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Problem 1

We will prove the following by induction for $n \geq 0$:

$$\sum_{i=1}^{n} \sinh^{-1}\left(\frac{1}{\sqrt{2^{i+2}+2}+\sqrt{2^{i+1}+2}}\right) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+1}}}\right)$$

For $n = 0$, clearly $\sum_{i=1}^{n} \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) = 0$, while $\sinh^{-1}\left(\frac{1}{\sqrt{2^{n+1}}}\right) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right)$, so the statement is true.

Now assume the statement is true for $n-1$. This means that

$$\begin{aligned}
&\sum_{i=1}^{n} \sinh^{-1}\left(\frac{1}{\sqrt{2^{i+2}+2}+\sqrt{2^{i+1}+2}}\right) \\
&= \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) + \sum_{i=1}^{n-1} \sinh^{-1}\left(\frac{1}{\sqrt{2^{i+2}+2}+\sqrt{2^{i+1}+2}}\right) \\
&= \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) + \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2^{n}}}\right)
\end{aligned}$$

However, since $\sinh^{-1}(u) - \sinh^{-1}(v) = \sinh^{-1}\left(u\sqrt{1+v^2} - v\sqrt{1+u^2}\right)$, we have:

$$\begin{aligned}
\sinh^{-1}\left(\frac{1}{\sqrt{2^{n}}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+1}}}\right) &= \sinh^{-1}\left(\frac{1}{\sqrt{2^{n}}}\sqrt{1+\frac{1}{2^{n+1}}} - \frac{1}{\sqrt{2^{n+1}}}\sqrt{1+\frac{1}{2^{n}}}\right) \\
&= \sinh^{-1}\left(\frac{1}{\sqrt{2^{2n+1}}}\sqrt{1+2^{n+1}} - \frac{1}{\sqrt{2^{2n+1}}}\sqrt{1+2^{n}}\right) \\
&= \sinh^{-1}\left(\frac{\sqrt{1+2^{n+1}} - \sqrt{1+2^{n}}}{\sqrt{2^{2n+1}}}\right) \\
&= \sinh^{-1}\left(\frac{2^{n+1}-2^{n}}{2^{n}\sqrt{2}(\sqrt{1+2^{n+1}}+\sqrt{1+2^{n}})}\right) \\
&= \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2})}\right)
\end{aligned}$$

Using this, by induction we have that for all $n \geq 0$:

$$\sum_{i=1}^{n} \sinh^{-1}\left(\frac{1}{\sqrt{2^{i+2}+2}+\sqrt{2^{i+1}+2}}\right) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+1}}}\right)$$

Then, since $\lim_{n\to\infty} \frac{1}{\sqrt{2^{n+1}}} = 0$ and $\sinh^{-1}$ is a continuous function, this means that:

$$\sum_{i=1}^{\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2^{i+2}+2}+\sqrt{2^{i+1}+2}}\right) = \lim_{n\to\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+1}}}\right) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right)$$