R$, thus equals $|\partial B_1(0)|u(x_0)$. We have thus proved

Proposition 2. *Suppose that $u \in C^2(U)$ is harmonic in U , and $B_R(x_0) \subset U$. Then for any $0 < r < R$,*

$$u(x_0) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(x_0 + r\omega) d\omega = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(x) d\sigma(x),$$

and

$$u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x) dx.$$

As a consequence of Proposition 2, we have

Theorem 1. *Suppose that $u \in C^2(U) \cap C^0(\bar{U})$ is harmonic in U . Then*

$$\min_{\partial U} u = \min_{\bar{U}} u \leq \max_{\bar{U}} u = \max_{\partial U} u.$$

Furthermore, if U is connected, u can not attain $\max_{\bar{U}}$ or $\min_{\bar{U}}$ u in U unless u is a constant in U .

There are versions of maximum principle for the so called subharmonic (superharmonic) functions.

Exercise 1. A $C^2(U)$ function is called subharmonic (superharmonic) in U , if $\Delta u \geq (\leq) 0$ in U .

(i). Prove that if $u \in C^2(U) \cap C(\bar{U})$ is subharmonic in U , then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii). Prove that if $u \in C^2(U) \cap C(\bar{U})$, then

$$\max_{\bar{U}} |u| \leq \max_{\partial U} |u| + C \max_{\bar{U}} |\Delta u|,$$

where C is a constant depending only on the diameter of U . (HINT: set $v = \max_{\partial U} |u| + \frac{1}{2}(d^2 - x_1^2) \max_{\bar{U}} |\Delta u|$, where we may assume that $U \subset \{x : 0 \leq x_1 \leq d\}$, then $\Delta(v \pm u) \leq 0$ in U , and $(v \pm u) \geq 0$ on ∂U , then apply (i).)

We then introduce the Green's representation formula, which states

$$u(x) = \int_U -\Delta u(y) \Phi(x-y) dy + \int_{\partial U} \left[\Phi(x-y) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \Phi(x-y)}{\partial n_y} \right] d\sigma(y), \quad (4)$$

for any $C^2(\bar{U})$ function u on a piecewise C^1 domain U , where

$$\Phi(x) = \begin{cases} \frac{1}{(n-2)|\mathbb{S}^{n-1}|} |x|^{2-n} & \text{if } n \geq 3, \\ \frac{1}{2\pi} \log \frac{1}{|x|} & \text{if } n = 2. \end{cases}$$

This is obtained from applying (3) to $u(y)$ and $\Phi(x-y)$ on $U \setminus \{B_\epsilon(x)\}$ for small $\epsilon > 0$ and sending $\epsilon \rightarrow 0$, noting that $\Delta_y \Phi(x-y) = 0$ in $U \setminus \{B_\epsilon(x)\}$, and $|\Phi(r)|r^{n-1} \rightarrow 0$, $\int_{\partial B_r(0)} \frac{\partial \Phi(r)}{\partial r} = -1$ as $r \rightarrow 0$. As a consequence (4), we have

Corollary. If $u \in C^2(U)$ is harmonic, then it is smooth in U .

Proof. For any proper subdomain V of U with C^1 boundary, we can use the Green's representation (4) on V to express $u(x)$, $x \in V$ as

$$\int_{\partial V} \left(\Phi(x-y) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \Phi(x-y)}{\partial n_y} \right) d\sigma(y).$$

Since the integrand is a smooth function of $x \in V$ for $y \in \partial V$, with any derivatives uniformly bounded for $y \in \partial V$ as long as $x \in V$ stays away from ∂V , this shows that u is smooth in V . \square

Corollary. If $u \in C^2(U)$ is harmonic, then it is real analytic in U .

Proof. This follows from the same representation formula: fix any $x_0 \in U$, then there exists $r_0 > 0$ such that

$$\Phi(x-y) = \sum_{\alpha} a_{\alpha}(x_0-y)(x-x_0)^{\alpha},$$

with uniform convergence for $|x - x_0| \leq r_0$ and $y \in \partial U$ (we should have chosen a subdomain V with C^1 boundary such that $x_0 \in V \subset\subset U$, as done in the proof of the previous corollary, but will neglect such issues). Thus for any $\epsilon > 0$, there is N such that

$$|\Phi(x - y) - \sum_{|\alpha| \leq N} a_\alpha(x_0 - y)(x - x_0)^\alpha| \leq \epsilon,$$

uniformly for $|x - x_0| \leq r_0$ and $y \in \partial U$. Multiplying by $\frac{\partial u(y)}{\partial n_y}$ and integrating over $y \in \partial U$, we obtain

$$|\int_{\partial U} \Phi(x - y) \frac{\partial u(y)}{\partial n_y} d\sigma(y) - \sum_{|\alpha| \leq N} A_\alpha(x_0)(x - x_0)^\alpha| \leq \epsilon \int_{\partial U} |\frac{\partial u(y)}{\partial n_y}| d\sigma(y),$$

with $A_\alpha(x_0) = \int_{\partial U} a_\alpha(x_0 - y) \frac{\partial u(y)}{\partial n_y} d\sigma(y)$. Similarly, the other integral is also analytic in $x \in U$, showing that u is analytic in U . \square

Remark 2. Since for each fixed y , $|x - y|^{2-n}$ is a harmonic function for $x \in \mathbb{R}^n (n \geq 3), x \neq y$, we may try to use the superposition principle to construct harmonic functions by

$$\int_E |x - y|^{2-n} f(y) dy$$

over some set E . The proofs of the above two corollaries already used this idea and indicated that it worked when E is taken to be ∂U . When E is taken to be U , however, $\int_U |x - y|^{2-n} f(y) dy$ is not necessarily a harmonic function of $x \in U$. This is because $\partial_{x_i x_j} |x - y|^{2-n}$ is no longer an integrable function of $y \in U$, so we can't differentiate twice in x on the integral and pass the differentiation inside the integral to conclude that $\int_U |x - y|^{2-n} f(y) dy$ is harmonic. We do have

Proposition 3. If $f \in C^1(\bar{U})$, then $\int_U \Phi(x - y) f(y) dy$ is a C^2 function of $x \in U$, and

$$-\Delta_x \int_U \Phi(x - y) f(y) dy = f(x), \quad \text{for } x \in U.$$

Remark 3. A proof for Proposition 3 will be provided later. With mere continuity of f , $\int_U \Phi(x - y) f(y) dy$ may not be C^2 function of $x \in U$. However, the regularity requirement of f can be weakened to the following: for some $0 < \alpha \leq 1$, and $C > 0$,

$$|f(y_1) - f(y_2)| \leq C |y_1 - y_2|^\alpha \quad \text{for all } y_1, y_2 \in U. \quad (5)$$

Functions satisfying (5) in U (therefore in \bar{U}) with $0 < \alpha < 1$ are said to be Hölder continuous in \bar{U} with exponent α , and the set of such functions is denoted as $C^\alpha(\bar{U})$; those functions satisfying (5) in U (therefore in \bar{U}) with $\alpha = 1$ are said to be Lipschitz continuous in \bar{U} , and the set of such functions is denoted as $Lip(\bar{U})$. $C^\alpha(U)$ (respectively $Lip(U)$) usually denotes the set of functions satisfying (5) on any compact subsets of U (the constant C may depend on the compact subset).

Remark 4. To solve the Dirichlet problem on U , one can still try to take some E disjoint from U , and use $\int_E \Phi(x-y)g(y)dy$ to construct harmonic functions in U . The question becomes: (i) whether such harmonic functions extends continuously to \bar{U} ? (ii) whether one can achieve all (continuous) boundary value functions on ∂U ? Such harmonic functions indeed extend continuously to \bar{U} when $E = \partial U$, ∂U is a piecewise C^1 hypersurface, and $g \in C(\partial U)$. However, for $\bar{x} \in \partial U$, in general,

$$\lim_{x \in U, x \rightarrow \bar{x}} \int_{\partial U} \Phi(x-y)g(y)d\sigma(y) \neq g(\bar{x}),$$

so even though we can construct a harmonic function $x \mapsto \int_E \Phi(x-y)g(y)dy$ for $x \in U$, this harmonic function may not solve (1) for the case $f \equiv 0$. In fact the map $g \in C(\partial U) \mapsto \int_{\partial U} \Phi(x-y)g(y)d\sigma(y) \in C(\partial U)$ is a compact linear map, so $\int_{\partial U} \Phi(x-y)g(y)d\sigma(y)$ can not possibly take on all continuous boundary value functions when g runs through $C(\partial U)$. It turns out a modification of this idea, using the so called double layer potential, $\int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n_y} g(y)d\sigma(y)$, one can solve the Dirichlet problem this way.

Exercise 2. (i). Prove that for a domain U with its boundary being a piecewise C^1 hypersurface and any bounded measurable f defined on ∂U , $\int_{\partial U} \Phi(x-y)f(y)d\sigma(y)$ defines a $C^2(U) \cap C^0(\bar{U})$ harmonic function.

(ii). Prove that for any bounded measurable f defined in a bounded domain U , $\int_U \Phi(x-y)f(y)dy$ defines a $C^{1,\alpha}(U) \cap C^1(\bar{U})$ function for every $0 < \alpha < 1$, and

$$D_{x_i} \int_U \Phi(x-y)f(y)dy = \int_U D_{x_i} \Phi(x-y)f(y)dy.$$

Exercise 3. For $x, y \in \partial U$, $x \neq y$, define $K(x, y) = \frac{\partial \Phi(x-y)}{\partial n_y}$. Prove that, if ∂U is assumed to be piecewise C^2 surface, then for any continuous function f defined on ∂U , $\int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n_y} f(y)d\sigma(y)$ extends continuously to \bar{U} , and for any $\bar{x} \in \partial U$,

$$\lim_{x \in U, x \rightarrow \bar{x}} \int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n_y} f(y)d\sigma(y) = \frac{1}{2}f(\bar{x}) + \int_{\partial U} K(\bar{x}, y)f(y)d\sigma(y).$$

We continue to explore the consequences of (4). Let $\phi(y)$ be a $C^2(U) \cap C^1(\bar{U})$ harmonic function in U , then applying the Green's theorem to ϕ and u on U , we have

$$0 = \int_U -\Delta u(y)\phi(y)dy + \int_{\partial U} \left(\phi(y) \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial \phi(y)}{\partial n_y} \right) d\sigma(y).$$

Combining this with (4), we have

$$\begin{aligned} u(x) &= \int_U -\Delta u(y) [\Phi(x-y) - \phi(y)] dy \\ &\quad + \int_{\partial U} \left\{ [\Phi(x-y) - \phi(y)] \frac{\partial u(y)}{\partial n_y} - u(y) \frac{\partial [\Phi(x-y) - \phi(y)]}{\partial n_y} \right\} d\sigma(y). \end{aligned}$$

If, for each $x \in U$, we can choose a harmonic function $\phi(y) = \phi^x(y)$ such that $\Phi(x - y) - \phi^x(y) \equiv 0$ for $y \in \partial U$ and $\phi(y) \in C^1(\bar{U})$, then we have

$$u(x) = \int_U -\Delta u(y) [\Phi(x - y) - \phi^x(y)] dy - \int_{\partial U} u(y) \frac{\partial [\Phi(x - y) - \phi^x(y)]}{\partial n_y} d\sigma(y).$$

That is, we can represent u in U by its Dirichlet data. When this is possible, we call $G(x, y) = \Phi(x - y) - \phi^x(y)$ the Green's function (for the Laplace operator with Dirichlet boundary condition) for the region U . We summarize the above discussion as

Proposition 4. *Suppose that, for each $x \in U$, there exists $\phi^x(y) \in C^1(\bar{U})$ such that*

$$\begin{cases} \Delta_y \phi^x(y) = 0, & \text{for } y \in U, \\ \phi^x(y) = \Phi(x - y), & \text{for } y \in \partial U. \end{cases}$$

Let $G(x, y) = \Phi(x - y) - \phi^x(y)$. Then

$$\begin{cases} \Delta_y G(x, y) = 0, & \text{for } y \in U \setminus \{x\}, \\ G(x, y) = 0, & \text{for } y \in \partial U, \\ \lim_{y \rightarrow x} |x - y|^{n-2} G(x, y) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|}, & \text{if } n \geq 3, \end{cases}$$

and for any $u \in C^2(\bar{U})$, there holds

$$u(x) = \int_U -\Delta u(y) G(x, y) dy - \int_{\partial U} u(y) \frac{\partial G(x, y)}{\partial n_y} d\sigma(y).$$

For some special domains, we can write out $G(x, y)$ explicitly. First, when $U = \mathbb{R}_+^n$, for each $x \in \mathbb{R}_+^n$, we define $\phi^x(y) = \Phi(x^* - y)$, where x^* is the mirror image of x in $\partial\mathbb{R}_+^n$. Then

$$\Phi(x - y) - \phi^x(y) \equiv 0 \quad \text{for } y \in \partial\mathbb{R}_+^n,$$

and

$$-\frac{\partial [\Phi(x - y) - \phi^x(y)]}{\partial n_y} = \frac{\partial [\Phi(x - y) - \phi^x(y)]}{\partial y_n} = \frac{2x_n}{|\mathbb{S}^{n-1}| |x - y|^n} := K(x, y),$$

for $x \in \mathbb{R}_+^n$ and $y \in \partial\mathbb{R}_+^n$. $K(x, y)$ is called the Poisson kernel for \mathbb{R}_+^n .

Theorem 2. *Assume $g \in C(\partial\mathbb{R}_+^n) \cap L^\infty(\partial\mathbb{R}_+^n)$. Define $u(x)$ for $x \in \mathbb{R}_+^n$ by*

$$u(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\mathbb{R}_+^n} \frac{2x_n}{|x - y|^n} g(y) d\sigma(y).$$

Then

- (i). $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$,
- (ii). $\Delta_x u(x) = 0$, in $x \in \mathbb{R}_+^n$,
- (iii). $\lim_{x \rightarrow \bar{x} \in \partial\mathbb{R}_+^n, x \in \mathbb{R}_+^n} u(x) = g(\bar{x})$.

And there is only one solution u satisfying (i)–(iii) above.

The last part of the above theorem is proved by maximum principle, and is left as an exercise. Notice that, although the Green's representation was derived under stricter requirement on u : $u \in C^2(\bar{U})$, we can apply the representation for any continuous boundary data, which produces a solution which may not be C^2 up to the boundary, but is C^2 in the interior of the domain, and is continuous up to the boundary.

Next, when $U = B_R(0)$ in \mathbb{R}^n . It turns out that for each $x \neq 0 \in B_R(0)$, if we define $x^* = \frac{R^2}{|x|^2}x$, then $\phi^x(y) = \Phi(\frac{|x|}{R}(x^* - y))$ would work, because $|y - x|/|y - x^*|$ is independent of $y \in \partial B_R(0)$, and equals $|x|/R$, so $\Phi(x - y) - \Phi(\frac{|x|}{R}(x^* - y)) = 0$ for all $y \in \partial B_R(0)$. It works out that for any $y \neq 0$ in $\bar{B}_R(0)$, $\phi^x(y) = \Phi(\frac{|x|}{R}(x^* - y)) \rightarrow \Phi(R)$ as $x \rightarrow 0$, so the construction of $\phi^x(y)$ continues to work in this case, and

$$-\frac{\partial [\Phi(x - y) - \phi^x(y)]}{\partial n_y} = \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}||x - y|^n} := K(x, y), \quad \text{for } x \in B_R(0) \text{ and } y \in \partial B_R(0).$$

$K(x, y)$ is called the Poisson kernel for $B_R(0)$. This generalizes our Poisson kernel on two dimensional discs, which was found based on separation of variables.

Theorem 3. Assume $g \in C(\partial B_R(0))$. Define $u(x)$ for $x \in B_R(0)$ by

$$u(x) = \int_{\partial B_R(0)} \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}||x - y|^n} g(y) d\sigma(y).$$

Then

- (i). $u \in C^\infty(B_R(0))$,
- (ii). $\Delta_x u(x) = 0$, in $x \in B_R(0)$,
- (iii). $\lim_{x \rightarrow \bar{x} \in \partial B_R(0), x \in B_R(0)} u(x) = g(\bar{x})$.

And there is only one solution u satisfying (i)–(iii) above.

Remark 5. Although the Poisson kernels are defined asymmetrically: x in the interior of the domain, and y on the boundary, the Green's function $G(x, y)$ is defined for both x and y inside the domain, $x \neq y$, and is a harmonic function in y . A useful property is

Proposition 5. For all $x, y \in U$, $x \neq y$, we have

$$G(x, y) = G(y, x).$$

Thus $G(x, y)$ is also a smooth function in $U \times U \setminus \{(z, z) : z \in U\}$, extends to a continuous function in $\bar{U} \times \bar{U} \setminus \{(z, z) : z \in \bar{U}\}$, and is harmonic in $x \in U$, for $x \neq y$.

Proposition 6. A C^0 function u in U which satisfies the mean value property for any balls $B \subset U$ is a smooth harmonic function.

Proof. Let u be such a C^0 function. For any ball $B \subset\subset U$, by Theorem 3, there is a harmonic function $v \in C(\bar{B}) \cap C^2(B)$ such that $v = u$ on ∂B . Since both u and v satisfy the maximum principle, we conclude now that $v = u$ in B , i.e., u is harmonic in B . Since this holds for any ball $B \subset\subset U$, we conclude that u is harmonic in U . \square

Another consequence of the Poisson representation formula on $B_R(0)$ is the gradient estimates and Harnack estimates.

Theorem 4. *If u is harmonic in $B_R(x_0)$, then, for some $C = C(n) > 0$,*

$$|\nabla u(x_0)| \leq \frac{C}{R} \max_{B_R(x_0)} |u|, \quad \text{and for } k > 1, \quad |\nabla^\alpha u(x_0)| \leq \frac{(Ck)^k}{R^k} \max_{B_R(x_0)} |u|,$$

for all α with $|\alpha| = k$.

Remark 6. *The gradient estimates can be proved in a simpler way: since we already proved that u will be smooth, so $u_{x_i}(x)$ is also harmonic. Then by the mean value property,*

$$u_{x_i}(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u_{x_i}(x) dx = \frac{1}{|B_r(x_0)|} \int_{\partial B_r(x_0)} u(x) n_i(x) d\sigma(x),$$

for all $0 < r < R$. Thus,

$$|\nabla u(x_0)| \leq \frac{1}{|B_r(x_0)|} \int_{\partial B_r(x_0)} |u(x)| d\sigma(x) \leq \frac{n}{r} \max_{B_r(x_0)} |u|.$$

Sending $r \rightarrow R$, we obtain

$$|\nabla u(x_0)| \leq \frac{n}{R} \max_{B_R(x_0)} |u|.$$

Theorem 5 (Harnack). *(Local Version) If u is a nonnegative harmonic function in $B_R(0)$, then*

$$R^{n-2} \frac{R - |x|}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq R^{n-2} \frac{R + |x|}{(R - |x|)^{n-1}} u(0), \quad \text{for all } x \in B_R(x_0).$$

In particular, for any $x \in B_{\frac{R}{2}}(0)$,

$$\frac{2^{n-2}}{3^{n-1}} u(0) \leq u(x) \leq 2^{n-2} 3 u(0).$$

(Global Version) For each connected open set $V \subset\subset U$, there exists a positive constant C depending on V and U , such that

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic function u in U .

Corollary (Liouville). *A bounded harmonic function on \mathbb{R}^n must be a constant. In fact, a harmonic function on \mathbb{R}^n bounded from below (or above) must be a constant.*

The first part can be proved by the gradient estimate, the second part can be proved by the Harnack estimate. The more powerful consequences of the gradient estimates are the convergence theorems.

Theorem 6. (i). Uniform limit of a sequence of harmonic functions is harmonic. (ii). Limit of a monotone sequence of harmonic functions is harmonic. (iii). A bounded sequence of harmonic functions on U must have a subsequence that converges on any compact subset of U to a harmonic function.

Theorem 7. If, for a sequence $g_k \in C(\partial U)$, there exists a (unique) solution u_k to

$$\begin{cases} \Delta u_k = 0, & \text{in } U, \\ u_k = g_k, & \text{on } \partial U, \end{cases}$$

and $g_k \rightarrow g$ uniformly on ∂U , then the Dirichlet problem with g as boundary value has a unique solution.

Next we discuss how to use the convergence theorems above to solve the Dirichlet problem on a ball without using the Poisson kernel. This consists of three steps. Let \mathcal{P}_k denote polynomials of degree $\leq k$.

Step 1. For any polynomial $p \in \mathcal{P}_{k-2}$, there exists a unique solution $v \in \mathcal{P}_k$ satisfying

$$\begin{cases} \Delta v = p, & \text{in } B_R(0), \\ v = 0, & \text{on } \partial B_R(0), \end{cases}$$

This follows from considering the linear map $T : \mathcal{P}_{k-2} \mapsto \mathcal{P}_{k-2}$ by $T(u) = \Delta [(R^2 - |x|^2)u(x)]$.

The maximum principle implies that T has trivial kernel, so must be an isomorphism from \mathcal{P}_{k-2} to \mathcal{P}_{k-2} .

Step 2. For any $p \in \mathcal{P}_{k-2}$ and $g \in \mathcal{P}_k$, there exists $u \in \mathcal{P}_k$ such that

$$\begin{cases} \Delta u = p, & \text{in } B_R(0), \\ v = g, & \text{on } \partial B_R(0), \end{cases} \quad (6)$$

This follows from Step 1. Since $\Delta g \in \mathcal{P}_{k-2}$, by Step 1, there exists $v \in \mathcal{P}_k$ solving

$$\begin{cases} \Delta v = p - \Delta g, & \text{in } B_R(0), \\ v = 0, & \text{on } \partial B_R(0), \end{cases}$$

The $u = v + g$ is the solution.

Step 3. For any $g \in C(\partial B_R(0))$, take a sequence of polynomials g_k such that $g_k \rightarrow g$ uniformly as $k \rightarrow \infty$. Then the maximum principle and convergence theorems provide a limit in $C(\overline{B_R(0)}) \cap C^2(B_R(0))$ harmonic function with g as boundary value.

We can attempt to solve (6) for more general right hand side, as we can already solve it for polynomials. Take f to be continuous on $\overline{B_R(0)}$ for instance. We can take polynomials p_k and g_k such that $p_k \rightarrow f$ uniformly in $\overline{B_R(0)}$, and $g_k \rightarrow g$ uniformly on $\partial B_R(0)$ as $k \rightarrow \infty$. Let u_k denote the corresponding (polynomial) solution. By Exercise 1, we have uniform bound on $\|u_k\|_{C(\overline{B_R(0)})}$. In fact, $\{u_k\}$ is Cauchy in $C(\overline{B_R(0)})$. But to prove the convergence of $\{u_k\}$ to a solution of (6), we need higher derivative estimates for $\{u_k\}$. This can be done using the Green's representations. But a more flexible method is the Bernstein's method.

Theorem 8. Let $u \in C^4(\overline{B_1})$ and denote Δu by f . Then for any $r \in (0, 1)$, there is a constant $N = N(n, r)$ such that in B_r

$$|u| + |\nabla u| + |\nabla^2 u| \leq N \left(\max_{B_1} |f| + \max_{B_1} |\nabla f| + \max_{B_1} |\nabla^2 f| + \max_{\partial B_1} |u| \right)$$

The idea of Bernstein is to verify that some auxiliary function involving u and its derivatives satisfies an appropriate differential inequality (subharmonic, for instance), and thus satisfies the maximum principle. We will illustrate the method by verifying that for any smooth cut-off function ζ supported in B_1 and identically equal to 1 on B_r , the function $w = \zeta^2 |\nabla u|^2 + Cu^2$ satisfies in B_1 , for C large depending on r ,

$$\Delta w \geq -\zeta^2 |\nabla f|^2 - Cf^2 - Cu^2 \geq -\tilde{N} \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{B_1} |u|^2 \right)$$

for some \tilde{N} depending on C . Thus a generalization of Exercise 2 implies that

$$\max_{B_1} w \leq N \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{B_1} |u|^2 \right) + \max_{\partial B_1} w$$

for some $N > 0$. Then using Exercise 2 again to estimate $\max_{B_1} |u|^2$ in terms of $\max_{B_1} |f|^2$ and $\max_{\partial B_1} |u|^2$, it follows that

$$\max_{B_r} |\nabla u|^2 \leq N \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{\partial B_1} |u|^2 \right).$$

A similar construction proves the second derivative estimate. The drawback of this method is the requirement on the higher derivatives of f . The advantage is that it is very flexible and applies even to some nonlinear equations.

Next we will use the maximum principle and the convergence theorems to find a harmonic function with prescribed Dirichlet data on fairly general domain. I will describe Poincaré's balayage method, which can be considered as a cousin of the Perron method. The latter is presented in most modern treatment.

Poincaré's method, as well as Perron's, depends on the maximum principle, the convergence theorems, and the solvability of Dirichlet problem on balls. The last we obtained by Poisson's kernel or the approximation method. We need to extend the notion of subharmonic functions to C^0 class.

Definition. A $C^0(U)$ function u is called subharmonic (superharmonic) in U , if for every ball $B \subset\subset U$ and every harmonic function h in B satisfying $u \leq (\geq)h$ on ∂B , we also have $u \leq (\geq)h$ inside B .

The maximum principle extends to these functions in the following way:

- (i). Suppose U is a bounded domain. If u is subharmonic in U and is continuous up to the boundary of U , then $\max_{\overline{U}} u \leq \max_{\partial U} u$. In fact, the strong maximum principle also holds. If v is superharmonic in U , and $u \leq v$ on ∂U , then in any connected component of U either $u < v$, or $u \equiv v$.

- (ii). If u is subharmonic in U and B is a ball strictly contained in U , then the harmonic lifting $h_B(u)$ of u in B is subharmonic in U , where $h_B(u)$ is defined as

$$h_B(u) = \begin{cases} \bar{u}(x), & \text{for } x \in B, \\ u(x), & \text{for } x \in U \setminus B, \end{cases}$$

with $\bar{u}(x)$ being the harmonic function in B satisfying $\bar{u}(x) = u(x)$ on ∂B .

Poincaré's method consists of several steps. Given a domain U and a continuous boundary function g .

Step 1. Assume first that there exists a subharmonic $u_0 \in C(\bar{U})$ such that $u_0|_{\partial U} = g$.

Step 2. Cover U by a countable number of balls $\{B_1, B_2, \dots\}$ such that each $B_i \subset\subset U$. Starting from u_0 , we will replace each with its harmonic lifting on successive balls to obtain a sequence of monotone increasing subharmonic functions that are bounded from above. Thus a limiting function u exists. We will prove this u is harmonic in U and will study its boundary behavior.

First let $B^{(1)} = B_1$ and define $u_1 = h_{B^{(1)}}(u_0)$. Then

- (a) $u_1(x) \geq u_0$, for all $x \in \bar{U}$.
- (b) u_1 is harmonic in $B^{(1)}$.
- (c) $u_1(x) \leq \max_{\partial U} g$, and $u_1(x) = g(x)$ on ∂U .
- (d) u_1 is still subharmonic in U .

Order the balls in the following way

$$\begin{array}{cccccccc} & B_1 & \rightarrow & B_2 & \leftarrow & & & \\ \leftarrow & B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \leftarrow & \\ \leftarrow & B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & B_4 & \leftarrow \\ \leftarrow & \dots & & & & & & & \end{array}$$

Define inductively, for each $i \geq 2$, $u_i(x)$ on U by $u_i = h_{B^{(i)}}(u_{i-1})$. Then

- (a) $u_i(x) \geq u_{i-1}$, for all $x \in \bar{U}$.
- (b) u_i is harmonic in $B^{(i)}$.
- (c) $u_i(x) \leq \max_{\partial U} g$, and $u_i(x) = g(x)$ on ∂U .
- (d) u_i is still subharmonic in U .

Therefore $\{u_i\}$ is a sequence of monotone increasing, subharmonic, continuous functions in U , and is bounded from above. Thus $u(x) = \lim_{i \rightarrow \infty} u_i(x)$ is well-defined for all $x \in U$. To prove u is harmonic in U , notice that for each $x \in U$, there is a ball $B_{i(x)}$ such that $x \in B_{i(x)}$. Notice also that because of the ordering of the balls, a subsequence of $\{u_i\}$ is actually monotone, *harmonic* in $B_{i(x)}$. Thus u is harmonic in $B_{i(x)}$ by the convergence theorem.

Step 3. The continuity in \bar{U} of the solution u in the above step is handled by the concept of barriers.

Definition. $w \in C(\bar{U})$ is called a *barrier function* at $\xi \in \partial U$ for the Dirichlet problem on U if

- a). w is superharmonic in U .
b). $w(\xi) = 0$, and $w(x) > 0$ for $x \in \bar{U} \setminus \{\xi\}$.

Suppose a barrier function w at $\xi \in \partial U$ exists. For any given $\epsilon > 0$, by the continuity of g at ξ , we can find $r_0 > 0$ such that $g(x) - \epsilon \leq g(x) \leq g(\xi) + \epsilon$ for all $x \in \partial U$ with $|x - \xi| \leq r_0$. There also exists $M > 0$ depending on w and g such that $g(x) - g(\xi) \leq Mw(x)$ for all $x \in \partial U$ with $|x - \xi| \geq r_0$. Thus

$$g(x) \leq g(\xi) + \epsilon + Mw(x), \quad \text{for all } x \in \partial U.$$

Note that $g(\xi) + \epsilon + Mw(x)$ is a superharmonic function on U . So in the steps above, we also have $u_0 \leq u_i \leq g(\xi) + \epsilon + Mw$ on \bar{U} . By the continuity of w and u_0 at ξ , we can find $r_* < r_0$ such that when $x \in U$ and $|x - \xi| \leq r_*$, $Mw(x) \leq \epsilon$ and $u_0(x) \geq g(\xi) - \epsilon$. Thus

$$g(\xi) - \epsilon \leq u_0(x) \leq u(x) \leq g(\xi) + 2\epsilon, \quad \text{for all } x \in U \text{ with } |x - \xi| \leq r_*,$$

proving the continuity of u at ξ .

Step 4. To remove that assumption in Step 1, we first argue that for any polynomial g given, $g_1 = g + Ax_1^2$ and $g_2 = Ax_1^2$ are subharmonic in U for $A > 0$ sufficiently large. Thus the first three steps can be applied to show that the Dirichlet problem has solutions with g_1 and g_2 as boundary values, thus it also has one with $g_1 - g_2 = g$ as boundary value. Finally the convergence theorems can be used to prove the existence of solution to the Dirichlet problem with any given continuous boundary value, provided the barrier argument in Step 3 can be carried out. That turns out to depend only on the geometry of the domain U .

An easily verified criterion for the existence of a barrier at $\xi \in \partial U$ is the existence of some exterior ball, *i.e.*, there exists a ball B such that $\bar{B} \cap \bar{U} = \{\xi\}$. Let x_0 be the center of this ball and r be its radius, then $w(x) = r^{2-n} - |x - x_0|^{2-n}$ defines a barrier at ξ .

Definition. A boundary point ξ is called regular with respect to the Laplacian if there exists a barrier at that point.

We can now summarize our discussion as

Theorem 9. The classical Dirichlet problem for the Laplacian in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.

Points on components of the boundary with codimension 2 or higher are not regular. For instance, we can't solve the Dirichlet boundary value on the domain $B \setminus \{0\}$ with prescribed value everywhere on $\partial\{B \setminus \{0\}\} = \partial B \cup \{0\}$, as the following theorem shows

Theorem 10. Suppose u is harmonic in $B \setminus \{0\}$, and satisfies $|u(x)| = o(|x|^{2-n})$ as $x \rightarrow 0$ (assume $n \geq 3$). Then u extends to a smooth harmonic function over B .

Proof. Let v be the unique solution in B to

$$\begin{cases} \Delta v = 0, & \text{in } B, \\ v = u, & \text{on } \partial B. \end{cases}$$

Then $w = u - v$ is still harmonic in $B \setminus \{0\}$, and satisfies $w(x) = o(|x|^{2-n})$ as $x \rightarrow 0$. Furthermore $w(x) \equiv 0$ on ∂B . We prove $w \equiv 0$ in B in the following way: for any $\epsilon > 0$, we can find $r > 0$ such that $B_r \subset\subset B$, and on ∂B_r , $|w(x)| \leq \epsilon(|x|^{2-n} - r_0^{2-n})$ (r_0 is the radius of B). Then $\pm w(x) + \epsilon(|x|^{2-n} - r_0^{2-n})$ is a harmonic function on $B \setminus B_r$, and is nonnegative on $\partial(B \setminus B_r)$. Thus by the maximum principle $\pm w(x) + \epsilon(|x|^{2-n} - r_0^{2-n}) \geq 0$ in $B \setminus B_r$, i.e., $|w(x)| \leq \epsilon(|x|^{2-n} - r_0^{2-n})$ for all $x \in B \setminus B_r$. For any fixed $\bar{x} \in B \setminus \{0\}$, \bar{x} is in $B \setminus B_r$ for all sufficiently small $\epsilon > 0$. Thus $|w(\bar{x})| \leq \epsilon(|\bar{x}|^{2-n} - r_0^{2-n})$, and by sending $\epsilon \rightarrow 0$, we conclude that $w(\bar{x}) = 0$. In conclusion, $w \equiv 0$ in B , and so $u \equiv v$, a smooth harmonic function in B . \square

The notion of local barrier is sometimes useful.

Definition. w is called a local barrier function at $\xi \in \partial U$ for the Dirichlet problem on U if there is a ball B centered at ξ such that $w \in C(\bar{U} \cap \bar{B})$ and

- a). w is superharmonic in $U \cap B$.
- b). $w(\xi) = 0$, and $w(x) > 0$ for $x \in (\bar{U} \cap \bar{B}) \setminus \{\xi\}$.

If a local barrier w at $\xi \in \partial U$ exists on $\bar{U} \cap \bar{B}$, it is easy to see that we can apply the barrier argument in Step 3 on $\bar{U} \cap \bar{B}$; or alternatively we can define $m = \min_{\partial B \cap U} w$ and

$$\tilde{w}(x) = \begin{cases} \min(w(x), m), & \text{for } x \in U \cap B, \\ m, & \text{for } x \in U \setminus B, \end{cases}$$

then \tilde{w} is a (global) barrier function at $\xi \in \partial U$. We now provide a proof for Proposition 3 and discuss how to solve the inhomogeneous problem (1).

Proof of Proposition 3. We will prove that, assuming (5) on U , the potential integral $u(x) = \int_U f(y)\Phi(x-y)dy$ is in $C^2(U)$, and

$$\frac{\partial u(x)}{\partial x_i} = \int_U \frac{\partial \Phi(x-y)}{\partial x_i} f(y) dy, \quad (7)$$

$$\frac{\partial^\alpha u(x)}{\partial x^\alpha} = \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy + f(x) \int_{\partial U} \nu_j(y) \partial_{y_i} \Phi(x-y) d\sigma(y) \quad (8)$$

for any $\alpha = (i, j)$ with $|\alpha| = 2$. It then follows that

$$\begin{aligned}
\Delta_x u(x) &= \int_U \Delta_x \Phi(x-y) [f(y) - f(x)] dy + f(x) \sum_{i=1}^n \int_{\partial U} \nu_i(y) \partial_{y_i} \Phi(x-y) d\sigma(y) \\
&= f(x) \left\{ \int_{\partial B_\epsilon(x)} \sum_{i=1}^n \nu_i(y) \partial_{y_i} \Phi(x-y) d\sigma(y) + \int_{U \setminus B_\epsilon(x)} \Delta_y \Phi(x-y) dy \right\} \\
&= f(x) \int_{\partial B_\epsilon(x)} \left(-\frac{\epsilon^{1-n}}{|\mathbb{S}^{n-1}|} \right) d\sigma(y) \\
&= -f(x),
\end{aligned}$$

which is the stated equality in Proposition 3.

The key difficulty in proving (7) and (8) is that the $\Phi(x-y)$ in the integrand is singular at $y = x$; more precisely, $\partial_x \Phi(x-y)f(y)$ is still integrable in $y \in U$, but $\partial_x^2 \Phi(x-y)f(y)$ is not integrable for $y \in U$ due to the strength of the singularity in $\partial_x^2 \Phi(x-y) \sim |x-y|^{-n}$. It is not hard to prove that

$$\partial_x \int_U \Phi(x-y)f(y) dy = \int_U \partial_x \Phi(x-y)f(y) dy,$$

for $x \in U$, although one can not seem to apply Lebesgue's dominated convergence theorem directly—see Hints to Exercise 2 below. A direct argument for differentiating twice under the integral sign is not that routine. A streamlined proof for (7) and (8) is to mollify the singularity in $\Phi(x-y)$ first: choose a smooth cut-off function $\eta(t)$ such that it is supported in $|t| \geq 1$ and is identically 1 for $|t| \geq 2$, and define $\eta_\epsilon(t) = \eta(t/\epsilon)$ for $\epsilon > 0$. Then $\Phi(x-y)\eta_\epsilon(|x-y|)$ is smooth in x . So if we set

$$u_\epsilon(x) = \int_U f(y)\Phi(x-y)\eta_\epsilon(|x-y|) dy,$$

then $u_\epsilon(x)$ is a smooth function of x , and

$$\frac{\partial^\alpha u_\epsilon(x)}{\partial x^\alpha} = \int_U f(y) \frac{\partial^\alpha}{\partial x^\alpha} [\Phi(x-y)\eta_\epsilon(|x-y|)] dy.$$

It's straight forward to see that $u_\epsilon(x) \rightrightarrows u(x)$ for $x \in U$. If we can prove that

$$\frac{\partial u_\epsilon(x)}{\partial x_i} \rightrightarrows \int_U \frac{\partial \Phi(x-y)}{\partial x_i} f(y) dy \quad \text{and} \quad (9)$$

$$\frac{\partial^\alpha u_\epsilon(x)}{\partial x^\alpha} \rightrightarrows \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy + f(x) \int_{\partial U} \nu_j(y) \partial_{y_i} \Phi(x-y) d\sigma(y) \quad (10)$$

uniformly for x in (compact subsets of) U for any $\alpha = (i, j)$ with $|\alpha| = 2$, then we have proved that $u \in C^2(U)$ and (7), (8) hold. Since

$$\frac{\partial [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i} = \frac{\partial \Phi(x-y)}{\partial x_i} \eta_\epsilon(|x-y|) + \Phi(x-y) \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_i},$$

and

$$\int_U \left| \Phi(x-y) \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_i} \right| dy \lesssim \int_{\epsilon \leq |y-x| \leq 2\epsilon} |x-y|^{2-n} \epsilon^{-1} dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

it follows that (9) holds.

Since we can write

$$\begin{aligned} & \int_U f(y) \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i \partial x_j} dy \\ &= \int_U [f(y) - f(x)] \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i \partial x_j} dy \\ & \quad + f(x) \int_U \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial y_i \partial y_j} d(y) \\ &= \int_U [f(y) - f(x)] \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i \partial x_j} dy \\ & \quad + f(x) \int_{\partial U} \frac{\partial [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial y_i} \nu_j(y) d\sigma(y) \\ &= \int_U [f(y) - f(x)] \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i \partial x_j} dy \\ & \quad + f(x) \int_{\partial U} \frac{\partial \Phi(x-y)}{\partial y_i} \nu_j(y) d\sigma(y), \end{aligned}$$

for $0 < 2\epsilon < \text{dist}(x, \partial U)$, as $\eta_\epsilon(|x-y|) = 1$ for $y \in \partial U$ in such a situation, and

$$\begin{aligned} & \frac{\partial^2 [\Phi(x-y)\eta_\epsilon(|x-y|)]}{\partial x_i \partial x_j} \\ &= \frac{\partial^2 \Phi(x-y)}{\partial x_i \partial x_j} \eta_\epsilon(|x-y|) + \frac{\partial \Phi(x-y)}{\partial x_i} \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_j} \\ & \quad + \frac{\partial \Phi(x-y)}{\partial x_j} \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_i} + \Phi(x-y) \frac{\partial^2 \eta_\epsilon(|x-y|)}{\partial x_i \partial x_j}, \end{aligned}$$

we find that

$$\begin{aligned} & \int_U [f(y) - f(x)] \frac{\partial \Phi(x-y)}{\partial x_i} \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_j} dy \\ & \lesssim \int_{\epsilon \leq |y-x| \leq 2\epsilon} |x-y|^{\alpha+1-n} \epsilon^{-1} dy \\ & \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0, \text{ due to (5)}. \end{aligned}$$

Likewise,

$$\int_U [f(y) - f(x)] \frac{\partial \Phi(x-y)}{\partial x_j} \frac{\partial \eta_\epsilon(|x-y|)}{\partial x_i} dy \rightarrow 0,$$

as $\epsilon \rightarrow 0$, and

$$\int_U [f(y) - f(x)] \Phi(x - y) \frac{\partial^2 \eta_\epsilon(|x - y|)}{\partial x_i \partial x_j} dy \rightarrow 0,$$

as $\epsilon \rightarrow 0$, thus proving (10).

The following is another proof for (7) and (8) by proving versions for nicer functions f first. First assume $f \in C_c^1(U)$, then

$$\int_U \Phi(x - y) f(y) dy = \int_{\mathbb{R}^n} \Phi(z) f(x - z) dz,$$

and

$$\begin{aligned} & \partial_{x_i} \int_U \Phi(x - y) f(y) dy \\ &= \int_{\mathbb{R}^n} \Phi(z) \partial_{x_i} f(x - z) dz \\ &= - \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} \Phi(z) \partial_{z_i} f(x - z) dz \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \int_{|z| \geq \epsilon} \partial_{z_i} \Phi(z) f(x - z) dz + \int_{|z| = \epsilon} \nu_i(z) \Phi(z) f(x - z) d\sigma(z) \right\} \\ &= \int_{\mathbb{R}^n} \partial_{z_i} \Phi(z) f(x - z) dz \\ &= \int_{\mathbb{R}^n} \partial_{x_i} \Phi(x - y) f(y) dy. \end{aligned}$$

Furthermore

$$\begin{aligned}
& \partial_{x_i x_j} \int_U \Phi(x-y) f(y) dy \\
&= \int_{\mathbb{R}^n} \partial_{z_i} \Phi(z) \partial_{x_j} f(x-z) dz \\
&= \int_U \partial_{x_i} \Phi(x-y) \partial_{y_j} f(y) dy \\
&= \lim_{\epsilon \rightarrow 0} \int_{U \setminus B_\epsilon(x)} \partial_{x_i} \Phi(x-y) \partial_{y_j} [f(y) - f(x)] dy \\
&= \lim_{\epsilon \rightarrow 0} \left\{ - \int_{U \setminus B_\epsilon(x)} \partial_{x_i y_j} \Phi(x-y) [f(y) - f(x)] dy - \int_{|y-x|=\epsilon} \nu_j(y) \partial_{x_i} \Phi(x-y) [f(y) - f(x)] d\sigma(z) \right. \\
&\quad \left. - \int_{\partial U} \nu_j(y) \partial_{x_i} \Phi(x-y) f(x) d\sigma(y) \right\} \\
&= \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy - f(x) \int_{\partial U} \nu_j(y) \partial_{x_i} \Phi(x-y) d\sigma(y) \quad (\text{since } f(y) = 0 \text{ for } y \in \partial U.) \\
&= \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy + f(x) \int_{\partial U} \nu_j(y) \partial_{y_i} \Phi(x-y) d\sigma(y).
\end{aligned}$$

The limit

$$\lim_{\epsilon \rightarrow 0} \int_{U \setminus B_\epsilon(x)} \partial_{x_i y_j} \Phi(x-y) [f(y) - f(x)] dy = - \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy$$

exists because the integrand $\Phi(x-y) [f(y) - f(x)]$ is integrable for $y \in U$ under our assumption (5). Next, for any $f \in C^1(\bar{U})$ and $x_0 \in U$, we can split $f(x) = f_1(x) + f_2(x)$, where $f_i \in C^1(\bar{U})$, but $f_1(x) = f(x)$ in a neighborhood of x_0 and has compact support in U . Thus $f_2(x) = 0$ in a neighborhood of x_0 . Then

$$\int_U \Phi(x-y) f(y) dy = \int_U \Phi(x-y) f_1(y) dy + \int_U \Phi(x-y) f_2(y) dy,$$

where $\int_U \Phi(x-y) f_2(y) dy$ is smooth in x near x_0 , and there $\partial_x^\alpha \int_U \Phi(x-y) f_2(y) dy = \int_U \partial_x^\alpha \Phi(x-y) f_2(y) dy$ holds for any α . We can apply the above argument to $\int_U \Phi(x-y) f_1(y) dy$ to conclude that it is a C^2 function of x near x_0 , and

$$\begin{aligned}
& \partial_{x_i x_j} \int_U \int_U \Phi(x-y) f_1(y) dy \\
&= \int_U \partial_{x_i x_j} \Phi(x-y) [f_1(y) - f_1(x)] dy + f_1(x) \int_{\partial U} \nu_j(y) \partial_{y_i} \Phi(x-y) d\sigma(y) + \int_U \partial_{x_i x_j} \Phi(x-y) f_2(y) dy \\
&= \int_U \partial_{x_i x_j} \Phi(x-y) [f(y) - f(x)] dy + f(x) \int_{\partial U} \nu_j(y) \partial_{y_i} \Phi(x-y) d\sigma(y)
\end{aligned}$$

near x_0 , as $f_1(x) = f(x)$ there.

To make the above argument work for an α -Hölder continuous function f , we can take a sequence $f_k \in C^1(\bar{U})$ such that $f_k \rightarrow f$ uniformly in \bar{U} . Because $f \in C^\alpha(\bar{U})$, we could prove that we can take $f_k \rightarrow f$ also in $C^\alpha(\bar{U})$. But it is easier to verify, using mollification for example, that the $C^\alpha(\bar{U})$ norm of $\{f_k\}$ is bounded, namely, for a common $M > 0$,

$$|f_k(x) - f_k(y)| \leq M|x - y|^\alpha \quad \text{for all } x, y \in U \text{ and } k.$$

If we set

$$u_k(x) = \int_U \Phi(x - y) f_k(y) dy, \quad \text{and} \quad w_k(x) = \partial_{x_i x_j} u_k(x),$$

then we have proved above that

$$w_k(x) = \int_U \partial_{x_i x_j} \Phi(x - y) [f_k(y) - f_k(x)] dy + f_k(x) \int_{\partial U} \nu_i(y) \partial_{y_j} \Phi(x - y) d\sigma(y).$$

Using the integral representation $u(x) = \int_U \Phi(x - y) f(y) dy$, it is easy to prove that $u_k \rightarrow u$ in $C(\bar{U})$; furthermore, $w_k = \partial_{x_i x_j} u_k(x)$ converges uniformly in \bar{U} to

$$w(x) = \int_U \partial_{x_i x_j} \Phi(x - y) [f(y) - f(x)] dy + f(x) \int_{\partial U} \nu_i(y) \partial_{y_j} \Phi(x - y) d\sigma(y).$$

The pointwise convergence of $w_k(x)$ to $w(x)$ can be justified by applying Lebesgue's dominated convergence theorem to $\partial_{x_i x_j} \Phi(x - y) [f_k(y) - f_k(x)]$ for $y \in U$, as they share an integrable dominating function. The uniform convergence can be checked as in the Hint for Exercise 2 below. \square

Hint to Exercise 2 (ii). Set $u(x) = \int_U \Phi(x - y) f(y) dy$. For any $x \in U$ and any fixed unit direction e , our goal is to prove

$$\left| \frac{u(x + he) - u(x)}{h} - \int_U \nabla_e \Phi(x - y) f(y) dy \right| \rightarrow 0,$$

as $h \rightarrow 0$. Although $\left| \frac{\Phi(x + he - y) - \Phi(x - y)}{h} - \nabla_e \Phi(x - y) \right| \rightarrow 0$, as $h \rightarrow 0$, for any fixed $y \in U$, $y \neq x$, there seems to be no straightforward way to construct a fixed integrable dominating function—note that the difference quotient term has two singularities at $y = x$ and at $y = x + he$. The key is to exploit the absolute integrability of $|\nabla \Phi(x - y)|$. Denoting $\left[\frac{\Phi(x + he - y) - \Phi(x - y)}{h} \right] f(y)$ by $F_h(y)$, we decompose

$$F_h(y) = F_h(y) \chi_{B_{2h}(x)}(y) + F_h(y) \chi_{B_{2h}^c(x)}(y) := F_h^{(1)}(y) + F_h^{(2)}(y).$$

Notice that when $y \in B_{2h}^c(x)$, for any point $z \in B_h(x)$, we have

$$|y - z| \geq |y - x| - |x - z| \geq \frac{1}{2}|y - x|.$$

Thus it is safe to say that when $y \in B_{2h}^c(x)$,

$$\left| \frac{\Phi(x + he - y) - \Phi(x - y)}{h} \right| = |\nabla_e \Phi(x + \theta he - y)| \leq C|x - y|^{1-n}, \quad \text{where } 0 < \theta < 1,$$

and we have $|F_h^{(2)}(y)| \leq C|x - y|^{1-n}|f(y)|$ for all $y \in U$. Since we have $F_h^{(2)}(y) \rightarrow \nabla_e \Phi(x - y)f(y)$, with the integrable dominating function $C|x - y|^{1-n}|f(y)|$ (it is here we need to assume U to be bounded. Instead of this assumption, we may also assume f to have the appropriate integrability, as $|x - y|^{1-n}$ is in L^p on any bounded ball with x as center for any $p < \frac{n}{n-1}$, while it is in L^p for any $p > \frac{n}{n-1}$ outside any ball with x as center), we can use Lebesgue's theorem on $F_h^{(2)}(y)$ to say

$$\lim_{h \rightarrow 0} F_h^{(2)}(y) = \int_U \nabla_e \Phi(x - y)f(y)dy.$$

What remains is to prove that $\int_U F_h^{(1)}(y)dy \rightarrow 0$ as $h \rightarrow 0$. This is because

$$\begin{aligned} & \int_U |F_h^{(1)}(y)|dy \\ & \leq h^{-1} \int_{B_{2h}(x)} \{\Phi(x - y) + \Phi(x + he - y)\} dy \\ & \leq h^{-1} \left\{ \int_{B_{2h}(x)} \Phi(x - y)dy + \int_{B_{3h}(x+he)} \Phi(x + he - y)dy \right\} \\ & \leq Ch^{-1} \left\{ \int_0^{3h} r^{2-n} r^{n-1} dr \right\} \\ & \leq C'h. \end{aligned}$$

Thus we conclude that

$$\frac{u(x + he) - u(x)}{h} = \int_U F_h^{(1)}(y)dy + \int_U F_h^{(2)}(y)dy \rightarrow \int_U \nabla_e \Phi(x - y)f(y)dy$$

as $h \rightarrow 0$.

Another approach is to establish the differentiation under integral sign for any smooth function f (or smooth kernel K). We will illustrate the idea by assuming the differentiation under integral sign has been proved for smooth f . Then for any bounded measurable f , let's take a sequence of smooth compactly supported f_j to approximate f in $L^p(U)$ for some $p > n$. Note that

$$\begin{aligned} |\nabla_\alpha u_j| &= \left| \int_U \nabla_\alpha \Phi(x - y)f_j(y)dy \right| \\ &\leq \left\{ \int_U |\nabla \Phi(x - y)|^{p'} dy \right\}^{1/p'} \left\{ \int_U |f_j(y)|^p dy \right\}^{1/p}, \quad \frac{1}{p'} + \frac{1}{p} = 1 \\ &\leq C(p, n, U) \|f_j\|_{L^p(U)}. \end{aligned}$$

Applying this estimate to $f_j - f_k$ implies that $\{\nabla u_j\}$ is Cauchy in $C(\bar{U})$. it is even simpler to verify that $\{u_j\}$ is Cauchy in $C(\bar{U})$ and converges uniformly to $u(x) = \int_U \Phi(x - y)f(y)dy$.

Thus we conclude that

$$\nabla_\alpha u(x) = \lim_{j \rightarrow \infty} \nabla_\alpha u_j(x) = \int_U \nabla_\alpha \Phi(x - y) f(y) dy.$$

The Hölder part can be verified based on the integral representation.