

Random Polygon

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An initial “random polygon” on the plane with n vertices is equivalent to the data of a vector $v \in \mathbb{C}^n$. The averaging process is equivalent to applying the matrix

$$A = \frac{1}{2}(I + T)$$

on v where I is the identity matrix and T is the standard shift matrix given as $T_{ij} = \delta_{i,j+1}$ where the indices are interpreted mod n . Therefore averaging N times simply amounts to taking the entries of $A^N v$ and plotting them on the complex plane.

A crucial role is played by the eigenvalues and eigenvectors of A . The diagonalization of A simply amounts to diagonalizing T which is well-known: One notes the matrix equation $T^n = 1$ which is actually the characteristic polynomial of T . Thus the eigenvalues satisfy $\omega^n = 1$, that is, they’re simply the n th roots of unity. The corresponding eigenvectors are also easy to write down. Let $\omega = e^{2\pi i/n}$. Then the (normalized) eigenvector corresponding to ω^k ($k = 0, \dots, n-1$) is simply

$$(e_k)_j = \frac{1}{\sqrt{n}} \omega^{kj}.$$

A will have the same eigenvectors e_k with corresponding eigenvalues given by

$$\lambda_k = \frac{1}{2}(1 + \omega^k) = e^{i\pi k/n} \cos\left(\frac{\pi k}{n}\right).$$

From this one can easily see that the largest eigenvalue is given by $k = 0$ when $\lambda_0 = 1$ and the eigenvector is simply given by $(e_0)_j = \frac{1}{\sqrt{n}}$. The others have magnitude strictly less than one with the second largest one corresponding to $k = \pm 1$. Now T being a unitary matrix means that the eigenvectors form an orthogonal basis for \mathbb{C}^n and so we can easily expand v in terms of this eigenbasis

$$v = \sum_{j=0}^{n-1} \langle e_j, v \rangle e_j$$

where we note that the j -th coefficient given by the inner product $\langle e_j, v \rangle$ is simply given by the discrete Fourier transform

$$x_j := \langle e_j, v \rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^n v_k e^{-\frac{2\pi i k j}{n}}.$$

The action of A^N now reads

$$A^N v = \sum_{j=0}^{n-1} x_j (\lambda_j)^N e_j := \sum_{j=0}^{n-1} (\lambda_j)^N \alpha_j$$

Let us now interpret each of the terms in decreasing absolute value of λ_j^N . First, α_0 is simply the vector with all entries $\frac{1}{n}(v_1 + \dots + v_n)$ which corresponds to the center of the initial random polygon. We can freely choose this to be zero. The subleading term i.e the term

$$E := (\lambda_1)^N \alpha_1 + (\lambda_{n-1})^N \alpha_{n-1}$$

is what will play the crucial role. Writing it out explicitly we have

$$E_m = \frac{1}{\sqrt{n}} \left(\cos\left(\frac{\pi}{n}\right) \right)^N \left(x_1 e^{\frac{i\pi N}{n}} e^{\frac{2\pi i m}{n}} + x_{n-1} e^{-\frac{i\pi N}{n}} e^{-\frac{2\pi i m}{n}} \right)$$

where x_1, x_{n-1} are components of the discrete Fourier transform as given above. The rest of the terms can be neglected for large N and so E is what should give us the sought-after ellipse. Indeed plotting this as in the program `E(L,n,N)` this traces out precisely the ellipse matching the program `IterPolygon1`. It should be possible to figure out the properties of the ellipse such as its eccentricity and orientation from the coefficients x_1 and x_{n-1} , which are the only remnants of the initial vector v , although I haven't had the time to do that.