640-Experimental Math
10/09/16

Homework 9

3) We will prove that

$$baS(h,d) = \frac{(h+1-d)(h+d)!}{d!(h+1)!}$$

for all (h,d) with $0 \leq d, h \leq h+1$ by induction on $d+h$. Suppose that $d+h=0$. Then, both $d,h=0$. We verify that $baS(h,d)=1$ and $\frac{(h+1-d)(h+d)!}{d!(h+1)!} = \frac{1}{1} = 1$. Now, suppose that $d=0$. Then, $baS(h,d)=1$, and $\frac{(h+1-d)(h+d)!}{d!(h+1)!} = \frac{(h+1)(h)!}{(h+1)!} = 1$. Next, suppose that $d=h+1$. Then, $baS(h,d)=0$ and $\frac{(h+1-d)(h+d)!}{d!(h+1)!} = \frac{0\cdot(h+d)!}{d!(h+1)!} = 0$. Finally, we are ready for the induction step. Suppose that the desired equality holds for all (h,d) pairs with $d+h \leq n-1$, and choose (h,d) such that $h+d=n$, assuming that $0 < d < h+1$. Then, by the recurrence we found in class, $baS(h,d) = baS(h-1,d) + baS(h,d-1)$. By our assumptions, we have $d-1, h-1 \geq 0$ and $d \leq h-1+1$, so we can apply the induction hypothesis to find that

$$\begin{aligned} baS(h,d) &= \frac{(h-d)(h-1+d)!}{d!(h)!} + \frac{(h+2-d)(h+d-1)!}{(d-1)!(h+1)!} \\ &= \frac{(h+1)(h-d)(h-1+d)! + d(h+2-d)(h+d-1)!}{d!(h+1)!} \\ &= \frac{((h+1)(h-d) + d(h+2-d))(h-1+d)!}{d!(h+1)!} = \frac{(h+1-d)(h+d)(h-1+d)!}{d!(h+1)!} \\ &\frac{(h+1-d)(h+d)!}{d!(h+1)!}, \end{aligned}$$

completing the proof by induction.

4) We will show a bijection f from the set of ordered complete binary trees (hereafter called binary trees or just trees) on n leaves to the set of ballot sequences of length $n-1$ on $\{H, D\}$ where, on every prefix, there are at least as many H's as D's. Suppose T is a binary tree. We first prove that T has exactly $2n-1$ vertices. A binary tree must have an odd number of vertices, because there exists a bijection between left-children and the right-children of the same parent vertex, and this bijection leaves only the root un-paired. Therefore, it suffices to show that every binary tree on $2n-1$ vertices has n leaves. We proceed by induction. Certainly a 1-vertex tree has 1 leaf, so suppose that every $2n-3$-vertex tree has n leaves and let R be a tree on $2n-1$ vertices. There must be a vertex of R whose children are both leaves, so remove these two children to obtain R'; a tree on $2n-3$ vertices with on fewer leaf than R. By the induction hypothesis, R' has $n-1$ leaves, so R has n leaves and the claim is proved.

Now, let $V(T) = \{v_1, ...v_{2n-1}\}$ be the vertex set of T. Order the vertices by assigning each one a string of 0s and 1s by tracing a path from the root to that vertex - every time the path goes to a left child add a 0 to the end of the string and every time it goes to a right child add a 1 to the end of the string. Then, order the vertices in lexicographical order by their corresponding strings. Define $f(T)$ by looking at the v_i's in order (from v_1 to v_{2n-2}) and recording an H if v_i has children and a D if v_i does not have children. We claim that this gives a valid ballot sequence. By way of contradiction, suppose that at some point the prefix of the sequence contains more D's than H's; we can consider the first such point to ensure that there is exactly one more D than H, so choose k such that there are k D's and $k-1$ H's. Now, each D corresponds to a leaf, so there are k leaves and $2k-1$ overall vertices, so this tree is already finished; there are no more vertices that have children whose children have not already been assigned. Therefore, $k = n$, and, since we don't look at v_{2n-1}, we should not have added the last D to the ballot sequence.

Next it is time to define the inverse map g. Let B be a ballot sequence, and construct a tree as follows. Beginning with the root, mark it with the first letter of B, and then do the following for as long as possible: go to the next vertex marked with an H (where the vertices are ordered as in the previous paragraph), and give it two children marking the left one with the next unused letter of B and the right one with the letter of B following that. Since we start with one vertex, and each H adds two vertices which can marked, we will always have more vertices that can be marked than vertices which have been marked as long as the number of H's exceeds the number of D's. To complete the tree, we pretend that there is one final D at the end of the ballot sequence to mark the $(2n-1)$th vertex. For Maple implementations of f and g, refer to my txt file.