A Summary of Enumerative and Algebraic Combinatorics

This article intruduces several vital methods for enumeration.

1. Enumeration

   Generally, the methods used for enumeration are the method of decompositions, refinements, recursions, generating functions, and so on.

2. Weight-Enumeration

   A weight-enumeration is a generalized enumeration of the cardinality of a set, which has a weight function on the base set A. Let $\alpha : A \to \mathbb{N}$ be the weight function on A. Then one can define its weighted generating function as the following.

   $f(x) := |A|_x := \sum_{a\in A} x^{\alpha(a)} = \sum_{n=0}^{\infty} a_n x^n$,

   where in the last term, a_n is the number of elements in A whose α equals n. From this point of view with some additional facts such as

   $|A \cup B|_x = |A|_x + |B|_x$, $|A \times B|_x = |A|_x \cdot |B|_x$,

   one can derive the explicit formular of $|A|_x$.

3. Enumeration Ansatzes

   This classifies the form of solutions for the generating functions according to the form of the sequence $\{a_n\}$.

4. Bijective Methods

   The nice way of proving that two sets have the same cardinality is to construct a bijection between the two sets. The existence of the bijection implies the equlity of the cardinalities of the two sets.

   This article introduces a classic example of the bijective methods; the proof of Euler's Odd=Distinct partition theorem.

5. Exponential Generating Functions

   As opposed to the fact that ordinary generating functions are suitable for counting ordered structures, exponential generating functions are useful for counting the number of the elements of the unordered structure, such as sets.

   In this section, the **Fundamental Theorem of Exponential Generating Functions** is given, which says that under a certain condition on two families A and B, $egf(B) = exp[egf(A)]$ is satisfied, followed by some interesting examples such as counting the number of set partitions and the number of permutations on n objects.

6. Polya-Redfield Enumeration

   Counting unlabeled objects is much harder than counting labeled objects. Polya's idea to count the unlabeled objects was to consider them as equivalence classes of labeled objects. This section is devoted to explain this method and provide a simple application.

7. The Principle of Inclusion-Exclusion and Mobius Inversion

   PIE is one of the most basic and essential methods in counting, and it is a special case of Mobius inversion on posets.