1. ON THE SET UP OF SOME PDE PROBLEMS

The goals of the first lecture are:

(1) to discuss the setting up process of some prototype PDEs, in particular, to become aware of the connections and differences between the physical and mathematical assumptions that enter into the setting up process;
(2) to discuss the need for notions of solutions with varying smoothness assumptions;
(3) to discuss typical boundary/initial conditions;
(4) to make an initial discussion on the well-posedness issue in general terms;
(5) to make some general comparisons with the known theory of ODEs, and to introduce the method of characteristics and separation of variables in the simplest setting.

Many PDEs that we encounter are set up by basically translating the relevant physical principle. However, most physical principles are not expressed as a PDE directly; we often need to make some smoothness assumptions on the relevant physical quantities to arrive at a PDE. Other times the direct, faithful translation of the physical principle may produce a PDE which is too difficult to manage, so we may make some simplifying physical or mathematical assumptions to obtain a simpler PDE as a first approximation to the physical process. Of course, whether the simplified PDE gives good description to the physical process is subject to verifications with data. But that is not our main task; our main task is to learn how to analyze the PDE. As we will learn, however, having some physical intuition will be of great help in the analysis of the PDEs.

Ex 1 (Equation for the motion of fluid along a one-dimensional tube)**.** *Here, the relevant physical quantities are: x = coordinate along the tube, $\rho(x,t)$ = density of fluid at location x and time t, $v(x,t)$= velocity (in the direction of x-axis here) of the fluid at location x and time t. The first relevant physical principle is the law of conservation of mass, which can be expressed as*

$$\begin{aligned}&\textit{rate of change of mass in any section of the tube}\\ &= \textit{rate of fluid flowing in across the ends of the tube}\\ &\quad - \textit{ rate of fluid flowing out across the ends of the tube.}\end{aligned}$$

More quantitatively, for any $x_1 < x_2$, and any moment t,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t). \tag{1}$$

(1) *is not a PDE yet, and is not easy to work with. We next make the* <u>*mathematical assumption*</u> *that* ρ *and* v *are* C^1 *functions of* x *and* t*, so that*

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \int_{x_1}^{x_2} \frac{\partial\,\rho(x,t)}{\partial t}\,dx,$$

$$\rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t) = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left[\rho(x,t)v(x,t)\right]\,dx.$$

Thus (1) *becomes*

$$\int_{x_1}^{x_2} \frac{\partial\,\rho(x,t)}{\partial t}\,dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left[\rho(x,t)v(x,t)\right]\,dx. \tag{2}$$

Since, at any moment t*,* (2) *holds for* any $x_1 < x_2$*, and we assumed the integrands to be continuous, we arrive at*

$$\frac{\partial\,\rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\rho(x,t)v(x,t)\right],$$

or often referred to as the equation of continuity, and written as

$$\frac{\partial\,\rho(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\rho(x,t)v(x,t)\right] = 0. \tag{3}$$

By a similar process using the <u>*law of conservation of momentum*</u>*, we arrive at*

$$\frac{\partial[\rho(x,t)v(x,t)]}{\partial t} + \frac{\partial}{\partial x}\left[\rho(x,t)v^2(x,t)\right] = -\frac{\partial p(x,t)}{\partial x}, \tag{4}$$

here, $p(x,t)$ *is the pressure at* (x,t)*, and we have made the assumption that there is no external force acting on the fluid. For an ideal isentropic fluid,* $p(x,t)$ *is determined through the density* $\rho(x,t)$*:* $p = p(\rho)$*. Then* (4) *is an equation in* ρ *and* v*. For many gases,* $p = A\rho^\gamma$ *for some* $\gamma \geq 1$*. As we will see later, the important property here is that* $p'(\rho) > 0$ *for* $\rho > 0$*.*

Ex 2 (Equation for heat conduction)**.** *Here, the relevant physical quantities are* $u(x,t)$ = *temperature at location* x *and time* t*, c=specific heat of the medium, i.e., the amount of heat needed for each unit temperature change per unit mass,* ρ = *density of the medium. Unless we deal with inhomogeneous medium, we assume* c *and* ρ *are constants across the medium. The basic law describing the conduction of heat is expressed as*

rate of change of heat in a region
= *heat flux in the region across the boundary of the region* + *heat source within the region.*

If we let $\vec{q}(x,t)$ *to denote the heat flux vector, i.e.,* $\vec{q}(x,t)\cdot\nu$ *gives the heat flux per unit area with unit normal vector* ν*, and* $f(x,t)$ *to denote the given heat source per unit mass within the region, then, for any region* Ω *with some regularity on its boundary, the above law translates into*

$$\int_\Omega c\rho\frac{\partial u}{\partial t}dx = -\int_{\partial\Omega} \vec{q}(x,t)\cdot\vec{n}(x)\,d\sigma + \int_\Omega \rho f(x,t)dx, \tag{5}$$

where $\vec{n}(x)$ denotes the exterior unit normal to $\partial\Omega$ at x, and $d\sigma$ denotes the area element of $\partial\Omega$. The heat flux vector is determined by Fourier's law: $\vec{q}(x,t) = -k\nabla u(x,t)$, where k is the thermal conductivity of the medium. To reduce (5) into a PDE, we make the mathematical assumption that $u(x,t)$ is twice continuously differentiable in x, so that

$$\begin{aligned} &-\int_{\partial\Omega} \vec{q}(x,t)\cdot \vec{n(x)}\, d\sigma \\ =&k\int_{\partial\Omega} \nabla u(x,t)\cdot \vec{n(x)}\, d\sigma \\ =&k\int_{\Omega} \Delta u(x,t) dx. \end{aligned}$$

Thus (5) reduces to

$$\int_\Omega c\rho\frac{\partial u}{\partial t}dx = \int_\Omega \left[k\Delta u + \rho f(x,t)\right] dx.$$

Since this relation holds on any region Ω and we assumed the integrands to be continuous, we arrive at

$$\frac{\partial u}{\partial t} = \gamma\Delta u + \frac{1}{c}f(x,t). \tag{6}$$

with $\gamma = \frac{k}{c\rho} > 0$ denoting the thermal diffusivity of the medium.

Remark 1. *If Fourier's law needs to be modified in a certain situation, such as assuming k to depend on $u(x,t)$: $k = k(u)$, or $\vec{q}(x,t) = -\mathbf{A}(x,t)\nabla u(x,t)$, where $\mathbf{A}(x,t)$ is an $n\times n$ matrix with certain properties, for example, positive definite, then (6) is modified in the first case as*

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\nabla\left(k(u)\nabla u(x,t)\right) + \frac{1}{c}f(x,t),$$

and in the second case as

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\nabla\left(\mathbf{A}(x,t)\nabla u(x,t)\right) + \frac{1}{c}f(x,t).$$

To determine $u(x,t)$ in a fixed region D, in addition to (6), we also need to know the initial data $u(x,0)$ for $x\in D$, and the boundary data $u(x,t)$ for $x\in\partial D$ and $t>0$. The simplest type of boundary condition is the homogeneous Dirichlet boundary condition $u(x,t)=0$ for $(x,t)\in\partial D\times\mathbb{R}^+$. One may also encounter a Neumann type boundary condition, prescribing $\frac{\partial u}{\partial\vec{n}}(x,t)$ along $(x,t)\in\partial D\times\mathbb{R}^+$. In some situations, we take D to be the entire $\mathbb{R}^n$, so there is only the initial condition but no explicit boundary condition (but for the heat equation, there will be implicit condition that the solution not grow too fast as $x\to\infty$). Such an initial value problem is called a Cauchy problem (compare with the Cauchy problem in ODE). For technical convenience, we often require (6) to hold only in the interior $D\times\mathbb{R}^+$ of the space-time region. This allows us to look for a solution $u\in C(\overline{D\times\mathbb{R}^+})$ with $u_t, \Delta u\in C(D\times\mathbb{R}^+)$.

Ex 3 (Equation of one dimensional waves)**.** *We will omit the derivation process here. Please see Strauss for a brief derivation and H. Weinberger,* A First Course in Partial Differential Equations, *for a thorough derivation. The equation takes the form*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{7}$$

where $u(x,t)$ denotes the displacement from x at time t. c^2 is determined by the property of the medium, and is equal to $\frac{T}{\rho}$ for describing the transverse vibration of a taut elastic string, with T being tension and ρ being the density. As we will learn soon c represents the speed of propagation of wave. (7) *comes from its integral form*

$$\frac{d}{dt} \int_{x_0}^{x_1} \rho u_t dx = T\left[u_x(x_1,t) - u_x(x_0,t)\right], \quad \text{for any } x_0 < x_1. \tag{8}$$

Ex 4 (Laplace equation)**.** *In both Examples 2 and 3, the equilibrium states satisfy $\Delta u = 0$, which is called the Laplace equation. This equation also arises from a great number of other situations. For instance, the real and imaginary parts of a complex analytic function satisfy the Laplace equation.*

Next we illustrate the method of the calculus of variations through the derivation for the equation of *Minimal Surfaces.*

Ex 5 (Equation of minimal surfaces)**.** *Fix a region Ω in $\mathbb{R}^n$ and fix a boundary value function $\phi(x)$ defined for $x \in \partial\Omega$. Among all graphs over Ω with the boundary value $\phi(x)$, we look for one with minimal area. More specifically, let*

$$M_\phi = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = \phi\right\}.$$

we want to solve

$$\min_{u \in M_\phi} \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

The existence of a graph attaining the minimum creates some issues. Suppose for now that such a graph u exists. Then for any $v \in M_0$, and any $\epsilon \in \mathbb{R}$ near 0, $u + \epsilon v \in M_\phi$, and

$$A(\epsilon) := \int_\Omega \sqrt{1 + |\nabla u + \epsilon \nabla v|^2} dx$$

has a minimum at $\epsilon = 0$ and is differentiable in ϵ, so

$$A'(0) = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}} dx = 0. \tag{9}$$

This is an integral version of the minimal surface equation. If we further assume that this u is $C^2(\Omega) \cap C^1(\overline{\Omega})$, then we can integrate by parts in the above to arrive at

$$-\int_\Omega div \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) v dx = 0.$$

—*(in fact, if we assume only $u \in C^2(\Omega)$ instead of $u \in C^2(\overline{\Omega})$, we need to require v to have compact support to justify the integration by parts properly; but this limitation on v is harmless for our argument.) Since this relation holds for arbitrary $v \in M_0$, we conclude that*

$$\begin{cases} div \left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0 & in\ \Omega, \\ u = \phi & on\ \partial\Omega. \end{cases} \tag{10}$$

Later in the course we will discuss that it is not easy to find directly a $C^2(\Omega)$ function achieving the minimum area, but it is relatively easy to find a less regular function satisfying (9)*. We will learn a theory which proves that any solution of* (9) *is automatically $C^2(\Omega)$ and satisfies* (10)*.*

In contrast to the equations in the Examples 2-3, (10) is nonlinear in the unknown u. Next we introduce the concept of linearization

Ex 6. *We define the operator $\mathcal{M}(u) = div \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right)$. Then the linearization of $\mathcal{M}$ at u is defined to be*

$$\mathcal{M}'_u[v] := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{M}(u+\epsilon v).$$

In our case, it turns out

$$\mathcal{M}'_u[v] = \sum_{i,j=1}^{n} \nabla_i \left[\frac{(1+|\nabla u|^2)\delta_{ij} - \nabla u_i \nabla u_j}{(1+|\nabla u|^2)^{3/2}} \nabla v_j \right],$$

and thus

$$\mathcal{M}'_0[v] = \Delta v.$$

For the equations of motion for ideal isentropic gas in one dimension, the left hand sides of the equations can be incorporated into the nonlinear operator

$$N(\rho, v) = \begin{pmatrix} \rho_t + (\rho v)_x \\ (\rho v)_t + (\rho v^2)_x + p'(\rho)\rho_x \end{pmatrix}.$$

Notice that for $\rho = \rho_0$, a constant, we have $N(\rho_0, 0) = \vec{0}$. The linearized operator of N at $(\rho_0, 0)$ is

$$N'_{(\rho_0,0)} \begin{pmatrix} \rho^* \\ v^* \end{pmatrix} = \begin{pmatrix} \rho^*_t + \rho_0 v^*_x \\ \rho_0 v^*_t + p'(\rho_0)\rho^*_x \end{pmatrix}.$$

So as a first approximation, if we want to solve $N'_{(\rho_0,0)} \begin{pmatrix} \rho^* \\ v^* \end{pmatrix} = \vec{0}$, we have

$$\begin{cases} \rho^*_t + \rho_0 v^*_x = 0, \\ \rho_0 v^*_t + p'(\rho_0)\rho^*_x = 0. \end{cases}$$

This is a <u>*system*</u> *of linear equations for* ρ^* *and* v^**, and one can easily eliminate one variable to arrive at*

$$\rho^*_{tt} - c^2 \rho^*_{xx} = 0, \qquad \text{with } c^2 = p'(\rho_0).$$

This confirms that small disturbances in a one dimensional ideal isentropic gas obeys the wave equation in Example 3.

2. Some Comparisons with ODEs

We make some general comments on the comparison between the behavior of ODEs and PDEs.

Remark 2. *The "general" solution of an ODE(system) depends on a finite number of parameters, which can often be adjusted to satisfy some additional side conditions such as initial or boundary conditions. In the case of a system of linear ODEs, this is particularly simple: the solution space is spanned by a finite number of base solutions. However, even for simple linear PDEs such as the ones we derived above, the solution space is infinite dimensional. One technique that we will learn is how to construct an infinite dimensional "basis" of solutions and use them to construct general solutions. One main task will be to make sense of the convergence.*

Ex 7. *Let's look for some sample solutions of* (6) *when* x *is one dimensional. We will use the method of separation of variables: look for solutions of the form* $u(x,t) = T(t)X(x)$ *and* <u>*reduce the construction of solutions of a PDE to some set of ODEs.*</u> *The direct application of separation of variables works when* $f = 0$*. For simplicity, we also take* $\gamma = 1$*. Then we need to solve*

$$T_t(t)X(x) = T(t)X_{xx}(x).$$

We deduce that $X_{xx}(x)/X(x)$ *must be a constant, independent of* (x,t)*. Call this constant* $-\lambda$*. Then we have a set of ODEs*

$$X_{xx} = -\lambda X(x), \tag{11}$$

$$T_t(t) = -\lambda T(t). \tag{12}$$

Note that for each ξ*,* $X(x) = e^{ix\xi}$ *is a solution of* (11) *with* $\lambda = \xi^2$*, and* $T(t) = e^{-\xi^2 t}$ *solves* (12) *for the same* λ*. So* $u_\xi = e^{ix\xi - \xi^2 t}$ *is a solution to* (11) *and* (12)*. By the superposition principle, any finite linear combination of such solutions*

$$\sum_{\xi \in \text{a finite set}} c(\xi) e^{ix\xi - \xi^2 t}$$

is a solution of (6)*.*

Question*: Does this construction generate all solutions to the Cauchy problem for the homogeneous heat equation* (6)*?*

To satisfy an arbitrary initial data, it is obviously not enough to take only finite linear combinations of solutions of the form $e^{ix\xi - \xi^2 t}$*. For an infinite linear combination, or an integral of the form* $\int c(\xi) e^{ix\xi - \xi^2 t} d\xi$*, one issue is the convergence of such integrals for* $(x,t) \in$

$\mathbb{R}^1 \times \mathbb{R}^+$; *another issue is how to choose* $c(\xi)$ *to satisfy a given initial data* $u(x,0)$. *Although for complex valued* ξ, $e^{ix\xi-\xi^2 t}$ *is also a solution of the homogeneous heat equation, but such solutions grow exponentially in* x *as* x *goes to one end of infinity. So we take* $\xi \in \mathbb{R}$. *At least formally, the factor* $e^{-\xi^2 t}$ *helps with the convergence of the integral for* $t > 0$, *but not for* $t < 0$; *also we expect*

$$u(x,0) = \int_{\xi\in\mathbb{R}} c(\xi) e^{ix\xi} d\xi. \tag{13}$$

We recognize that this is possible if we choose $c(\xi)$ *to be the Fourier transform of* $u(x,0)$, *up to a constant. We will make this argument rigorous later on. The purpose here is to illustrate that the homogeneous heat equation has an infinite parameter family of solutions, and how this family may be used to construct a general solution.*

If one is interested in solving the homogeneous heat equation on a finite interval $[0,l]$ *with the simplest boundary conditions* $u(0,t) = u(l,t) = 0$ *for all* $t > 0$, *for instance, then in our sample solutions, we want* $X(0) = X(l) = 0$. *So the choice of* $X(x)$ *is limited to satisfy a boundary value problem of ODEs:*

$$X_{xx} = -\lambda X(x), \tag{14}$$

$$X(0) = X(l) = 0. \tag{15}$$

The choice of λ *is now limited to be a discrete set of numbers* $\lambda = \left(\frac{n\pi}{l}\right)^2$, *for* $n = 1, 2, \cdots$, *and* $X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$. *So*

$$\sum_{n\in \textit{a finite set}} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

is a genuine solution of the homogeneous heat equation with the boundary condition $u(0,t) = u(l,t) = 0$ *for all* $t > 0$. *To satisfy an arbitrarily given initial data* $u(x,0)$, *we again need to make an infinite sum. Formally, we need to choose* c_n *such that*

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = u(x,0) \quad \textit{on } [0,l]. \tag{16}$$

Again, the factor $e^{-\left(\frac{n\pi}{l}\right)^2 t}$ *will help with the convergence for* $t > 0$, *but not for* $t < 0$.

This brings up the issue of wellposedness.

Remark 3. *For an ODE system of the form* $\frac{du}{dt} = f(u(t), t)$, *the Cauchy-Picard theorem gives us the existence of a unique solution on a (maybe short) time interval* $[t_0 - \delta, t_0 + \delta]$, *for right hand side* f *that is Lipschitz in* u; *and the solution depends continuously on the initial data. For a boundary/initial value problem for PDE, we also need to address the same issues. We say a boundary/initial value problem for PDE is wellposed if*

(1) *there exists a solution for all data in a reasonable (often closed) set of a function space;*
(2) *the solution is unique (or at least has only on a finite number of free parameters);*

(3) *the solution depends on the data in a continuous way.*

The meaning of "continuous way" has some flexibility in the sense that we may need to measure the variation of solution in different norms.

Our calculation for the sample solutions of the heat equation above suggests that the Cauchy problem (or the boundary/initial value problem) is probably wellposed in the above sense for forward time $t > 0$, but not for backward time $t < 0$. The latter can be seen easily: all the solutions $e^{ix\xi-\xi^2 t}$ have norm in L^∞ equal to 1 at time $t = 0$, yet at any negative time $t = -\delta$, the L^∞ norm is equal to $e^{\xi^2\delta}$, which can be made as large as one wants by taking large ξ. This phenomenon of wellposedness only in one direction is very different from the behavior of ODEs.

Remark 4. *There is no direct generalization of the Cauchy-Picard theorem for a general Cauchy problem for a PDE. For the Cauchy problem for PDEs which are analytic in the unknowns and have analytic initial data on an analytic initial surface which is non-characteristic, there is a partial generalization of the Cauchy-Picard theorem, called Cauchy-Kowalevskaya theorem. We will discuss this theorem and the relevant notions of characteristics of PDEs later on in the course.*

Remark 5. *Our examples earlier may suggest that PDEs (at least linear ones) may have too big a solution space that we have to impose appropriate boundary/initial conditions to have a well posed problem. So it was a surprise when,in 1957, H. Lewy constructed the first linear PDE that has no solution anywhere.*

We now make some more definite statements about the solutions to (6) obtained above through separation of variables and superposition principle—both lead to Fourier's method, one to Fourier series, the other to Fourier transforms. First, you are invited to verify

Exercise 1. *For any $c(\xi) \in L^p(\mathbb{R})$, $1 \le p \le \infty$, $\int_{\mathbb{R}} c(\xi) e^{ix\xi-\xi^2 t} d\xi$ defines a smooth function for $x \in \mathbb{R}$, and $t > 0$, and satisfies $\frac{\partial u}{\partial t} = \Delta u$.*

The main issue now is to verify that, if we choose $c(\xi)$ appropriately, namely, $c(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} u(y,0)e^{-iy\xi}\, dy$, then the $u(x,t)$ defined above takes on the initial value $u(x,0)$ as $t \to 0$. One situation where this can be checked easily is when we assume $u(x,0)$ can be represented as in (13) with $c(\xi) \in L(\mathbb{R})$. Then $u(x,t) \to u(x,0)$ as $t \to 0+$ by Lebesgue's dominated convergence theorem. But this assumption on $u(x,0)$ is through its Fourier transform, and would imply some global condition on $u(x,0)$. It turns out we can verify the continuity of $u(x,t)$ up to $t = 0$ under much weaker condition on $u(x,0)$. We proceed by deducing a further representation formula for $u(x,t)$ as follows.

$$\begin{aligned}
&u(x,t)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}u(y,0)e^{-iy\xi}\,dy\right\}e^{ix\xi-\xi^2 t}d\xi\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}u(y,0)\left\{\int_{\mathbb{R}}e^{i(x-y)\xi-\xi^2 t}d\xi\right\}dy\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}u(y,0)e^{-\frac{|x-y|^2}{4t}}\sqrt{\frac{\pi}{t}}dy \quad \text{using contour integration and } \int_{\mathbb{R}}e^{-u^2}du=\sqrt{\pi}\\
=&\int_{\mathbb{R}}u(y,0)\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}dy,
\end{aligned}\tag{17}$$

here, to justify the interchange of integrals, you may well assume $u(y,0)$ to have compact support. The integral kernel $K(x-y,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}$, called the heat kernel, enjoys the following properties:

(AP1). $\int_{\mathbb{R}} K(x-y,t)dy=1$, for any x and $t>0$.
(AP2). For any $\delta>0$, $\int_{|x-y|>\delta} K(x-y,t)dy\to 0$ as $t\to 0$.
(AP3). $K(x-y,t)\geq 0$.

So $\{K(x-y,t)\}_{t>0}$ forms a family of *approximation to identity.* With these properties above, one can easily prove

Theorem 1. *Define $u(x,t)$ in terms of $u(x,0)$ through (17).*

(1) *For any $u(x,0)\in L^p(\mathbb{R})$, $1\leq p<\infty$, $u(x,t)$ is a C^∞ solution to $u_t-u_{xx}=0$ on $(x,t)\in\mathbb{R}\times(0,\infty)$, and $||u(x,t)-u(x,0)||_{L^p}\to 0$, as $t\to 0$.*
(2) *Suppose $u(x,0)$ is continuous at x_0, then $u(x,t)$ is continuous at $u(x_0,0)$.*

Remark 6. *A consequence of the conclusion of the above Theorem is that, instead of solving the Cauchy problem which demands that the initial value be taken on pointwise, it is reasonable to demand the initial value be taken on in the L^p sense.*

Notice also that even if the initial data has discontinuity, the solution given by (17) instantly becomes a smooth function of (x,t) for $t>0$.

One advantage of the above representation is that it provides a solution (possibly on a finite interval) even for initial data $u(x,0)$ that has growth slower than $e^{a|x|^2}$ for some $a>0$, even though the formula was derived under much stricter requirement on $u(x,0)$. In fact, this kind of approach will be used repeatedly: on a first try, one does not pay too close attention to justify every step, only after one obtains some useful results, does one go back to check every step, or verify the conclusion by other means.

We now make some comments on Fourier series which arise from solving the initial value problem for the heat equation with boundary conditions. Recall that we can define c_n in (16) for any $u(x,0)\in L^p[0,l]$. There are three facts that are used often (see, for instance, Theorems 2-4 in 5.4 of Strauss):

(i). If $u(x,0) \in L^2[0,l]$, then (16) holds in the sense of L^2.

(ii). The pointwise convergence of the LHS of (16) at some x_0 depends on the local behavior of $u(x,0)$ near x_0, and if $u(x,0)$ is piecewise locally Lipschitz continuous at x_0, then it converges to $[u(x_0+,0)+u(x_0-,0)]/2$.

(iii). If $u(0,0)=u(l,0)=0$ and $u_x(x,0) \in L^2[0,l]$, then the LHS of (16) converges of $u(x,0)$ uniformly for $x \in [0,l]$.

One can formulate a criterion for the question of pointwise convergence of

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

to $u(x,0)$ as $t \to 0$, similar to the one for the heat equation on the entire line, namely, by assuming $u(x,0)$ is such that its Fourier coefficients form an absolutely convergent series: $\sum_{n=1}^{\infty} |c_n| < \infty$. The argument is the same. One sufficient condition on $u(x,0)$ which would imply $\sum_{n=1}^{\infty} |c_n| < \infty$ is as stated in (iii) above. In such cases,

$$c_n = \frac{2}{l}\int_0^{2\pi} u(x,0)\sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{n\pi}\int_0^{2\pi} u_x(x,0)\cos\left(\frac{n\pi x}{l}\right) dx.$$

The Fourier coefficients of $u_x(x,0)$ is l^2 summable by Parseval relation. So by Cauchy-Schwarz inequality $\sum_{n=1}^{\infty} |c_n| < \infty$.

However, it is not that routine to prove the analogue of Theorem 1 except in the case of $p=2$. The convergence results in (i) and (ii) above are not readily helpful on this issue. Here we can still convert the sum above into a convolution of $u(x,0)$ with a kernel function. But it is not easy to directly verify the properties (AP1-3) as those for the heat kernel on $\mathbb{R}$. We will come back to this issue later.

Nonetheless, if we reformulate the problem slightly, we have a satisfactory answer.

Theorem 2. *For any $g \in L^2[0,l]$, there is a unique solution $u \in C^\infty([0,l]\times(0,\infty)) \cap C([0,\infty), L^2[0,l])$ to*

$$\begin{cases} u_t - u_{xx} = 0, & \text{for} \quad (x,t) \in (0,l)\times(0,\infty), \\ u(0,t) = u(l,t) = 0, & \text{for} \quad t > 0, \\ u(x,0) = g(x) & \text{for} \quad x \in [0,l]. \end{cases} \tag{18}$$

Here the space $C([0,\infty), L^2[0,l])$ means $u(\cdot,t)$ is continuous as elements of $L^2[0,l]$.

The uniqueness part will be addressed in an exercise at the end of this section. For the existence part, we simply take c_n to be the Fourier coefficients of g, and construct

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}.$$

Then, as we discussed earlier, the only remaining issue is the continuity of $u(\cdot,t)$ in $L^2[0,l]$, which follows as a consequence of the Parseval identity for the Fourier coefficients. Note that, unless further assumptions on g is made, the solution $u(x,t)$ may not be continuous at

$(0,0)$ and $(l,0)$. At issue here is the compatibility of the given initial and boundary date at these two corner points.

We now apply similar method to find a solution formula for the Laplace equation on a round disk in $\mathbb{R}^2$. We could write the Laplace operator in polar coordinates and apply separation of variables to look for solutions of the form $u = R(r)\Theta(\theta)$. But based on our knowledge of complex analytic functions, we know directly that for any $n \geq 0$, $z^n = r^n e^{in\theta}$ and $\bar{z}^n = r^n e^{-in\theta}$ are harmonic functions on $B_1(0) \subset \mathbb{R}^2$. So

$$\sum_{n \in \text{a finite set}} \left\{ a_n r^n e^{in\theta} + b_n r^n e^{-in\theta} \right\}$$

provides a harmonic function on $B_1(0)$ with boundary value

$$\sum_{n \in \text{a finite set}} \left\{ a_n e^{in\theta} + b_n e^{-in\theta} \right\},$$

which is a trigonometric polynomial in θ. In fact, we now have a solution to the Dirichlet problem on $B_1(0)$ with any trigonometric polynomial as boundary value. To deal with the problem of arbitrary boundary data (continuous, say), we form the infinite sum

$$u = \sum_{n=0}^{\infty} \left\{ a_n r^n e^{in\theta} + b_n r^n e^{-in\theta} \right\}$$

and hope to be able to choose a_n and b_n to obtain arbitrary boundary data. The issues that need to be justified are dealt with in the following:

Exercise 2. (1) *Prove that for any bounded sequence $\{a_n\}$ and $\{b_n\}$, the series defining u above converges uniformly on any smaller disk $\{r \leq r_0 < 1\}$, and gives rise to a smooth harmonic function in $B_1(0)$.*

(2) *If we choose $\{a_n\}$ and $\{b_n\}$ to be the Fourier coefficients of the boundary value function $g(\theta)$:*

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-in\theta} d\theta, \qquad \text{for } n \geq 0,$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{in\theta} d\theta, \qquad \text{for } n > 0, \quad \text{and} \quad b_0 = 0,$$

then $u(r,\theta) = \int_0^{2\pi} g(\phi) P(\theta - \phi, r) d\phi$, where $P(\theta, r) = \frac{1-r^2}{2\pi(1+r^2-2r\cos\theta)}$. This is called Poisson formula.

(3) *Prove that $\{P(\theta,r)\}_{r\to 1}$ forms an approximation of identity.*

(4) *Prove that for any continuous function $g(\theta)$ on $\partial B_1(0)$, the u above provides a harmonic function in $B_1(0)$ that is continuous on $\overline{B_1(0)}$ and takes on $g(\theta)$ as boundary value.*

(5) *Verify that if we write $X = (r\cos\theta, r\sin\theta)$, and $Y = (\cos\phi, \sin\phi)$, then $P(\theta-\phi, r) = \frac{1-|X|^2}{2\pi|X-Y|^2}$.*

Exercise 3. (1) *Apply separation of variables/Fourier transforms to construct solutions to the one dimensional wave equation* $u_{tt} - c^2 u_{xx} = 0$ *on* $(x,t) \in \mathbb{R}^2$.

(2) *Apply separation of variables/Fourier series to construct solutions to the initial/boundary value problem to the one dimensional wave equation*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l]. \end{cases} \tag{19}$$

We next explore another approach related to the method of ODEs to find a solution formula for the one dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ for $(x,t) \in \mathbb{R}^2$. One ingredient is to recognize that $u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u$. So if we set $(\partial_t + c\partial_x)u = v(x,t)$, then we are to solve a system of first order equations

$$(\partial_t + c\partial_x)u = v(x,t), \tag{20}$$

$$(\partial_t - c\partial_x)v = 0. \tag{21}$$

A second ingredient is to think of the actions of the first order differential operators in (20)-(21) at each point (x,t) as a directional derivative. Thus, for (21), there is a vector-field which happens to take value $(-c, 1)$ at every (x,t), along the integral curve of which v remains a constant. These integral curves are called the characteristic curves for the wave equation. Here, the integral curves satisfy $\frac{dx}{dt} = -c$, or $x = -ct + x_0$. Thus $v(-ct + x_0, t)$ is a constant of t and is equal to $v(x_0, 0)$. In other words, $v(x,t) = v(x + ct, 0)$. Plugging this into (20), and applying similar method, we find $\frac{d}{dt}u(ct + x_1, t) = v(ct + x_1, t) = v(2ct + x_1, 0)$. Thus

$$u(ct + x_1, t) = u(x_1, 0) + \int_0^t v(2c\tau + x_1, 0)d\tau = u(x_1, 0) + \int_{x_1}^{x_1+2ct} v(y,0)dy/2c.$$

Or setting $x = ct + x_1$, we see $x_1 = x - ct$, and

$$\begin{aligned} u(x,t) &= u(x-ct, 0) + \int_{x-ct}^{x+ct} v(y,0)dy/2c \\ &= u(x-ct,0) + \int_{x-ct}^{x+ct} \left[(\partial_t + c\partial_y)u(y,0)\right] dy/2c \\ &= u(x-ct,0) + \int_{x-ct}^{x+ct} u_t(y,0)dy/2c + \left[u(x+ct,0) - u(x-ct,0)\right]/2 \\ &= \left[u(x+ct,0) + u(x-ct,0)\right]/2 + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(y,0)dy. \end{aligned}$$

This formula is called D'Alembert's formula.

Remark 7. *The derivation process here also proves that a C^2 solution is unique, while that is not the case for the solution formula for the heat equation or for the Poisson formula.*

Theorem 3. *For any $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$, there is a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$ solving*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad \text{in} \quad \mathbb{R} \times [0,\infty), \\ u(x,0) = g(x), \\ u_t(x,t) = h(x). \end{cases}$$

Remark 8. (i). *The D'Alembert's formula exhibits that the solution at (x,t) depends on its initial data only in the range $[x-ct, x+ct]$, which is called the interval of dependence of (x,t) on $t=0$. Viewed from a different perspective, this means that signals travel with a finite speed, which in this case is c. Recall that the solution to the heat equation as provided by (17) shows that the value of $u(x,t)$ is influenced by its initial value at every location, in other words, there is infinite speed of propagation for the heat equation.*

(ii). *The derivation process here showed that the general solution to the homogeneous one dimensional wave equation is $G(x-ct) + H(x+ct)$ for some C^2 functions G and H. This exhibits the wave character of the solutions. In fact, even if G or H are not necessarily C^2, but are only continuous, with piecewise continuous derivatives, $G(x-ct) + H(x+ct)$ should still be regarded as a solution to the wave equation, although it may not satisfy its differential form* (7)*. One can verify the integral form* (8) *for such solutions. Suppose x_0 is, say, the only point of discontinuity for the derivative (or higher derivative) of G, then $G(x-ct)$ is a smooth solution to the wave equation except along the characteristic line $x - ct = x_0$, along which the discontinuity of the derivative persists. This is different from the behavior of solutions to the heat equation, which smoothes out any discontinuity instantly.*

3. Duhamel's principle and energy methods for the wave equation

We first explain the Duhamel's principle, which gives a procedure for constructing the solution to non-homogeneous equations based on the construction of solutions to the corresponding homogeneous equations.

Theorem 4. *For each parameter s, let $U(x,t;s)$ be the unique solution to*

$$\begin{cases} U_{tt}(x,t;s) - c^2 U_{xx}(x,t;s) = 0, \quad \text{in} \quad \mathbb{R} \times [s,\infty), \\ U(x,s;s) = 0, \\ U_t(x,s;s) = f(x,s). \end{cases} \tag{22}$$

Then

$$u(x,t) = \int_0^t U(x,t;s)\,ds$$

solves

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad \text{in} \quad \mathbb{R}\times[0,\infty), \\ u(x,0) = 0, \\ u_t(x,0) = 0. \end{cases} \tag{23}$$

Remark 9. *The Duhamel principle applies in any space dimension. In dimenion* 1*,* $U(x,t;s)$ *can be written out explicitly as*

$$U(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy,$$

so

$$u(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2c} f(y,s)dyds.$$

Combining Theorems 1 and 2, we have

Theorem 5. *For any* $g \in C^2(\mathbb{R})$*,* $h \in C^1(\mathbb{R})$*, and* $f \in C^1(\mathbb{R}\times[0,\infty))$*, there is a unique solution* $u(x,t) \in C^2(\mathbb{R}\times[0,\infty))$ *solving*

$$\begin{cases} u_{tt} - c^2 u_{xx} = f, \quad \text{for} \quad (x,t)\in\mathbb{R}\times[0,\infty), \\ u(x,0) = g(x), \quad \text{for} \quad x\in\mathbb{R} \\ u_t(x,t) = h(x), \quad \text{for} \quad x\in\mathbb{R}. \end{cases} \tag{24}$$

Exercise 4. *This problem deals with the Duhamel principle applied to the Cauchy problem for the heat equation:*

$$\begin{cases} u_t - u_{xx} = f(x,t), \quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ u(x,0) = 0, \quad \text{for } x\in\mathbb{R}. \end{cases} \tag{25}$$

Prove that, if f *is* C^1 *in its variables, then* $u(x,t) = \int_0^t K(x-y,t-\tau)f(y,\tau)dyd\tau \in C(\mathbb{R}\times\overline{\mathbb{R}^+})\cap C^2(\mathbb{R}\times\mathbb{R}^+)$ *is a solution to* (25)*, where* $K(x,t)$ *is the heat kernel to the heat equation as introduced in* (17)*. (*Remark: Differentiation under the integral sign can not be justified if one merely assumes f to be continuous. However, as we will discuss later, some Hölder continuity on f is enough.*)*

Next we introduce the energy method for the wave equation. Based on physical considerations, we define

$$E[u(\cdot,t)] = \frac{1}{2}\int_{\mathbb{R}} \left(u_t^2(x,t) + c^2 u_x^2(x,t)\right) dx$$

to be the energy of the solution u at time t. Then

$$\begin{aligned}\frac{dE[u(\cdot,t)]}{dt} &= \int_{\mathbb{R}} \left(u_t u_{tt} + c^2 u_x u_{xt}\right) dx \\ &= \int_{\mathbb{R}} \left(u_t u_{tt} - c^2 u_{xx} u_t\right) dx \\ &= 0.\end{aligned}$$

The integration by parts can be justified if we assume $u(x,0)$ and $u_t(x,0)$ to have compact support. For, then, the solution will remain compactly supported by the finite speed of propagation. To make the argument not to depend on the separate argument for the finite speed of propagation, we modify the physical argument in the following way. Notice that

$$u_t(u_{tt} - c^2 u_{xx}) = (\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2)_t - (c^2 u_x u_t)_x.$$

So for a solution u to the homogeneous wave equation, the vector-field

$$(P,Q) = (-c^2 u_x u_t, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2)$$

is divergence free. For any given (X,T), we form the triangle with vertices (X,T), $(X-cT,0)$, and $(X+cT,0)$, and also consider the trapezoid

$$D_0^\tau(X,T) = \{(x,t) : 0 \le t \le \tau, |x - X| \le c(T-t)\}$$

for any $0 < \tau < T$. Integrating the above divergence free vector-field (P,Q) on $D_0^\tau(X,T)$, we obtain

$$0 = \int_{\partial D_0^\tau(X,T)} (P,Q) \cdot (n_x, n_t) ds.$$

On the $t = \tau$ portion of $\partial D_0^\tau(X,T)$, $X - c(T-\tau) \le x \le X + c(T-\tau)$, and

$$(P,Q) \cdot (n_x, n_t) = \frac{1}{2}u_t^2(x,\tau) + \frac{c^2}{2}u_x^2(x,\tau),$$

while on the $t = 0$ portion of $\partial D_0^\tau(X,T)$, $X - cT \le x \le X + cT$,

$$(P,Q) \cdot (n_x, n_t) = -\left[\frac{1}{2}u_t^2(x,0) + \frac{c^2}{2}u_x^2(x,0)\right].$$

On the lateral portion of $\partial D_0^\tau(X,T)$, $(x,t) = (X \pm c(T-t), t)$, and

$$(P,Q) \cdot (n_x, n_t) = \frac{c}{2\sqrt{c^2+1}} \left[u_t(X \pm c(T-t), t) \mp c u_x(X \pm c(T-t), t)\right]^2.$$

The key feature is that $(P,Q)\cdot(n_x,n_t)\geq 0$ along $\partial D_0^\tau(X,T)$, as long as we take $n_t\geq 0$, so

$$\begin{aligned}
&\int_{X-c(T-\tau)}^{X+c(T-\tau)}\left[\frac{1}{2}u_t^2(x,\tau)+\frac{c^2}{2}u_x^2(x,\tau)\right]dx\\
&+\int_0^\tau \frac{c}{2}\left[u_t(X+c(T-t),t)-cu_x(X+c(T-t),t)\right]^2dt\\
&+\int_0^\tau \frac{c}{2}\left[u_t(X-c(T-t),t)+cu_x(X-c(T-t),t)\right]^2dt\\
&=\int_{X-cT}^{X+cT}\left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right]dx,
\end{aligned}$$

and we have the local version of the energy estimate

$$\int_{X-c(T-\tau)}^{X+c(T-\tau)}\left[\frac{1}{2}u_t^2(x,\tau)+\frac{c^2}{2}u_x^2(x,\tau)\right]dx\leq\int_{X-cT}^{X+cT}\left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right]dx.$$

Thus, if

$$\int_{X-cT}^{X+cT}\left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right]dx=0,$$

then

$$\int_{X-c(T-\tau)}^{X+c(T-\tau)}\left[\frac{1}{2}u_t^2(x,\tau)+\frac{c^2}{2}u_x^2(x,\tau)\right]dx=0,$$

for all $0<\tau<T$, proving that the speed of propagation is not faster than c. A direct consequence of the energy estimate is the uniqueness property: if $u(x,0)\equiv u_t(x,0)\equiv 0$, then $u(x,t)\equiv 0$—one simply applies the above energy estimate on any finite trapezoid as in the proof above.

We can also apply the same method to deal with initial/boundary value problem such as (19), for whose solutions u, we can easily prove

$$\int_0^l\left[\frac{1}{2}u_t^2(x,t)+\frac{c^2}{2}u_x^2(x,t)\right]dx=\int_0^l\left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right]dx.$$

The uniqueness to the initial boundary value problem (19) follows from this estimate.

Corollary. *In the class $C^2((0,l)\times(0,T))\cap C^1([0,l]\times[0,T])$, there is at most one solution to* (19).

Another consequence of the energy estimates is the continuous dependence in L^2 norm of the (derivatives of) solution on data. Such energy estimates on the solutions can also be used to prove existence of solutions. For instance, when the initial data $g(x)$ and $h(x)$ are trigonometric sine polynomials, the Fourier series method readily provides a genuine solution to (19). For more general initial data, we can try to use trigonometric sine polynomials $g_n(x)$ and $h_n(x)$ to approximate $g(x)$ and $h(x)$, respectively, in the following L^2 norm:

$$||g_n - g||_{\dot{H}^1} \stackrel{\text{def}}{=} \left[\int_0^l |\partial_x g_n(x) - \partial_x g(x)|^2 dx\right]^{1/2} \to 0,$$

and

$$||h_n - h||_{L^2} = \left[\int_0^l |h_n(x) - h(x)|^2 dx\right]^{1/2} \to 0,$$

as $n \to \infty$. Let $u_n(x,t)$ denote the corresponding unique solution to (19) with $u_n(x,0) = g_n(x)$ and $\partial_t u_n(x,0) = h_n(x)$, then the energy estimate above says $\{u_n(x,t)\}$ is a Cauchy sequence in the above L^2 norm, so we expect to find an appropriate limit, which should satisfy the equation in some form. If g and h are such that we can take the approximating sine polynomials g_n and h_n with the further property that $\{\partial_x h_n\}$ and $\{\partial^2_{xx} g_n\}$ are also Cauchy in $L^2[0,l]$, then we can apply the energy estimates to $\partial_t u_n(x,t)$ and $\partial_x u_n(x,t)$ to obtain that $\{\partial^2_{xx} u_n\}$, $\{\partial^2_{xt} u_n\}$, and $\{\partial^2_{tt} u_n\}$ are also Cauchy in $C([0,T], L^2[0,l])$. In such cases, we expect the limit to have some notion of second derivatives. This approach will lead to the L^2 weak derivatives and weak solutions. The sacrifice is that we may not get a C^2 limit as a solution. But many natural properties for the wave equation suggest that L^2 space is a more natural space to work with for the wave equation. We will explore the ideas here in more detail later on.

Remark 10. *The energy method is the first instance for us to work out the properties of solutions directly from the equations, instead of relying on a representation formula. Most of the more robust methods we will develop later on also have this feature.*

Summary. *Here are a few key features that have emerged.*

(i). *It is natural and fruitful to approach a PDE by first finding formal, exploratory sample solutions, and then try to build more general solutions. When the equation has a nice structure (for instance, with constant coefficients, and is homogeneous), and the domain has nice geometry, separation of variables is often effective. This method often reduces the problem to an eigenvalue boundary value problem for an ODE. The key for building a general solution for a linear PDE through superposition is to find the right notion of convergence.*

(ii). *The convergence of the (sample or later approximate) solutions can be established if we have appropriate a priori estimates for smooth solutions such as the engergy estimates for solutions of the wave equation. Note that we don't have to have an explicit formula for the solution to derive a useful estimate. We also learned that it is not fruitful to always insists on pointwise or uniform convergence, that convergence in other (often integral) norms are often forced upon us by the structure of the problem.*

(iii). *For non-homogeneous linear PDE, in addition to the Duhamel principle, we can also adapt the result of seperation of variables method to form the method of eigenfunction expansions. We illustrate this method via* (19) *with a non-homogeneous right hand* *side $f(x,t)$. Since any $L^2[0,l]$ function can be expanded as a series in $\{\sin(\frac{n\pi x}{l})\}$ which is convergent in $L^2[0,l]$, we may suppose to look for a solution $u(x,t)$ such that at any t, $u(\cdot,t)$, $u_{tt}(\cdot,t)$, and $u_{xx}(\cdot,t) \in L^2[0,l]$:*

$$
\begin{aligned}
u(\cdot,t) &= \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{l}),\\
u_{tt}(\cdot,t) &= \sum_{n=1}^{\infty} \alpha_n(t)\sin(\frac{n\pi x}{l}),\\
u_{xx}(\cdot,t) &= \sum_{n=1}^{\infty} \beta_n(t)\sin(\frac{n\pi x}{l}).
\end{aligned}
$$

We also assume that

$$f(\cdot,t) = \sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi x}{l}).$$

Thus

$$\alpha_n(t) - c^2\beta_n(t) = f_n(t), \qquad \textit{for all } n.$$

But

$$\alpha_n(t) = \frac{2}{l}\int_0^l u_{tt}(x,t)\sin(\frac{n\pi x}{l})dx = u_n''(t),$$

and

$$
\begin{aligned}
\beta_n(t) &= \frac{2}{l}\int_0^l u_{xx}(x,t)\sin(\frac{n\pi x}{l})dx\\
&= \frac{2}{l}u_x(x,t)\sin(\frac{n\pi x}{l})|_{x=0}^{x=l} - \frac{2}{l}\int_0^l \frac{n\pi}{l}u_x(x,t)\cos(\frac{n\pi x}{l})dx\\
&= -\frac{2}{l}\frac{n\pi}{l}u(x,t)\cos(\frac{n\pi x}{l})|_{x=0}^{x=l} - \frac{2}{l}\int_0^l \left(\frac{n\pi}{l}\right)^2 u(x,t)\sin(\frac{n\pi x}{l})dx\\
&= -\left(\frac{n\pi}{l}\right)^2 u_n(t) \qquad \textit{using the condition } u(0,t) = u(l,t) = 0
\end{aligned}
$$

Thus $u_n(t)$ satisfies

$$u_n''(t) + c^2\left(\frac{n\pi}{l}\right)^2 u_n(t) = f_n(t).$$

We can use the initial conditions to obtain

$$u_n(0) = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx, \quad u_n'(0) = \frac{2}{l}\int_0^l h(x)\sin(\frac{n\pi x}{l})dx.$$

Thus our problem has been reduced to solving an infinite system of ODEs. See Strauss' text for more illustrations of this method. To complete the construction of a solution, *the central issue is again the convergence fo the series. Here it is still most easily handled by the energy estimates.*

As we move forward, here are a few questions that we should keep in mind.

Questions. *We have been able to solve the three prototype equations when the domain has special geometry. How to deal with the situation when the domain has no special geometry? We have been been able to obtain solvability for these prototype equations with non-homogeneous right hand side when it is sufficiently nice. Can we reduce the smoothness assumptions on it? How do we solve problems which are variations of the prototype equations (with some additional terms added, or with variable coefficients)? Given a general PDE which does not bear much resemblance to any of our prototype equations, how do we know whether it is reasonable (well-posed?) to study the boundary value or initial value problem for it? how do we know whether its solution is behaving like those of the heat equation or wave equation, or something completely different?*

Exercise 5. *In this problem we consider the energy method for the heat equation. Consider any solution u to* (18).

(1) *Prove that*

$$\frac{d}{dt}\int_0^l \frac{u^2(x,t)}{2}dx = -\int_0^l u_x^2(x,t)dx, \text{ for } t > 0.$$

(2) *Prove the uniqueness to* (18) *in the class $C^1([0,l]\times[0,T]) \cap C^2((0,l)\times(0,T))$.*

Exercise 6. *Here is a version of the energy estimates for solutions to the wave equation with non-homogeneous right hand side. Let $u \in C^2(\mathbb{R}\times(0,T) \cap C^1(\mathbb{R}\times[0,T])$ be a solution to* (24)*. Then there exists $M = M(T) > 0$ such that*

$$\int_{\mathbb{R}} \left[u_t^2(x,t) + c^2u_x^2(x,t)\right] dx \le M\left\{\int_{\mathbb{R}} \left[c^2g_x^2(x) + h^2(x)\right] dx + \iint_{\mathbb{R}\times[0,t]} f^2(x,\tau)dxd\tau\right\},$$

and

$$\iint_{\mathbb{R}\times[0,T]} \left[u_t^2(x,t) + c^2u_x^2(x,t)\right] dx\, dt \le MT\left\{\int_{\mathbb{R}} \left[c^2g_x^2(x) + h^2(x)\right] dx + \iint_{\mathbb{R}\times[0,T]} f^2(x,\tau)dxd\tau\right\}.$$

Exercise 7. *This problem illustrates that the energy method can be adapted to deal with certain issues involving variable coefficient wave equations. Prove that if u is a C^2 solution to*

$$u_{tt} - c^2u_{xx} + \alpha(x,t)u_t + \beta(x,t)u_x + \gamma(x,t)u = 0,$$

with $u(x,0) = u_t(x,0) = 0$ for $|x-x_0| \le R$, then $u(x,t) = 0$ for $|x-x_0| \le R - ct$ for $0 < t < R/c$. (Hint: formulate and prove a version of the energy estimates.)

Exercise 8.

In this problem we consider the relation between the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{26}$$

and the variational problem

$$I[u] = \min_{w \in X_g} I[w], \tag{27}$$

where $I[w] = \int_\Omega \left(\frac{1}{2}|\nabla w(x)|^2 - f(x)w(x)\right) dx$, *and* $X_g = \left\{w \in C^1(\overline{\Omega}) : w = g \text{ on } \partial\Omega\right\}$.

(1) *Prove that* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *is a solution of* (26), *iff it is a solution of* (27).

(2) *One can use the variational principle above to derive the expression for* Δu *in non-rectangular (or general Riemannian) coordinates. For instance, in two dimension,* $|\nabla w|^2 = w_r^2 + \frac{1}{r^2}w_\theta^2$, *so the variation of* $\int \frac{1}{2}|\nabla w|^2$ *in the direction of* v *is*

$$\begin{aligned} &\int \left\{ w_r v_r + \frac{1}{r^2} w_\theta v_\theta \right\} r dr\, d\theta \\ = &-\int v \left\{ \frac{1}{r}(r w_r)_r + \frac{1}{r^2} w_{\theta\theta} \right\} r dr\, d\theta \end{aligned}$$

So in polar coordinates (r, θ), $\Delta w = \frac{1}{r}(r w_r)_r + \frac{1}{r^2} w_{\theta\theta}$. *Derive an expression for* Δu *in spherical coordinates in* $\mathbb{R}^3$, *and produce some examples of harmonic functions (radial and non-radial) that are singular at the origin.*

4. One dimensional wave equation on the half line: reflection method and compatibility conditions

We use the one dimensional wave equation on the half line to illustrate the reflection method and introduce the issue of compatibility conditions. Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & \text{for} \quad (x,t) \in \mathbb{R}^+ \times [0, \infty), \\ u(x,0) = g(x), & \text{for} \quad x \geq 0, \\ u_t(x,t) = h(x), & \text{for} \quad x \geq 0, \\ u(0,t) = 0, & \text{for} \quad t > 0. \end{cases}$$

We want to use the D'Alembert's formula to construct a solution to the above problem. In order to satisfy the boundary condition $u(0,t) = 0$, for $t > 0$, it is natural to do an odd extension of the initial data and the the right hand side of the equation:

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if } x \geq 0; \\ -g(-x), & \text{if } x < 0. \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x), & \text{if } x \geq 0; \\ -h(-x), & \text{if } x < 0. \end{cases}$$

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & \text{if } x \geq 0; \\ -f(-x,t), & \text{if } x < 0. \end{cases}$$

Then

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+ct) + \tilde{g}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(y)dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \tilde{f}(y,\tau)dy \quad (28)$$

is the solution formula provided by the D'Alembert's formula. Due to the odd symmetry, $\tilde{u}(0,t) = 0$ for all $t > 0$. In the region $x > ct$, $\tilde{u}(x,t)$ is expressed in terms of the given initial data and $f(x,t)$ on $x > 0$:

$$\tilde{u}(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)dy.$$

In the region $x < ct$, we obtain

$$\tilde{u}(x,t) = \frac{1}{2}\left[g(x+ct) - g(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} h(y)dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \tilde{f}(y,\tau)dy$$

where the last integral can also be expressed in terms of integrals of $f(y,t)$ in intervals of $y > 0$. From Theorem 5, in order for $\tilde{u}(x,t)$ to be a C^2 solution for all (x,t), a sufficient condition is to assume $\tilde{g}$ to be C^2, and $\tilde{h}$, $\tilde{f}$ to be C^1. This amounts to

$$\begin{aligned} g(0) = 0, \qquad & g''(0) = 0; \\ h(0) &= 0; \\ f(0,t) = 0, \qquad & \text{for all } t > 0. \end{aligned}$$

But these conditions are too restrictive. Assuming only that $g \in C^2[0,\infty)$, $h \in C^1[0,\infty)$, and $f \in C^1([0,\infty)\times[0,\infty))$, the formula in (28) provides a function which is C^2 in $(x,t) \in [0,\infty)\times[0,\infty)$ with possibly the exception along $x = ct$. In order for $\tilde{u}$ to be continuous at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+}\tilde{u}(x,0) = \lim_{t\to 0+}\tilde{u}(0,t)$, *i.e.*, $g(0) = 0$. In order for $\tilde{u}$ to be C^1 at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+}\tilde{u}_t(x,,0) = \lim_{t\to 0+}\tilde{u}_t(0,t)$, *i.e.*, $h(0) = 0$. In order for $\tilde{u}$ to be C^2 at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+}\tilde{u}_{xx}(x,,0) = \lim_{t\to 0+}\tilde{u}_{xx}(0,t)$. Since $\lim_{x\to 0+}\tilde{u}_{xx}(x,,0) = g''(0)$, but $\lim_{t\to 0+}\tilde{u}_{xx}(0,t)$ can be calculated through the equation as $\lim_{t\to 0+}\left[u_{tt}(0,t) - f(0,t)\right]/c^2 = -f(0,0)/c^2$, we have $g''(0) = -f(0,0)/c^2$. These are called compatibility conditions. It turns out that these compatibility conditions are also sufficient conditions for $\tilde{u}$ to be a C^2 solution in $(x,t) \in [0,\infty)\times[0,\infty)$.

Remark 11. *From both physical and mathematical points of view, it is too restrictive to demand to deal only with C^2 solutions. Examination of the above discussion shows that if we assume $g(0) = 0$, then $\tilde{u}(x,t)$ will be continuous in $[0,\infty)\times[0,\infty)$; and if we we assume $h(0) = 0$, in addition to $g(0) = 0$, then $\tilde{u}(x,t)$ will be C^1 in $[0,\infty)\times[0,\infty)$. Both should be regarded as generalized solutions.*

We now add a few more words on the concept of generalized solutions. Here are the guiding principles for the concept of generalized solutions:

(A). Classical solutions must be generalized solutions;
(B). Uniqueness should hold for the generalized solutions, and there should be some kind of continuous dependence (in appropriate norms) of the generalized solutions on data.

As a consequence, limits of classical solutions (in appropriate norms) should be generalized solutions. Generalized solutions may allow the solutions to have some singularities (discontinuity of the solutions or their derivatives, or the size of the solutions or their derivatives become infinite on some part of the region). Depending on the problem, there may be different kinds of singularities that are relevant. For instance, there may be situations where discontinuity in the solutions need to be considered; while in other situations, one may need to consider solutions that are continuous, but have discontinuous derivatives. For problems coming from physical background, one often goes back to the physical principles to find the appropriate notion of generalized solutions. Later in the course, we will discuss the jump conditions across the interface of discontinuity for piecewise continuous solutions to equations of conservation laws. For the moment we will use the wave equation to introduce another often used method for finding a reasonable notion of generalized solutions. Discarding the boundary condition $u(0,t) = 0$ for the moment, we can formulate notions of generalized solutions to the one dimensional wave equation in the following way. We start from the following relations

$$
\begin{aligned}
&(u_{tt} - c^2 u_{xx})\zeta \\
=& - u_t\zeta_t + c^2 u_x \zeta_x + [u_t\zeta]_t - [c^2 u_x \zeta]_x \\
=& u(\zeta_{tt} - c^2\zeta_{xx}) + [u_t\zeta - u\zeta_t]_t + [c^2 u\zeta_x - c^2\zeta u_x]_x
\end{aligned}
$$

for any C^2 functions u and ζ. Assume u is a C^2 solution to $u_{tt} - c^2 u_{xx} = f$ and ζ is a C^2 function with compact support on $\mathbb{R}\times[0,\infty)$, then integrating the above relations, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)dxdt \\
=&\iint_{\mathbb{R}\times[0,\infty)} [-u_t\zeta_t + c^2 u_x\zeta_x]dxdt - \int_{\mathbb{R}} u_t(x,0)\zeta(x,0)dx \\
=&\iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt} - c^2\zeta_{xx})]dxdt - \int_{\mathbb{R}} [u_t(x,0)\zeta(x,0) - u(x,0)\zeta_t(x,0)]dx.
\end{aligned}
$$

Conversely if $u \in C^2(\mathbb{R}\times[0,\infty))$ satisfies either of the above integral relations for <u>all test functions</u> ζ which are C^2 with compact support on $\mathbb{R}\times[0,\infty)$, it is routine to argue that u solves (24). Notice that the integral relations above make sense under weaker requirements on u. Since we have not introduced the concept of generalized derivatives yet, we will formulate a notion of generalized solution to the wave equation in the following way.

Definition. *We say* $u \in L^2_{local}(\mathbb{R} \times [0,\infty))$ *is a generalized solution to* (24) *if*

$$\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)dxdt = \iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt}-c^2\zeta_{xx})]dxdt - \int_{\mathbb{R}} [u_t(x,0)\zeta(x,0)-u(x,0)\zeta_t(x,0)]dx$$

for all $\zeta \in C^2(\mathbb{R}\times[0,\infty))$ *with compact support.*

Here is an easy argument for the uniqueness of generalized solution to (24). It suffices to prove that if

$$\iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt} - c^2\zeta_{xx})]dxdt = 0,$$

for all $\zeta \in C^2(\mathbb{R}\times[0,\infty))$ with compact support, then $u \equiv 0$. Fix any $T > 0$, then there exists a sequence $u_n \in C^1_c(\mathbb{R}\times(0,T))$ such that $\lim_{n\to\infty} ||u - u_n||_{L^2(\mathbb{R}\times(0,T))} = 0$. For each u_n, we can construct a classical solution ζ_n to

$$\begin{cases} \zeta_{tt} - c^2\zeta_{xx} = u_n, & \text{for } (x,t) \in \mathbb{R}\times(0,T), \\ \zeta(x,T) = 0, \\ \zeta_t(x,T) = 0. \end{cases}$$

We also know that ζ_n will have compact support in $\mathbb{R}\times[0,T]$. Thus we can use these ζ_n as test function to obtain

$$\iint_{\mathbb{R}\times(0,T)} uu_n = 0, \quad \text{for each } n.$$

Sending $n \to \infty$, we obtain $\int_{\mathbb{R}\times(0,T)} u^2dxdt = 0$. Thus $u = 0$.

Exercise 9. *Prove that, if* u *is a (generalized) solution to* (24) *or* (19) *with a non-homogeneous right hand side* $f(x,t)$, *then for any* $T > 0$, *there is a constant* $M = M(T) > 0$ *such that*

$$||u||^2_{L^2(\mathbb{R}\times(0,T))} \le M\left[||f||^2_{L^2(\mathbb{R}\times(0,T))} + ||u(\cdot,0)||^2_{L^2(\mathbb{R})} + ||u_t(\cdot,0)||^2_{L^2(\mathbb{R})} + ||u_x(\cdot,0)||^2_{L^2(\mathbb{R})}\right]$$

With the help of **Exercises** 6 and 9, we can give an argument for the existence of a generalized solution to (24) when $f \in L^2_{\text{local}}(\mathbb{R}\times[0,\infty))$, g, g', and $h \in L^2(\mathbb{R})$. For any $T > 0$, we can take a sequence $f_n \in C^\infty_c(\mathbb{R}\times[0,T])$ approximating f in $L^2(\mathbb{R}\times[0,T])$, and a sequence g_n, $h_n \in C^\infty_c(\mathbb{R})$ such that $g_n - g \to 0$, $g'_n - g' \to 0$, and $h_n - h \to 0$ in $L^2(\mathbb{R})$. Then we can use D'lambert's formula to construct a classical solution $u_n(x,t)$ on $\mathbb{R}\times[0,T])$ to (24) with f_n, g_n, h_n substituing for f, g, h, respectively. Then **Exercises** 6 and 9 imply that $\{u_n\}$, $\{\partial_t u_n\}$, and $\{\partial_x u_n\}$ are Cauchy in $L^2(\mathbb{R}\times[0,T])$. So there is a limit $u \in L^2(\mathbb{R}\times[0,T])$, $v \in L^2(\mathbb{R}\times[0,T])$, and $w \in L^2(\mathbb{R}\times[0,T])$ such that

$$u_n \to u, \quad \partial_t u_n \to v, \quad \text{and} \quad \partial_x u_n \to w \quad \text{in } L^2(\mathbb{R}\times[0,T]).$$

For u_n, we have

$$\iint_{\mathbb{R}\times[0,\infty)} f_n(x,t)\zeta(x,t)dxdt = \iint_{\mathbb{R}\times[0,\infty)} [u_n(\zeta_{tt}-c^2\zeta_{xx})]dxdt - \int_{\mathbb{R}} [\partial_t u_n(x,0)\zeta(x,0)-u_n(x,0)\zeta_t(x,0)]dx,$$

and

$$\iint_{\mathbb{R}\times[0,\infty)} u_n(x,t)\zeta_t(x,t)dxdt = -\iint_{\mathbb{R}\times[0,\infty)} \partial_t u_n(x,t)\zeta(x,t)dxdt - \int_{\mathbb{R}} u_n(x,0)\zeta(x,)dx,$$

$$\iint_{\mathbb{R}\times[0,\infty)} u_n(x,t)\zeta_x(x,t)dxdt = -\iint_{\mathbb{R}\times[0,\infty)} \partial_x u_n(x,t)\zeta(x,t)dxdt,$$

for all $\zeta \in C^2(\mathbb{R}\times[0,\infty))$ with compact support. Taking limit as $n\to\infty$, we obtain that

$$\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)dxdt = \iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt}-c^2\zeta_{xx})]dxdt - \int_{\mathbb{R}} [\partial_t u(x,0)\zeta(x,0)-u(x,0)\zeta_t(x,0)]dx,$$

and

$$\iint_{\mathbb{R}\times[0,\infty)} u(x,t)\zeta_t(x,t)dxdt = -\iint_{\mathbb{R}\times[0,\infty)} v(x,t)\zeta(x,t)dxdt - \int_{\mathbb{R}} u(x,0)\zeta(x,0)dx,$$

$$\iint_{\mathbb{R}\times[0,\infty)} u(x,t)\zeta_x(x,t)dxdt = -\iint_{\mathbb{R}\times[0,\infty)} w(x,t)\zeta(x,t)dxdt.$$

Thus we obtain a generalized solution u to (24), and we can identify v and w to be the generalized derivatives $\partial_t u$, and $\partial_x u$, respectively.