ON NUMBERS AND GAMES

SECOND EDITION

J. H. CONWAY

John von Neumann Professor of Mathematics,
Princeton University

A K Peters, Ltd.
Natick, Massachusetts

Editorial, Sales, and Customer Service Office

A K Peters, Ltd.
63 South Avenue
Natick, MA 01760

Library of Congress Cataloging-in-Publication Data

Conway, John Horton.
On numbers and games / John H. Conway.–2nd ed.
p.cm.
Includes bibliographical references and index.
ISBN 1-56881-127-6 (alk. paper)
1. Number theory. 2. Game theory. I. Title.

QA241 .C69 2000
519.3–dc21 00-046927

Printed in the United States of America
05 04 03 02 10 9 8 7 6 5 4 3 2

Prologue

Just over a quarter of a century ago, for seven consecutive days I sat down and typed from 8:30 am until midnight, with just an hour for lunch, and ever since have described this book as "having been written in a week."

Not entirely honest, because there were loose ends still to be tied up, and Chapter 16 was written just before the book appeared, while Chapter 13 was largely copied from a paper, "Hackenbush, Welter and Prune", that had been written a year earlier. But also not entirely dishonest.

Why the rush? Because ONAG, as the book is familiarly known, was getting in the way of writing *Winning Ways* (WW). Now that both books are happily being republished by A K Peters, Onagers (a word that also means "Wild Asses"!) can be told just how it came about before they surrender themselves to pure pleasure (as "Onag" means in Hebrew!).

A few years previously, Elwyn Berlekamp, Richard Guy and I had agreed to write a book on mathematical games, by which at that time we meant the Nim-like theory developed independently by Roland Sprague and Peter Michael Grundy for sums of impartial games—those for which the two players have exactly the same legal moves.

I had long intended to see what would become of the theory when this restriction was dropped, but only got around to doing so when the then British Go Champion became a member of the Cambridge University Pure Mathematics Department. Astonishingly, it was the resulting attempt to understand "Go" that led to the discovery of the Surreal Numbers! This happened because the typical "Go" endgame was visibly a sum of games in the sense of this book, making it clear that this notion was worthy of deep study in its own right. The Surreal Numbers then emerged as the simplest domain to which it applies!

However, their theory rapidly burgeoned in ways that made it inappropriate for the book that later became *Winning Ways*. A busy term was approaching, and it seemed that this "transfinite" material just had to be got out of the way before that term started if *Winning Ways* was ever to be published. So I sat down for that week and wrote this book, and then confessed the fact to my co-authors.

The most surprising immediate result was a threat of legal action from Elwyn Berlekamp! But somehow we must have patched this up, because both ONAG and WW appeared in the next few years, and we remain good friends.

In fact, the Surreal Numbers "surfaced" before ONAG appeared, partly through my 1970 lectures at Cambridge and Cal. Tech., but mostly through the wide circulation of Donald Knuth's little book, *Surreal Numbers*. I am very grateful to Knuth for inventing this name—the original version of ONAG said "Because of the generality of this Class, we shall simply describe its members as numbers, without adding any restricting adjective." "Surreal Numbers" is much better!

I am very happy and grateful that A.K. Peters have agreed to publish millennial editions of both this book and *Winning Ways*.

Ariel Jaffee and Kathryn Maier were responsible for handling the changes to this edition. This is also the place to acknowledge Richard Guy's considerable contributions to the original edition. In particular, he designed and drew a number of the original figures and computed, or recomputed several of the tables.

I have called this a Prologue rather than a Preface because it is usually understood that the Preface to a later edition of a book should contain a description of the changes in the book and its subject since its first edition. Some of these functions are addressed in the Epilogue.

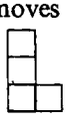
John H. Conway

Preface

This book was written to bring to light a relation between two of its autho favourite subjects—the theories of transfinite numbers and mathemati games. A few connections between these have been known for some ti but it appears to be a new observation that we obtain a theory at o simpler and more extensive than Dedekind's theory of the real numb just by *defining* numbers as the strengths of positions in certain gam When we do this the usual properties of order and the arithmetic operati follow almost immediately from definitions that are naturally suggested, that it was quite an amusing exercise to write the zeroth part of the book if these definitions had arisen instead from an attempt to genera Dedekind's construction!

However, we suspect that there will be many readers who are m interested in playing games than philosophising about numbers. For th readers we offer the following words of advice. Start reading Chapter on playing several games at once, and find an interested friend with whom play a few games of the domino game described there. In this it's easy

see why  and  give Left one and two moves advantage respectively

when you feel you vaguely understand why  gives him just half o move's advantage, you might like to read Chapter 0, which explains how simplest numbers arise. You should then find no difficulty in reading the r of the book without knowing any more about numbers than that "ordina are a certain kind of (usually infinite) whole number, and that the Author strange idiosyncracies which make him use capital letters for certain v large infinite collections.

Many friends have helped me to write this book, often without bei aware of the fact. I owe an especial debt to Elwyn Berlekamp and Richa Guy, with whom I am currently preparing a more extended book on mati matical games which should overlap this one in several places. The bo would never have appeared without the repeated gentle proddings that ca from Anthony Watkinson of Academic Press; it would have contain

The most surprising immediate result was a threat of legal action from Elwyn Berlekamp! But somehow we must have patched this up, because both ONAG and WW appeared in the next few years, and we remain good friends.

In fact, the Surreal Numbers "surfaced" before ONAG appeared, partly through my 1970 lectures at Cambridge and Cal. Tech., but mostly through the wide circulation of Donald Knuth's little book, *Surreal Numbers*. I am very grateful to Knuth for inventing this name—the original version of ONAG said "Because of the generality of this Class, we shall simply describe its members as numbers, without adding any restricting adjective." "Surreal Numbers" is much better!

I am very happy and grateful that A.K. Peters have agreed to publish millennial editions of both this book and *Winning Ways*.

Ariel Jaffee and Kathryn Maier were responsible for handling the changes to this edition. This is also the place to acknowledge Richard Guy's considerable contributions to the original edition. In particular, he designed and drew a number of the original figures and computed, or recomputed several of the tables.

I have called this a Prologue rather than a Preface because it is usually understood that the Preface to a later edition of a book should contain a description of the changes in the book and its subject since its first edition. Some of these functions are addressed in the Epilogue.

John H. Conway

Preface

This book was written to bring to light a relation between two of its author's favourite subjects—the theories of transfinite numbers and mathematical games. A few connections between these have been known for some time, but it appears to be a new observation that we obtain a theory at once simpler and more extensive than Dedekind's theory of the real numbers just by *defining* numbers as the strengths of positions in certain games. When we do this the usual properties of order and the arithmetic operations follow almost immediately from definitions that are naturally suggested, so that it was quite an amusing exercise to write the zeroth part of the book as if these definitions had arisen instead from an attempt to generalise Dedekind's construction!

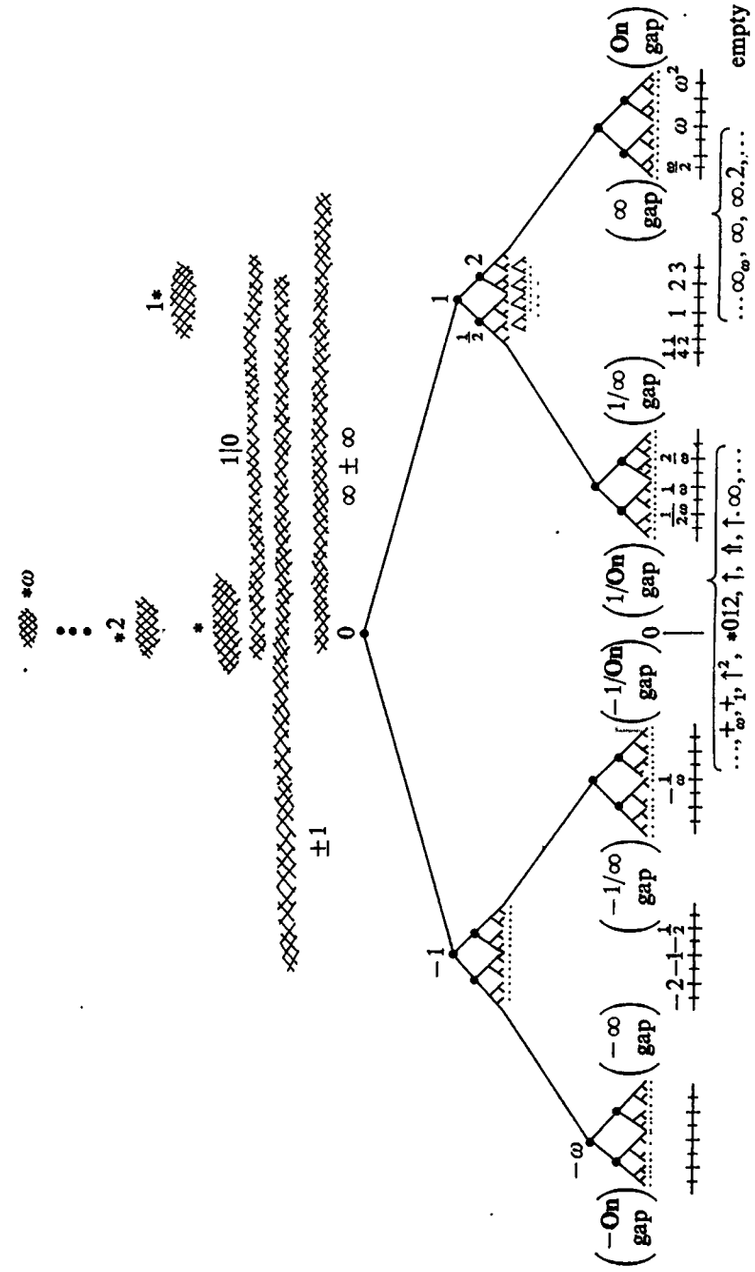
However, we suspect that there will be many readers who are more interested in playing games than philosophising about numbers. For these readers we offer the following words of advice. Start reading Chapter 7, on playing several games at once, and find an interested friend with whom to play a few games of the domino game described there. In this it's easy to see why  and  give Left one and two moves advantage respectively—when you feel you vaguely understand why  gives him just half of a move's advantage, you might like to read Chapter 0, which explains how the simplest numbers arise. You should then find no difficulty in reading the rest of the book without knowing any more about numbers than that "ordinals" are a certain kind of (usually infinite) whole number, and that the Author has strange idiosyncracies which make him use capital letters for certain very large infinite collections.

Many friends have helped me to write this book, often without being aware of the fact. I owe an especial debt to Elwyn Berlekamp and Richard Guy, with whom I am currently preparing a more extended book on mathematical games which should overlap this one in several places. The book would never have appeared without the repeated gentle proddings that came from Anthony Watkinson of Academic Press; it would have contained

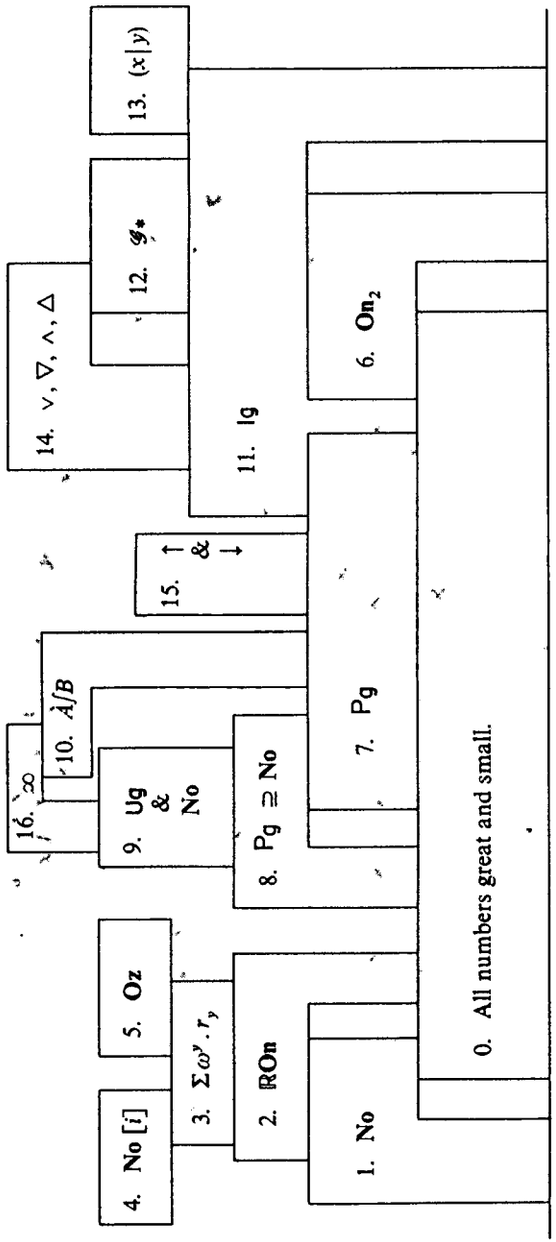
many errors were it not for the careful reading of Paul Cohn as editor, and the quality of the printing and layout could never have been so high without the detailed attentions of Ron Hitchings and the staff of the printers at Page Bros of Norwich. Others whose comments have affected more than one page are Mike Christie, Aviezri Fraenkel, Mike Guy, Peter Johnstone, Donald Knuth and Simon Norton. The varied nature of the advice they gave is neatly encapsulated in the following lines from Bunyan's *Apology for his Book (Pilgrim's Progress)*:

Some said 'John, print it'; others said, 'Not so.'
Some said 'It might do good'; others said, 'No.'

October 1975 J.H.C.



Frontispiece. The tree of numbers, and the positions of some games.



Dependence of Chapters. (Not on oath!)

Contents

ZEROTH PART

ON NUMBERS . . .

A Hair, they say, divides the False and True;
Yes; and a single Alif were the clue,
Could you but find it—to the Treasure-house,
And peradventure to The Master too!

Edward Fitzgerald's
"Rubaiyat of Omar Khayyam"

CHAPTER 0

All Numbers Great and Small

Whatever is not forbidden, is permitted.
J. C. F. von Schiller, Wallensteins Lager

This book is in two = {zero, one | } parts. In this zeroth part, our topic is the notion of *number*. As examples we have the finite numbers $0, 1, 2, \ldots, -1, \frac{1}{2}, \sqrt{2}, \pi, \ldots$; infinite numbers such as ω (the first infinite ordinal); and also infinitesimal numbers such as $1/\omega$. If we were to adopt the axiom of choice, then the infinite cardinal numbers like $\aleph_0$ could be identified with the least corresponding ordinal numbers, so that we can regard these too as part of our system (although the arithmetic is different).

In the system of "Surreal Numbers" we shall describe, every number has its own unique name and properties and many remarkable numbers, such as

$$\sqrt[3]{(\omega + 1)} - \frac{\pi}{\omega}$$

appear. But the "number" $i = \sqrt{-1}$ will not arise in the same way (though we add it in Chapter 4), since there is no property enjoyed by i which is not shared by $-i$. In fact we reply to questions about "the square root of -1" by simply asking exactly which square root of -1 is meant?

Let us see how those who were good at constructing numbers have approached this problem in the past.

Dedekind (and before him the author—thought to be Eudoxus—of the fifth book of Euclid) constructed the real numbers from the rationals. His method was to divide the rationals into two sets L and R in such a way that no number of L was greater than any number of R, and use this "section" to define a new number $\{L \mid R\}$ in the case that neither L nor R had an extremal point.

His method produces a logically sound collection of real numbers (if we ignore some objections on the grounds of ineffectivity, etc.), but has been criticised on several counts. Perhaps the most important is that the rationals are supposed to have been already constructed in some other way, and yet

are "reconstructed" as certain real numbers. The distinction between the "old" and "new" rationals seems artificial but essential.

Cantor constructed the infinite ordinal numbers. Supposing the integers $1, 2, 3, \ldots$ given, he observed that their *order-type* ω was a new (and infinite) number greater than all of them. Then the order-type of $\{1, 2, 3, \ldots, \omega\}$ is a still greater number $\omega + 1$, and so on, and on, and on. The similar objections to Cantor's procedure have already been met by von Neumann, who observes that it is unnecessary to suppose $1, 2, 3, \ldots$ given, and that it is natural to start at 0 rather than 1. He also takes each ordinal as the *set* (rather than the order-type) of all previous ones. Thus for von Neumann, 0 is the empty set, 1 the set $\{0\}$, 2 the set $\{0, 1\}, \ldots, \omega$ the set $\{0, 1, 2, \ldots\}$, and so on.

In this chapter we shall show that these two methods are part of a simpler and more general one by which we can construct the very large Class **No** of "Surreal Numbers," which includes both the real numbers and the ordinal numbers, as well as others like those mentioned above. Inside this book we shall usually omit the adjective "surreal," coined by Donald Knuth, and simply call these things "numbers." It turns out that **No** is a Field (i.e., a field whose domain is a proper Class)—in general we shall capitalise the initial letter of any "big" concept, on the grounds that proper Classes, like proper names, deserve capital letters. So, for instance, the word *Group* will mean any group whose domain is a proper class.

CONSTRUCTION

If L, R are any two sets of numbers, and no member of L is $\geqslant$ any member of R, then there is a number $\{L \mid R\}$. All numbers are constructed in this way.

CONVENTION

If $x = \{L \mid R\}$ we write x^L for the typical member of L, and x^R for the typical member of R. For x itself we then write $\{x^L \mid x^R\}$.

$x = \{a, b, c, \ldots \mid d, e, f, \ldots\}$ means that $x = \{L \mid R\}$, where $a, b, c, \ldots$ are the typical members of L, and $d, e, f, \ldots$ the typical members of R.

DEFINITIONS

Definition of $x \geqslant y$, $x \leqslant y$.
We say $x \geqslant y$ iff (no $x^R \leqslant y$ and $x \leqslant$ no y^L), and $x \leqslant y$ iff $y \geqslant x$.
We write $x \nleqslant y$ to mean that $x \leqslant y$ does not hold.

Definition of $x = y$, $x > y$, $x < y$.
$x = y$ iff $(x \geqslant y$ and $y \geqslant x)$. $x > y$ iff $(x \geqslant y$ and $y \ngeqslant x)$.
$x < y$ iff $y > x$.

Definition of $x + y$.
$$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}.$$

Definition of $-x$.
$$-x = \{-x^R \mid -x^L\}.$$

Definition of xy.
$$xy = \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid \\ x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L\}.$$

It is remarkable that these few lines already define a real-closed Field with a very rich structure.

We now comment on the definitions. A most important comment whose logical effects will be discussed later is that *the notion of equality is a defined relation.* Thus apparently different definitions will produce the same number, and we must distinguish between the *form* $\{L \mid R\}$ of a number and the number itself.

All the definitions are inductive, so that to decide, for instance, whether $x \geqslant y$ we must consider a number of similar questions about the pairs x^R, y and x, y^L, but these problems are all simpler than the given one. It is perhaps not quite so obvious that the inductions require no basis, since ultimately we are reduced to problems about members of the empty set.

In general when we wish to establish a proposition $P(x)$ for all numbers x, we will prove it inductively by deducing $P(x)$ from the truth of all the propositions $P(x^L)$ and $P(x^R)$. We regard the phrase "all numbers are constructed in this way" as justifying the legitimacy of this procedure. When proving propositions $P(x, y)$ involving two variables we may use *double induction*, deducing $P(x, y)$ from the truth of all propositions of the form $P(x^L, y)$, $P(x^R, y)$, $P(x, y^L)$, $P(x, y^R)$ (and, if necessary, $P(x^L, y^L)$, $P(x^L, y^R)$, $P(x^R, y^L)$, $P(x^R, y^R)$). Such multiple inductions can be justified in the usual way in terms of repeated single inductions.

We shall allow ourselves to use certain expressions $\{L \mid R\}$ that are not numbers, since they do not satisfy the condition that no member of L shall be $\geq$ any member of R. In general we may write down any expression $\{L \mid R\}$ and even discuss inequalities between such expressions before establishing that they are numbers, but if we wish such an expression to represent a number we must establish the condition on L and R. In the more general theory developed in the next part of the book, we show that when the condition on L and R is omitted we obtain the more general notion of a *game*.

Our next comments concern the motives for these particular definitions. Now it is our intention that each new number x shall lie between the numbers x^L (to the left) and x^R (to the right), and that $\geqslant, +, -, \cdot$, etc., shall have their usual properties. So that if (say) $y \geqslant$ some x^R we would not have $x \geqslant y$, for

then $x \geqslant x^R$. Similarly, we could not allow $x \geqslant y$ if $x \leqslant$ some y^L. So we define $x \geqslant y$ in all other cases. (This conforms with our motto, and helps to ensure that numbers are totally ordered.)

The spirit of the definitions is to ask what we know already (i.e. by the answers to *simpler* questions) about the object being defined, and to make the answers part of our definition. Thus if addition is to have nice properties and if x is between x^L and x^R, and y between y^L and y^R, then we know "already" that $x + y$ must lie between both $x^L + y$ and $x + y^L$ (on the left) and $x^R + y$ and $x + y^R$ (on the right), which yields the definition of $x + y$. Similarly $-x$ will lie between $-x^R$ (on the left) and $-x^L$ (on the right), which suffice to define $-x$.

It is not nearly so easy to find exactly what we "already" know about xy. It might seem, for instance, that we know that xy lies between x^Ly and xy^L (on the left) and x^Ry and xy^R (on the right), which would yield the definition

$$xy = \{x^Ly, xy^L \mid x^Ry, xy^R\}.$$

But this fails in two ways. Firstly, what we "knew" here is sometimes false (consider negative numbers), and secondly, even when it is true it need not be the strongest information we "already" know. In fact, of course, this defines the same function as $x + y$.

It takes a great deal of thought to find the correct definition, which comes from the observation that (for instance) from $x - x^L > 0$ and $y - y^L > 0$ we can deduce $(x - x^L)(y - y^L) > 0$, so that we must have $xy > x^Ly + xy^L - x^Ly^L$. Since all the products here are simpler ones, and since we regard addition and subtraction as already defined, we can regard this inequality as already known when we come to define xy, and the other inequalities in the definition are similar. [Note that for positive numbers x and y the inequality $xy > x^Ly + xy^L - x^Ly^L$ is stronger than both inequalities $xy > x^Ly$, $xy > xy^L$.]

We can summarise our comments by saying that the definitions of the various operations and relations are just the simplest possible definitions that are consistent with their intended properties. In the next chapter, we shall verify that these intended properties really hold of all numbers, but in the rest of this chapter we shall simply explore the system in a more informal way. To simplify the notation, when L is the set $\{a, b, c, \ldots\}$ and R the set $\{\ldots, x, y, z\}$, we shall simply write $\{a, b, c, \ldots \mid \ldots, x, y, z\}$ for $\{L \mid R\}$.

EXAMPLES OF NUMBERS, AND SOME OF THEIR PROPERTIES

The number 0

According to the construction, every number has the form $\{L \mid R\}$, where

L and R are two sets of earlier constructed numbers. So how can the system possibly get "off the ground", since initially there will be no earlier constructed numbers?

The answer, of course, is that even before we have any numbers, we have a certain *set* of numbers, namely *the empty set* $\varnothing$! So the earliest constructed number can only be $\{L \mid R\}$ with both $L = R = \varnothing$, or in the simplified notation, the number $\{\mid\}$. This number we call 0.

Is 0 a number? Yes, since we cannot have any inequality of the form $0^L \geqslant 0^R$, for there is neither a 0^L nor a 0^R!

Is $0 \geqslant 0$? Yes, for we can have no inequality of the form $0^R \leqslant 0$ or $0 \leqslant 0^L$. So by the definition, and happily, we have $0 = 0$. We also see from the definitions that $-0 = 0 + 0 = 0$, since there is no number of any of the forms $-0^R, -0^L, 0^L + 0, 0 + 0^L, 0^R + 0, 0 + 0^R$. A slightly more complicated observation of the same type is that $x0 = 0$, since in every one of the terms defining xy there is a mention of y^L or y^R, so that when $y = 0$ no term is needed and the expression for xy reduces to $\{\mid\} = 0$. So the number 0 has at least some of the properties we know and love.

The numbers 1 and -1

We can now use the sets $\{\}$ and $\{0\}$ for L and R, obtaining hopefully the numbers $\{\mid\}$, $\{0 \mid\}$, $\{\mid 0\}$, $\{0 \mid 0\}$. But since we have already proved that $0 \geqslant 0$, $\{0 \mid 0\}$ is *not* a number, and we have only two new cases, which we call $1 = \{0 \mid\}$ and $-1 = \{\mid 0\}$. Note that -1 is indeed a case of the definition $-x$.

Is $0 \geqslant 1$? This will be true unless there is 0^R with $0^R \leqslant 1$ (there isn't) or 1^L with $0 \leqslant 1^L$ (there is, namely $1^L = 0$). So we do *not* have $0 \geqslant 1$.

Is $1 \geqslant 0$? This is true unless there is 1^R with "..." or 0^L with "..." (whatever "..." is, there plainly can't be). So we have $1 \geqslant 0$, and so $1 > 0$.

By symmetry, we have $-1 < 0$, and so if inequalities "behave", then we should have $-1 < 1$. We check this:

Is $-1 \geqslant 1$? This will happen unless there is $(-1)^R \leqslant 1$ or ... (there is, namely $(-1)^R = 0$). So we do not have $-1 \geqslant 1$.

Is $1 \geqslant -1$? This will happen unless there is 1^R with ... or $(-1)^L$ with ... (there isn't). So $1 \geqslant -1$, so $1 > -1$, as we hoped.

We can generalise a part of this last argument. If there is no x^R and no y^L, then $x \geqslant y$ holds vacuously.

We forgot to check that $1 \geqslant 1$. Why not do this yourself?

The numbers 2, $\frac{1}{2}$, and their negatives

We now have three numbers $-1 < 0 < 1$, and so a whole battery of

particular sets

$$\{\ \},\ \{-1\},\ \{0\},\ \{1\},\ \{-1,0\},\ \{-1,1\},\ \{0,1\},\ \{-1,0,1\}$$

to use for L and R. But the condition that no member of L should be $\geqslant$ any member of R restricts us to the possibilities

$$\{\,|\,R\},\ \{L\,|\,\},\ \{-1\,|\,0\},\ \{-1\,|\,0,1\},\ \{-1\,|\,1\},\ \{0\,|\,1\},\ \{-1,0\,|\,1\}.$$

If our hopes are fulfilled, we should have $\{1\,|\,\} > 1$ and $0 < \{0\,|\,1\} < 1$. So we anticipate their probable values, and define $\{1\,|\,\} = 2$, $\{0\,|\,1\} = \frac{1}{2}$. We then have, of course, $\{\,|\,-1\} = -2$, and $\{-1\,|\,0\} = -\frac{1}{2}$, from the definition of negation.

Before we justify these names, let us ask about some of the other possibilities. For example, what about the number $x = \{0, 1\,|\,\}$? This x is presumably restricted by the conditions $0 < x$, $1 < x$. But since $0 < 1$, if inequalities behave (and we shall suppose from now on that they do), the condition $1 < x$ already implies $0 < x$, so in some sense the entry 0 isn't telling us anything. We can therefore hope that $x = \{0, 1\,|\,\} = \{1\,|\,\} = 2$. We test this, supposing $2 > 1 > 0$.

Is $x \geqslant 2$? This is so unless there is $x^R \leqslant 2$ (no) or $x \leqslant$ some 2^L (no, because the only 2^L is 1, and we believe $x > 1$). So we think that $x \geqslant 2$.

Is $2 \geqslant x$? Yes, unless some $2^R \ldots$ (no) or $2 \leqslant$ some x^L (no, since the only x^L are 1 and 0). So indeed $x = 2$, if all our expectations are fulfilled.

In a similar way, we should expect all the equalities in the table:

$$\begin{aligned}
-2 &= \{\,|\,-1\} = \{\,|\,-1,0\} = \{\,|\,-1,1\} = \{\,|\,-1,0,1\}\\
-1 &= \{\,|\,0\} = \{\,|\,0,1\}\\
-\tfrac{1}{2} &= \{-1\,|\,0\} = \{-1\,|\,0,1\}\\
0 &= \{\,|\,\} = \{-1\,|\,\} = \{\,|\,1\} = \{-1\,|\,1\}\\
\tfrac{1}{2} &= \{0\,|\,1\} = \{-1,0\,|\,1\}\\
1 &= \{0\,|\,\} = \{-1,0\,|\,\}\\
2 &= \{1\,|\,\} = \{0,1\,|\,\} = \{-1,1\,|\,\} = \{-1,0,1\,|\,\}.
\end{aligned}$$

Clearly we need some way of automating our expectations. Let us ask when the number $X = \{y, x^L\,|\,x^R\}$ obtained by adding a new entry y to the left of x is still equal to x.

Is $X \geqslant x$? Yes, unless some $X^R \leqslant x$ (no, since every X^R is an x^R) or $X \leqslant$ some x^L (no, since every x^L is an X^L).

Is $x \geqslant X$? Yes, unless some $x^R \leqslant X$ (no, since every x^R is an X^R) or $x \leqslant$ some X^L (and so $x \leqslant y$, since every other X^L is an x^L). We conclude that provided $y \ngeqslant x$, we can add y to the left of x in this way without affecting

x. This justifies all the equalities in the table. (We allow also, of course, y to be inserted on the right if $y \nleqslant x$.)

[In the case $\{-1\,|\,1\}$ we need to use the process twice. Thus since $-1 \ngeqslant 0 = \{\,|\,\}$, we have $0 = \{-1\,|\,\}$. Then since $1 \nleqslant 0 = \{-1\,|\,\}$, we have $0 = \{-1\,|\,1\}$.]

It is not hard to check the inequalities

$$-2 < -1 < -\tfrac{1}{2} < 0 < \tfrac{1}{2} < 1 < 2,$$

which shows that at least these numbers have the right order properties. What else do we require to justify their names?

According to the definition

$$1 + 1 = \{0 + 1, 1 + 0\,|\,\},$$

since 0 is the only 1^L, and there is no 1^R. So provided $0 + 1$ and $1 + 0$ behave as expected, we have $1 + 1 = 2$, as we might hope. But provided $x^L + 0 = x^L$ and $x^R + 0 = x^R$, we have

$$x + 0 = \{x^L + 0\,|\,x^R + 0\} = \{x^L\,|\,x^R\} = x,$$

and similarly $0 + x = x$. Since we already know $0 + 0 = 0$, this shows that $1 + 0 = 0 + 1 = 1$, as we wanted for the proof of $1 + 1 = 2$, but in fact it gives us an inductive proof that $x + 0 = 0 + x = x$ for all x.

It is much harder to show that $\frac{1}{2} + \frac{1}{2} = 1$, justifying the name of $\frac{1}{2}$. From the definition (supposing that $x + y = y + x$ for all x, y, which is quite easy to prove inductively) we see that

$$\tfrac{1}{2} + \tfrac{1}{2} = \{\tfrac{1}{2}\,|\,1\tfrac{1}{2}\},$$

where we are using $1\frac{1}{2}$ as a temporary name for $1 + \frac{1}{2}$.

Is $\frac{1}{2} + \frac{1}{2} \geqslant 1$? Yes, unless $1\frac{1}{2} \leqslant 1$ or $\frac{1}{2} + \frac{1}{2} \leqslant 0$. Oh my, we have to do these first. Let's get on with it.

Is $1 \geqslant 1\frac{1}{2}$? Yes, unless (empty) or $1 \leqslant$ some $1\frac{1}{2}^L$. But one of the $(1 + \frac{1}{2})^L$ is $1 + 0 = 1$, so $1 \ngeqslant 1\frac{1}{2}$.

Is $0 \geqslant \frac{1}{2} + \frac{1}{2}$? Yes, unless (empty) or $0 \leqslant$ some $(\frac{1}{2} + \frac{1}{2})^L$. But since $0 \leqslant \frac{1}{2} + 0$, we have $0 \ngeqslant \frac{1}{2} + \frac{1}{2}$. So (at last) $\frac{1}{2} + \frac{1}{2} \geqslant 1$.

Now is the time to leave the question

"is $1 \geqslant \frac{1}{2} + \frac{1}{2}$?"

to the reader. He should conclude that indeed $\frac{1}{2} + \frac{1}{2} = 1$.

In most of our examples x^L and x^R have been fairly close to each other, so that there was an obvious candidate for $\{x^L\,|\,x^R\}$. When they are far apart, there will be many simple numbers in between—which one of these will $\{x^L\,|\,x^R\}$ be? We consider $x = \{-1\,|\,2\}$.

Is $x \geqslant 0$? Yes, unless $2 \leqslant 0$ (false) or $x \leqslant$ some 0^L (false). So in this case we have $x \geqslant 0$.

Is $0 \geqslant x$? Yes, unless some $0^R \leqslant x$ (false) or $0 \leqslant -1$ (false). So in fact $x = 0$.

More generally, the argument proves that if every $x^L < 0$ and every $x^R > 0$, then $x = 0$, so for instance $\{-1, -\frac{1}{2} \mid 2, 3\} = 0$.

But when we have defined $2\frac{1}{2}$ and 17 we shall have to decide about $\{2\frac{1}{2} \mid 17\}$. A first guess might be their mean, $9\frac{3}{4}$, but since we have just seen that the mean rule does not always hold, this seems unlikely. A clue is given in the form of the preceding argument—since we must ask the questions "is $x = y$?" for the various possible y in order of simplicity, the answer should be the *simplest* y that is not prohibited. This rule will be established in Chapter 2, and it implies, for instance, that $\{2\frac{1}{2} \mid 17\} = 3$, and $\{\frac{1}{4} \mid 1\} = \frac{1}{2}$.

The numbers $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{2}$, 3, and so on

Once we have settled all the trivialities like $x \geqslant x$ for all x (which we have begun to take for granted), we can proceed a little faster. For instance, if L and R are sets of numbers chosen from those we already have, then since we suspect these numbers are totally ordered, in any expression $x = \{x^L \mid x^R\}$ we need only consider the greatest x^L (if any) and the least x^R (ditto). This gives us for the next "day" only the numbers

$$0 < \{0 \mid \tfrac{1}{2}\} < \tfrac{1}{2} < \{\tfrac{1}{2} \mid 1\} < 1 < \{1 \mid 2\} < 2 < \{2 \mid \}$$

and their negatives. What are the proper names for these numbers? We suspect that $\{2 \mid \} = 3$, and indeed we can verify that

$$1 + 1 + 1 = \{0 + 1 + 1, 1 + 0 + 1, 1 + 1 + 0 \mid \} = \{2 \mid \}.$$

The equation $\{1 \mid 2\} = 1\frac{1}{2}$ is almost as easy to guess and verify. So we shall make $1\frac{1}{2}$ a permanent name for this number.

The two likely guesses for $\{0 \mid \frac{1}{2}\}$ are $\frac{1}{3}$ and $\frac{1}{4}$. If anything, the first might seem the better guess, since otherwise it's hard to see what $\frac{1}{3}$ will be. But in fact it turns out that $\{0 \mid \frac{1}{2}\}$ is $\frac{1}{4}$—at least we can verify that twice this number is $\frac{1}{2}$. In a similar way, $\{\frac{1}{2} \mid 1\}$ turns out to be $\frac{3}{4}$ rather than $\frac{2}{3}$.

It is now easy to guess the pattern for the numbers which take only finitely much work to define. Let us imagine the numbers created on successive "days", in such a way that on day number n we create all numbers $x = \{L \mid R\}$ for which every member of each of the two sets L, R has already been constructed. We number the day on which 0 was created with the number 0 itself, so that our creation story begins (or began?) on *the zeroth day*.

Then the numbers seem to form a tree, as shown in Fig. 0. Each node of the tree has two "children", namely the first later numbers born just to the left

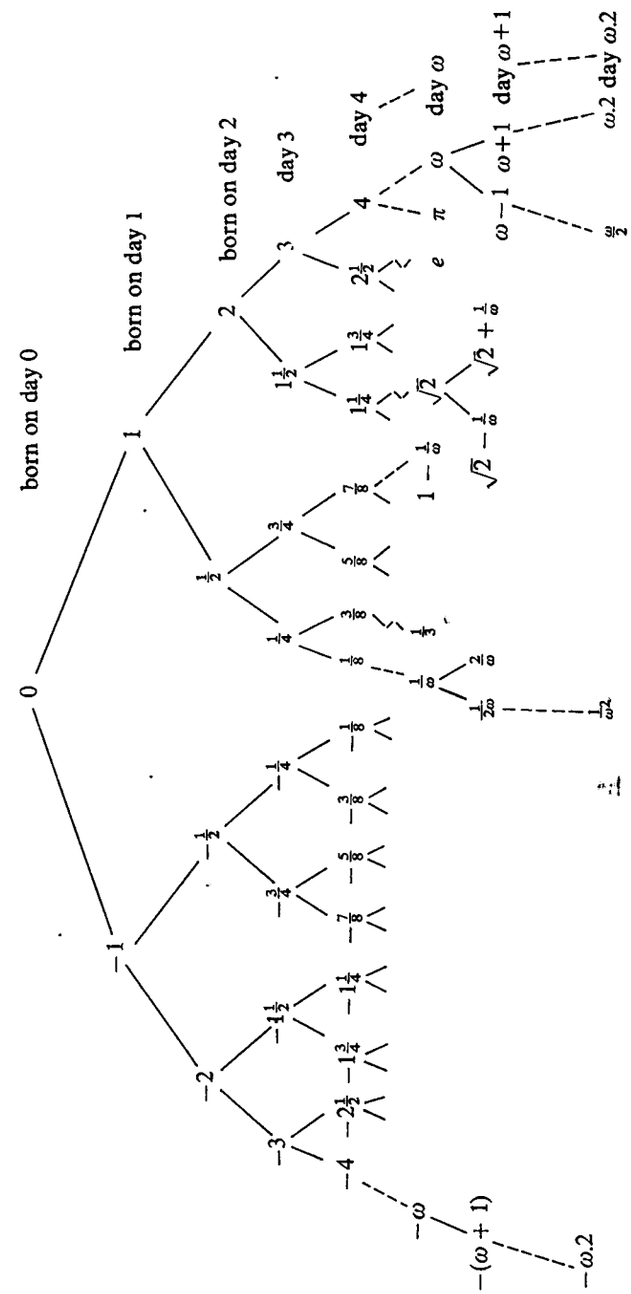


FIG. 0. When the first few numbers were born.

and right of it. We guess that on the n'th day the extreme numbers to be born are n and $-n$, and that each other number is the arithmetic mean of the numbers to the left and right of it. Happily, of course, this turns out to be the case. Supposing all this, we know all numbers born on finite days.

The numbers born on day ω

Of course the process doesn't stop with these numbers. The next day we call day ω. Let's consider some of the numbers born then. The largest number is the number ω itself, defined as $\{0, 1, 2, 3, \ldots \mid \}$. Of course, ω has many other forms, for instance $\omega = \{1, 2, 4, 8, 16, \ldots \mid \}$, or even $\omega = \{\text{all numbers } (m/2^n) \mid \}$. But since the collection of numbers to the left of ω has no largest member in these expressions, we cannot simply eliminate all but one of the numbers appearing on the left.

Of course the most negative number born on day ω will be

$$-\omega = \{ \mid 0, -1, -2, -3, \ldots\}.$$

The smallest positive number born on this day is the number $\{0 \mid 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, which turns out to be $1/\omega$, surprisingly and fortunately.

But besides these strange new numbers, some quite ordinary numbers are born at the same time. For instance, we have

$$\tfrac{1}{4} < \tfrac{1}{4} + \tfrac{1}{16} < \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} < \ldots < \tfrac{1}{3} < \ldots < \tfrac{1}{2} - \tfrac{1}{8} < \tfrac{1}{2},$$

so we might expect the number

$$\{\tfrac{1}{4}, \tfrac{1}{4} + \tfrac{1}{16}, \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64}, \ldots \mid \tfrac{1}{2}, \tfrac{1}{2} - \tfrac{1}{8}, \ldots\} = x, \text{ say}$$

to be $\frac{1}{3}$, and behold, it can in fact be proved that $x + x + x = 1$! In a similar way, all of the real numbers defined by Dedekind, including in particular all the remaining rational numbers can be defined as "Dedekind sections" of the dyadic rational numbers (by which we mean the numbers of the form $m/2^n$, m and n integers), rather than as sections of *all* rationals. So $\sqrt{2}$, e, and π are all born on day ω.

It is rather nice that our definition of equality ensures automatically that the number (for example)

$$\{\text{dyadic rationals} < \tfrac{3}{8} \mid \text{dyadic rationals} > \tfrac{3}{8}\}$$

turns out to be the same as the number $\frac{3}{8} = \{\frac{1}{4} \mid \frac{1}{2}\}$, so that the dyadic rationals "recreated" on day ω are "the same" as those created before.

It is also rather nice that Cantor's ordinal numbers (as modified by von Neumann) fit smoothly into our system. Thus we have

$$0 = \{ \mid \}, \quad 1 = \{0 \mid \}, \quad 2 = \{0, 1 \mid \}, \ldots, \quad \omega = \{0, 1, 2, 3, \ldots \mid \}, \quad \alpha = \{\beta < \alpha \mid \},$$

where von Neumann has

$$0 = \{\}, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \ldots, \quad \omega = \{0, 1, 2, \ldots\}, \quad \alpha = \{\beta < \alpha\}, \ldots.$$

In other words, the ordinal numbers are those we obtain by requiring always that the set R be empty. We may say that Cantor was only interested in moving ever rightwards, whereas Dedekind stopped to fill in the gaps, so that R was *always* empty for Cantor, *never* empty for Dedekind. It is remarkable that by dropping these restrictions we obtain a theory that is both more general and more easy to work with. (Compare the theory developed in the next chapter with the classical foundation for the real numbers in which we must first construct or postulate the "natural numbers", then rationals as equivalence classes of ordered pairs, then reals as sections of rationals, with negative numbers being introduced at some stage in the process.)

Some more numbers

After ω, the number $\{0, 1, 2, 3, \ldots, \omega \mid \} = \omega + 1$ need come as no surprise, but perhaps the number $\{0, 1, 2, 3, \ldots \mid \omega\}$ will. This number, call it x, should satisfy $n < x < \omega$ for all finite integers n, in other words, x should be an infinite number less than the "least" infinite number ω. Adding 1 to x, we find the number

$$\{1, 2, 3, \ldots, x \mid \omega + 1\} = y, \text{ say.}$$

Here, since $x < \omega$, and $\omega + 1 \nleq \omega$, we see that $y = \omega$, for the new entries x on the left and $\omega + 1$ on the right have made no difference. So $x + 1 = \omega$, $x = \omega - 1$.

Check that we get the same result on subtracting 1 from ω.

In a similar way, we find successively that

$$\omega - 2 = \{0, 1, 2, 3, \ldots \mid \omega, \omega - 1\}, \ldots,$$
$$\omega - n = \{0, 1, 2, 3, \ldots \mid \omega, \omega - 1, \omega - 2, \ldots, \omega - (n - 1)\}.$$

Plainly the next number to consider is

$$z = \{0, 1, 2, 3, \ldots \mid \omega, \omega - 1, \omega - 2, \ldots\} = \{n \mid \omega - n\}, \text{ say.}$$

It should not take the reader too long to verify that $z = \omega/2$. When he has done this, and defined $\omega/4$, $\omega/8, \ldots$ as well, he should be in a position to define $\omega/3$ (for instance), and to verify our assertion that

$$\{0, 1, 2, 3, \ldots \mid \omega, \omega/2, \omega/4, \omega/8, \ldots\}$$

is a square root of ω.

Other easy exercises are

$$\left\{0 \,\middle|\, \frac{1}{\omega}\right\} = \frac{1}{2\omega}, \qquad \left\{\frac{1}{\omega} \,\middle|\, 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots\right\} = \frac{2}{\omega}, \qquad \left\{0 \,\middle|\, \frac{1}{\omega}, \frac{1}{2\omega}, \frac{1}{4\omega}, \ldots\right\} = \frac{1}{\omega^2},$$

and so on.

If the reader prefers to try his hand at "constructing" new numbers rather than examining values of those given here, let him try to find definitions for $\sqrt[3]{\omega}$, $\omega^{1/\omega}$, $\omega + \pi$, $(\omega + 1)^{-1}$, $\sqrt{(\omega - 1)}$, and to show, making any reasonable assumptions, that they have the properties we should expect.

In the next chapter, we shall prove that the Class of all numbers really is a Field, making no use of any of the supposed "facts" from this chapter. It will be some time before we see so many particular numbers mentioned again. In the third chapter, we shall produce a "canonical form" for numbers, and learn how to manipulate them a little more freely, and in the process will see exactly how general our class of numbers turns out to be.

CHAPTER 1

The Class No is a Field

Ah! why, ye Gods, should two and two make four?

Alexander Pope, "The Dunciad"

PRELIMINARY COMMENTS

There are two problems that arise in the precise treatment which need special comment. The first is that it is necessary to have an expression $\{L \mid R\}$ *existing* even before we have proved that it is a number. The second concerns the fact that equality is a defined relation, which must initially be distinguished from identity.

Games. The construction for numbers generalises immediately to the following construction for what we call *games*.

Construction. If L and R are any two sets of games, then there is a game $\{L \mid R\}$. All games are constructed in this way.

Although games are properly the subject of the first part of this book (where the name will be justified), it is logically necessary to introduce them before numbers. Order-relations and arithmetic operations on games are defined by the same definitions as for numbers. The most important distinction between numbers and general games is that numbers are totally ordered, but games are not—there exist games x and y for which we have neither of $x \geqslant y$, $y \geqslant x$.

To show that a game $x = \{x^L \mid x^R\}$ is a number, we must show *firstly* that all of the games x^L, x^R are numbers, and *secondly*, that there is no inequality of the form $x^L \geqslant x^R$.

IDENTITY AND EQUALITY

We shall call games x and y *identical* ($x \equiv y$) if their left and right sets are identical—that is, if every x^L is identical to some y^L, every x^R identical to

some y^R, and vice versa. Recall that x and y are defined to be *equal* ($x = y$) if and only if we have both $x \geqslant y$ and $y \geqslant x$. The distinction causes no great problems until we come to multiplication, where the trouble is that there can exist equal games x and y for which xz and yz are unequal. But all goes well as long as we restrict ourselves to the multiplication of numbers.

Finally, we note that almost all our proofs are inductive, so that, for instance, in proving something about the pair (x, y) we can suppose that thing already known about all pairs (x^L, y), (x^R, y), (x, y^L), (x, y^R). After a time we feel free to suppress all references to these inductive hypotheses. We remind the reader again that since ultimately we are reduced to questions about members of the empty set, no one of our inductions will require a "basis". The games x^L, x^R will be called the Left, Right *options* of x.

PROPERTIES OF ORDER AND EQUALITY

Recall that $x \geqslant y$ if we have no inequality of form $x^R \leqslant y$ or $x \leqslant y^L$.

THEOREM 0. *For all games x we have*

(i) $x \ngeqslant x^R$,
(ii) $x^L \ngeqslant x$,
(iii) $x \geqslant x$,
(iv) $x = x$.

Proof. (i) Taking y as x^R in the definition of $\geqslant$, and using the inductively true relation $x^R \leqslant x^R$, we see that we cannot have $x \geqslant y$.

(ii) is similar.

(iii) Taking y as x, we now know that we have no $x^R \leqslant y$ and $x \leqslant$ no y^L, whence $x \geqslant y$.

(iv) from $x \geqslant x$ and $x \leqslant x$, we deduce $x = x$.

THEOREM 1. *If $x \geqslant y$ and $y \geqslant z$, then $x \geqslant z$.*

Proof. Since $x \geqslant y$, we cannot have $x^R \leqslant y$, and so by induction we cannot have $x^R \leqslant z$. Similarly we cannot have $x \leqslant z^L$, and so we must have $x \geqslant z$.

Summary. We now know that $\geqslant$ is a partial order relation on games, and that $=$ has the right properties (for instance $x = y$ and $x < z$ imply $y < z$).

THEOREM 2. *For any number x we have $x^L < x < x^R$ for all x^L, x^R. Also, for any two numbers x and y we must have $x \leqslant y$ or $x \geqslant y$.*

Proof. (i) Since we know $x \ngeqslant x^R$, it suffices to prove $x^R \geqslant x$. This will be true unless some $x^{RR} \leqslant x$ or $x^R \leqslant$ some x^L. But the former inductively implies $x^R < x^{RR} \leqslant x$, a contradiction, and the latter is prohibited by the definition of number.

(ii) The inequality $x \ngeqslant y$ implies either some $x^R \leqslant y$ or $x \leqslant$ some y^L, whence either $x < x^R \leqslant y$ or $x \leqslant y^L < y$.

Summary. Numbers are totally ordered.

PROPERTIES OF ADDITION

Definition. $0 = \{\,|\,\}$.
We recall that $x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$.

THEOREM 3. *For all x, y, z we have*

$$x + 0 \equiv x, \qquad x + y \equiv y + x, \qquad (x + y) + z \equiv x + (y + z).$$

Proof.

$$x + 0 \equiv \{x^L + 0 \mid x^R + 0\} \equiv \{x^L \mid x^R\} \equiv x$$

$$x + y \equiv \{x^L + y, x + y^L \mid x^R + y, x + y^R\} \equiv$$
$$\equiv \{y + x^L, y^L + x \mid y + x^R, y^R + x\} \equiv y + x.$$

$$(x + y) + z \equiv \{(x + y)^L + z, (x + y) + z^L \mid \ldots\} \equiv$$
$$\equiv \{(x^L + y) + z, (x + y^L) + z, (x + y) + z^L \mid \ldots\} \equiv$$
$$\equiv \{x^L + (y + z), x + (y^L + z), x + (y + z^L) \mid \ldots\} \equiv$$
$$\equiv \ldots \equiv x + (y + z).$$

In each case the middle identity follows from the inductive hypothesis. Proofs like these we call *1-line proofs* even when as here the "line" is too long for our page. We shall meet still longer 1-line proofs later on, but they do not get harder—one simply transforms the left-hand side through the definitions and inductive hypotheses until one gets the right hand side.

Summary. Addition is a commutative Semigroup operation with 0 as zero, even when we demand identity rather than equality.

PROPERTIES OF NEGATION

Recall the definition $-x = \{-x^R \mid -x^L\}$.

THEOREM 4. (i) $-(x + y) \equiv -x + -y$
(ii) $-(-x) \equiv x$
(iii) $x + -x = 0$

Proof. (i) and (ii) have easy 1-line proofs. Note that (iii) is an equality rather than an identity. If, say, $x + -x \not\geqslant 0$, we should have some $(x + -x)^R \leqslant 0$, that is, $x^R + -x \leqslant 0$ or $x + -x^L \leqslant 0$. But these are false, since we have by induction $x^R + -x^R \geqslant 0$, $x^L + -x^L \geqslant 0$.

Summary. With equality rather than identity, addition is a commutative Group operation, with 0 for zero, and $-x$ for the negative of x. All this is true for general games.

PROPERTIES OF ADDITION AND ORDER

THEOREM 5. *We have $y \geqslant z$ iff $x + y \geqslant x + z$.*

Proof. If $x + y \geqslant x + z$, we cannot have

$$x + y^R \leqslant x + z \text{ or } x + y \leqslant x + z^L,$$

and so by induction we cannot have $y^R \leqslant z$ or $y \leqslant z^L$, so that $y \geqslant z$.

Now supposing $x + y \not\geqslant x + z$, we must have one of

$$x^R + y \leqslant x + z, \quad x + y^R \leqslant x + z, \quad x + y \leqslant x^L + z, \quad x + y \leqslant x + z^L,$$

and if we further suppose $y \geqslant z$, we deduce one of

$$x^R + y \leqslant x + y, \quad x + y^R \leqslant x + y, \quad x + z \leqslant x^L + z, \quad x + z \leqslant x + z^L,$$

all of which imply contradictions by cancellation.

Theorem 5 implies in particular that we have $y = z$ iff $x + y = x + z$, justifying replacement by equals in addition.

THEOREM 6. (i) 0 *is a number,*
(ii) *if x is a number, so is $-x$,*
(iii) *if x and y are numbers, so is $x + y$.*

Proofs. (i) we cannot have $0^L \geqslant 0^R$, since there exists neither a 0^L nor a 0^R.
(ii) From $x^L < x < x^R$ and x^L, x^R numbers, we inductively deduce $-x^R < -x < -x^L$ and $-x^R, -x^L$ numbers.
(iii) We deduce inductively that each of

$$x^L + y, x + y^L < x + y < \text{each of } x^R + y, x + y^R,$$

all of $x^L + y$, etc., being numbers.

Summary. Numbers form a totally ordered Group under addition.

PROPERTIES OF MULTIPLICATION

Definition. $1 = \{0 \mid \}$

We recall the definition of multiplication

$$xy = \{x^Ly + xy^L - x^Ly^L, \quad x^Ry + xy^R - x^Ry^R \mid$$
$$\mid x^Ly + xy^R - x^Ly^R, x^Ry + xy^L - x^Ry^L\}.$$

THEOREM 7. *For all x, y, z we have the identities*

$$x0 \equiv 0, \quad x1 \equiv x, \quad xy \equiv yx, \quad (-x)y \equiv x(-y) \equiv -xy,$$

and the equalities

$$(x + y)z = xz + yz, \quad (xy)z = x(yz).$$

Proof. The identities have easy 1-line proofs. The equalities also have 1-line proofs, as follows:

$$\begin{aligned}(x + y)z &\equiv \{(x + y)^Lz + (x + y)z^L - (x + y)^Lz^L, \ldots \mid \ldots\} \equiv \\ &\equiv \{(x^L + y)z + (x + y)z^L - (x^L + y)z^L, \\ &\quad (x + y^L)z + (x + y)z^L - (x + y^L)z^L, \ldots \mid \ldots\} = \\ &= \{(x^Lz + xz^L - x^Lz^L) + yz, \; xz + (y^Lz + yz^L - y^Lz^L), \ldots \mid \ldots\} \\ &\equiv xz + yz.\end{aligned}$$

[This fails to yield an identity since the law $x + -x = 0$ is invoked.]

The central expression for xyz has four expressions like

$$x^Lyz + xy^Lz + xyz^L - x^Ly^Lz - x^Lyz^L - xy^Lz^L + x^Ly^Lz^L$$

(with perhaps some even number of x^L, y^L, z^L replaced by x^R, y^R, z^R) on the left, and four similar expressions (with an odd number of such replacements) on the right.

Note. We now have the more illuminating form

$$\{xy - (x - x^L)(y - y^L), \quad xy - (x^R - x)(y^R - y) \mid$$
$$\mid xy + (x - x^L)(y^R - y), \quad xy + (x^R - x)(y - y^L)\}$$

for the product xy.

THEOREM 8. (i) *If x and y are numbers, so is xy*
(ii) *If $x_1 = x_2$, then $x_1y = x_2y$*
(iii) *If $x_1 \leqslant x_2$, and $y_1 \leqslant y_2$, then $x_1y_2 + x_2y_1 \leqslant x_1y_1 + x_2y_2$, the conclusion being strict if both the premises are.*

Proof. We shall refer to the inequality of (iii) as $P(x_1, x_2 : y_1, y_2)$. Note that if $x_1 \leqslant x_2 \leqslant x_3$, then we can deduce $P(x_1, x_3 : y_1, y_2)$ from the inequalities $P(x_1, x_2 : y_1, y_2)$ and $P(x_2, x_3 : y_1, y_2)$ by adding these and cancelling common terms from the two sides.

Now to prove (i), we observe first that inductively, all options of xy are numbers, so that we have only to prove a number of inequalities like

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_2}y + xy^R - x^{L_2}y^R.$$

But if $x^{L_1} \leqslant x^{L_2}$ we have

$$x^{L_1}y + xy^L - x^{L_1}y^L \leqslant x^{L_2}y + xy^L - x^{L_2}y^L < x^{L_2}y + xy^R - x^{L_2}y^R$$

(these two inequalities reducing respectively to $P(x^{L_1}, x : y^L, y)$ and $P(x^{L_2}, x : y^L, y^R)$), while if $x^{L_2} \leqslant x^{L_1}$ we have instead

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_1}y + xy^R - x^{L_1}y^R \leqslant x^{L_2}y + xy^R - x^{L_2}y^R.$$

(these being $P(x^{L_1}, x : y^L, y^R)$ and $P(x^{L_2}, x^{L_1} : y, y^R)$).

Now to prove (ii). This implication follows immediately from the fact that every Left option of either is strictly less than the other, and every Right option strictly greater, the relevant inequalities all being easy.

If $x_1 = x_2$ or $y_1 = y_2$ we can use (ii) to show that the terms on the Left of (iii) are equal to those on the Right.

So we need only consider the case $x_1 < x_2$, $y_1 < y_2$. Since $x_1 < x_2$, we have either $x_1 < x_1^R \leqslant x_2$ or $x_1 \leqslant x_2^L < x_2$, say the former. But then $P(x_1, x_2 : y_1, y_2)$ can be deduced from $P(x_1, x_1^R : y_1, y_2)$ and $P(x_1^R, x_2 : y_1, y_2)$, of which the latter is strictly simpler than the original. A similar argument now reduces our problem to proving strict inequalities of the four forms

$$P(x^L, x : y^L, y), \quad P(x^L, y : y, y^R), \quad P(x, x^R : y^L, y), \quad \text{and} \quad P(x, x^R : y, y^R)$$

which merely assert that xy has the right order relations with its options.

THEOREM 9. *If x and y are positive numbers, so is xy.*

Proof. This follows from $P(0, x{:}0, y)$.

Summary. Numbers form a totally ordered Ring. Note that in view of Theorem 8 and the distributive law, we can assert, for example, that $x \geqslant 0$, $y \geqslant z$ together imply $xy \geqslant xz$, and that if $x \neq 0$, we can deduce $y = z$ from $xy = xz$.

PROPERTIES OF DIVISION

We have just shown that if there is any number y such that $xy \doteq t$, then y is uniquely determined by x and t provided that $x \neq 0$. We must now show how to produce such a y. It suffices to show that for positive x there is a number y such that $xy = 1$. We first put x into a sort of standard form.

LEMMA. *Each positive x has a form in which 0 is one of the x^L, and every other x^L is positive.*

Proof. Let y be obtained from x by inserting 0 as a new Left option, deleting all negative Left options. Then it is easy to check that y is a number, and that $y = x$.

We write $x = \{0, x^L \mid x^R\}$ in this section, and restrict use of the symbol x^L to the positive Left options of x. (Note that all the x^R are automatically positive.)

Now we shall define a number y, explain the definition, and prove that y is a number and that $xy = 1$.

Definition

$$y = \left\{0, \frac{1 + (x^R - x)y^L}{x^R}, \frac{1 + (x^L - x)y^R}{x^L} \,\middle|\, \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R}\right\}$$

Note that expressions involving y^L and y^R appear in the definition of y. It is this that requires us to "explain" the definition. The explanation is that we regard these parts of the definition as defining new options for y in terms of old ones. So even the definition of this y is an inductive one.† [This is in addition to the "other" induction by which we suppose that inverses for the x^L and x^R have already been found.]

THEOREM 10. *We have* (i) $xy^L < 1 < xy^R$ *for all* y^L, y^R.
(ii) *y is a number.*
(iii) $(xy)^L < 1 < (xy)^R$ *for all* $(xy)^L, (xy)^R$.
(iv) $xy = 1$.

Proof. We observe that the options of y are defined by formulae of the form

$$y'' = \frac{1 + (x' - x)y'}{x'}$$

where y' is an "earlier" option of y, and x' some non-zero option of x. This formula can be written

$$1 - xy'' = (1 - xy')\frac{x' - x}{x'}$$

which shows that y'' satisfies (i) if y' does. Plainly 0 does. Part (ii) now follows,

† To see how the definition works, take $x = \{0, 2 \mid \} = 3$. Then there is no x^R and the only x^L is 2, so $x^L - x = -1$ and the formula for y becomes $y = \{0, \frac{1}{2}(1 - y^R) \mid \frac{1}{2}(1 - y^L)\}$. The initial value $y^L = 0$ gives us $\frac{1}{2}(1 - 0) = \frac{1}{2}$ for a new y^R, whence $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$ as a y^L, then $\frac{1}{2}(1 - \frac{1}{4}) = \frac{3}{8}$ for a y^R, and so on, yielding $y = \{0, \frac{1}{4}, \frac{5}{16}, \ldots \mid \frac{1}{2}, \frac{3}{8}, \ldots\}$, which certainly looks like $\frac{1}{3}$.

since we cannot have any inequality $y^L \geqslant y^R$. The typical form of an option of xy is $x'y + xy' - x'y'$, which can be written as $1 + x'(y - y'')$ with the above definition of y'', and this suffices to prove (iii). For (iv), we observe first that $z = xy$ has a left option 0 (take $x^L = y^L = 0$), and that (iii) asserts that $z^L < 1 < z^R$ for all z^L, z^R. Then

$z \geqslant 1$, since no $z^R \leqslant 1$, and $z \leqslant$ no 1^L (since some $z^L = 0$), and also

$1 \geqslant z$, since no $1^R \leqslant z$, and $1 \leqslant$ no z^L,

so that indeed $z = 1$.

Summary. The Class **No** of all numbers forms a totally ordered Field.

Clive Bach has found a similar definition for the square root of a non-negative number x. He defines

$$\sqrt{x} = y = \left\{\sqrt{x^L}, \frac{x + y^L y^R}{y^L + y^R} \,\middle|\, \sqrt{x^R}, \frac{x + y^L y^{L*}}{y^L + y^{L*}}, \frac{x + y^R y^{R*}}{y^R + y^{R*}}\right\}$$

where x^L and x^R are non-negative options of x, and y^L, y^{L*}, y^R, y^{R*} are options of y chosen so that no one of the three denominators is zero. We shall leave to the reader the easy inductive proof that this is correct.

Martin Kruskal has pointed out that the options of $1/x$ can be written in the form

$$\frac{1 - \Pi\left(1 - \dfrac{x}{x_i}\right)}{x}$$

where the denominator x cancels formally, the x_i denote positive options of x, and the product may be empty. This is a Left option of $1/x$ just when an *even* number of the x_i are Left options of x. There is a similar closed form for Bach's definition of $\sqrt{x}$.

CHAPTER 2

The Real and Ordinal Numbers

Don't let us make imaginary evils, when you know we have so many real ones to encounter.

Oliver Goldsmith, "The Good-Natured Man"

The following theorem gives us a very easy way of evaluating particular numbers. We call it *the simplicity theorem.*

THEOREM 11. *Suppose for* $x = \{x^L \mid x^R\}$ *that some number* z *satisfies* $x^L \ngeqslant z \ngeqslant x^R$ *for all* x^L, x^R, *but that no option of* z *satisfies the same condition. Then* $x = z$.

[Note; this holds even when x is only given to be a game.]

Proof. We have

$$x \geqslant z \text{ unless some } x^R \leqslant z \text{ (no!) or } x \leqslant \text{ some } z^L.$$

But from $x \leqslant z^L$, we can deduce $x^L \ngeqslant x \leqslant z^L < z \ngeqslant x^R$ for all x^L, x^R, from which we have $x^L \ngeqslant z^L \ngeqslant x^R$, contradicting the supposition about z. So $x \geqslant z$, similarly $z \geqslant x$, and so $x = z$.

The main assertion of the theorem is that when x is given as a number, it is always the *simplest* number lying between the x^L and the x^R, where *simplest* means *earliest created.* [For if z is this simplest number, the simpler numbers z^L, z^R cannot satisfy the same condition.] But the exact version presented above has several advantages, since it holds when x is given as a game not necessarily known to equal a number, and it is perhaps not quite obvious exactly what is meant by "the simplest number such that . . .". In the applications below, there is never any problem.

THEOREM 12. *If* x *is a rational number whose denominator divides* 2^n, *then* $x = \{x - (1/2^n) \mid x + (1/2^n)\}$.

Proof. For $n = 0$ the theorem holds, since it asserts that x is the simplest