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First, hamon toda for the wonderful gift: the three issues of the AAM for my 65-th birthday.

Here is a problem (probably classical) which might involve bijections.

Let $P(n)$ denote the set of the partitions of n. As usual, for $\lambda \in P(n)$, $\ell(\lambda)$ is the number of parts of λ. Let $SP(n) \subset P(n)$ denote the partitions of n of odd distinct parts. For example, $SP(8) = \{(7,1),(5,3)\}$.

Fact:
1. $|P(n) \setminus SP(n)|$ is always even, and moreover,
2. for exactly half of the partitions $\lambda \in P(n) \setminus SP(n)$, $\ell(\lambda)$ is even.

I can prove this *purely combinatorial* fact using the character theory of S_n and A_n. Is there a direct - purely combinatorial - proof?

Here is a "bijective" proof that $|P(n) \setminus SP(n)|$ is always even:
Let $P_{sym}(n) = \{\lambda \vdash n \mid \lambda = \lambda'\}$, so $p_{sym}(n) = |P_{sym}(n)|$. Let $\lambda = \lambda'$, write it in the Frobenius notation $\lambda = (a_1, \dots, a_k \mid a_1, \dots, a_k)$ then map it

$$h : (a_1, \dots, a_k \mid a_1, \dots, a_k) \to (2a_1 + 1, \dots, 2a_k + 1) \in SP(n),$$

then $h : P_{sym}(n) \to SP(n)$ is a bijection, so $|P_{sym}(n)| = |SP(n)|$. This yields a bijection between $P(n) \setminus SP(n)$ and $P(n) \setminus P_{sym}(n)$. But $P(n) \setminus P_{sym}(n) = \{\lambda \vdash n \mid \lambda \neq \lambda'\}$ is a union of the pairs λ, λ', hence $|P(n) \setminus P_{sym}(n)| = |P(n) \setminus SP(n)|$ is even.

We would get a proof of 2. if we can extend the bijection $h : P_{sym}(n) \to SP(n)$ into a bijection $h^* : P(n) \to P(n)$ – therefore also $h^* : P(n) \setminus P_{sym}(n) \to P(n) \setminus SP(n)$ – such that to each pair λ, λ', exactly one partition of the pair $h^*(\lambda), h^*(\lambda')$ is of even length So, the question is how to find h^* for all n.

lehitraot,

Amitai