SOLUTIONS to MATH 250 (1), Dr. Z., Exam 1, Thurs., Oct. 14, 2010, 8:40-10:00am, SEC 202

This Version of Dec. 7, 2010 [Thanks to Sarita Paul who won $2 for correcting a mistake in the answer of #5].

Previous Version (Dec. 6, 2010) [Thanks to Sarita Paul who won $2 for correcting a mistake in the answer of #4].

1. (10 pts. altogether) (a) (7 pts) What is the rank of the matrix

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1
\end{bmatrix}
\]

Sol. to 1(a): You apply the first phase of Gaussian elimination. The elementary row operations \( r_3 - r_1 \to r_3 \) and \( r_4 - 2r_1 \to r_4 \) will get everything below the \((1, 1)\) entry to be 0:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1
\end{bmatrix}
\]

The elementary row operation \( r_4 - r_2 \to r_4 \) will get everything below the \((2, 2)\) to be 0:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

The elementary row operation \( r_3 - r_4 \to r_3 \) will yield

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Now it is in row-echelon form. We see that there are 3 pivots (or equivalently, three rows that are not all-zero). So the rank is 3.

Ans. to 1(a): 3.

(b) (3 points) Using part (a) find the nullity of \( A \).

Sol. to 1(b)): The nullity is the number of columns \((n)\) minus the rank. So it is \(4 - 3 = 1\).

Ans. to 1(b): 1.
2. (10 pts.) Let 

\[ S = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -10 \end{bmatrix} \} \]

determine whether the set \( S \) is linearly independent or linearly dependent. In case it is linearly dependent, write the zero vector \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) explicitly as a non-trivial linear combination of the vectors in \( S \).

**Sol. of 2:** This is so simple that we can do it by **inspection**. The second vector is \(-5\) times the first one, so:

\[ \begin{bmatrix} -5 \\ -10 \end{bmatrix} = (-5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} . \]

Moving everything to the left, we get

\[ (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \]

**Ans. to 2:** \( S \) is **linearly dependent** and the expression of 0 as a non-trivial linear combination of the vectors of \( S \) is:

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -10 \end{bmatrix} . \]

Note: there are many (infinitely many other) ways to do this, for example:

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (10) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} -5 \\ -10 \end{bmatrix} , \]

and the **general way** is:

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (5c) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (c) \begin{bmatrix} -5 \\ -10 \end{bmatrix} , \quad c \neq 0 . \]

If you can’t do it by inspection, you form the matrix whose columns are the two vectors

\[ \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \]

and then you can use Gaussian elimination. The elementary row operation \( r_2 - 2r_1 \rightarrow r_2 \) will yield

\[ \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \]

From here you see that the second column is \(-5\) the first column and by the **column-correspondence property** you get the same answer.
3. (10 pts altogether) Let

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
\]

Calculate the following matrix products, if they are defined, or explain why they don’t make sense.

(a) (5 points) \(AB\)

**Sol. to 3a):**

\[
AB = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 \\
0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 \\
1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
1 & 0
\end{bmatrix}
\]

(b) (3 points) \(AB^T\)

**Sol. to 3b):** undefined. You can’t multiply a \(3 \times 3\) matrix by a \(2 \times 3\) matrix.

(c) (2 points) \(C^2\)

**Sol. to 3c):**

\[
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
1 \cdot 1 + 1 \cdot -1 & 1 \cdot 1 + 1 \cdot 1 \\
-1 \cdot 1 + 1 \cdot -1 & -1 \cdot 1 + 1 \cdot 1
\end{bmatrix} = \begin{bmatrix}
0 & 2 \\
-2 & 0
\end{bmatrix}.
\]
4. (10 pts.) For the matrix 

\[
A = \begin{bmatrix}
1 & -1 \\
2 & 1
\end{bmatrix}
\]

compute the matrix \(A^8\).

**Sol. of 4):** (corrected Dec. 6, 2010, thanks to Sarita Paul).

\[
A^2 = \begin{bmatrix}
1 & -1 \\
2 & -1
\end{bmatrix}^2 = \begin{bmatrix}
-1 & -2 \\
4 & -1
\end{bmatrix}
\]

\[
A^4 = \begin{bmatrix}
-1 & -2 \\
4 & -1
\end{bmatrix}^2 = \begin{bmatrix}
-7 & 4 \\
-8 & -7
\end{bmatrix}
\]

\[
A^8 = \begin{bmatrix}
-7 & 4 \\
-8 & -7
\end{bmatrix}^2 = \begin{bmatrix}
17 & -56 \\
112 & 17
\end{bmatrix}
\]

**Note:** Some (not too many) people computed \(A^3, A^4, A^5, A^6, A^7, A^8\). Some even got the correct answer. I had to give these people full credit, since their method is correct, and they got the right answer. But their method is inefficient. If they had to compute \(A^{1024}\) with their way, they would need to do 1023 matrix multiplication, whereas with repeated squaring, we only need 10 operations.
5. (10 pts.) For the following matrix $A$ finds its **reduced-row-echelon form**, $R$, and find an invertible matrix $P$ such that $PA = R$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

**Sol. of 5:** (added Dec. 7, 2010: I thank Sarita Paul for spotting a misprint, the previous last row of $P$ was erroneous)

We first bring the matrix to **reduced row echelon form**, taking careful note of the elementary row operations:

$$\begin{align*}
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} & \xrightarrow{r_3 - r_1 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
& \xrightarrow{r_3 - r_2 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}$$

Now it is in reduced-row-echelon form. This is the first part of the **answer**, $R$. TO get $P$ we apply the above elementary row operations to the identity matrix:

$$\begin{align*}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \xrightarrow{r_3 - r_1 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\
& \xrightarrow{r_3 - r_2 \rightarrow r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.
\end{align*}$$

**Ans. to 5:**

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

**Note:** Since $A$ is not invertible ($R$ is not $I_3$), there are more than one correct $P$. The $R$ is always the same, regardless of the choice of the order of elementary row operations, but the $P$ may be different. That’s why it is good to check that $PA = R$, because there is more than one correct $P$ that makes it come true.
6. (10 pts. altogether) In each case below, give an \( m \times n \) matrix \( R \) in reduced row echelon form satisfying the given condition, or explain why it is impossible to do so.

(a) (4 pts) \( m = 2, \, n = 3 \) and the equation \( R \mathbf{x} = \mathbf{c} \) has a solution for all \( \mathbf{c} \).

\textbf{Sol. to 6a):} There many “correct solutions”. One of them is:

\[
R = \begin{bmatrix} 
1 & 0 & 2 \\
0 & 1 & 3 
\end{bmatrix}
\]

\textbf{Explanation:} The system \( R \mathbf{x} = \mathbf{c} \), in high-school language is:

\[
\begin{align*}
x_1 + 2x_3 &= c_1, \\
x_2 + 3x_3 &= c_2.
\end{align*}
\]

Obviously you can solve it for any choice of real numbers \( c_1, \, c_2 \). \( x_3 \) is a free variable, and the general solution is \( x_1 = c_1 - 2x_3, \, x_2 = c_2 - 3x_3, \, x_3 = x_3 \), so in this system there are infinitely many solutions for all \( \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), but this is not what the question demanded, only that there is at least one solution.

(b) (4 pts) \( m = 2, \, n = 2 \) and the equation \( R \mathbf{x} = \mathbf{c} \) has a unique solution for all \( \mathbf{c} \).

\textbf{Sol. to 6b):}

\[
R = \begin{bmatrix} 
1 & 0 \\
0 & 1 
\end{bmatrix}
\]

\textbf{Explanation:} The system \( R \mathbf{x} = \mathbf{c} \), in high-school language is:

\[
\begin{align*}
x_1 &= c_1, \\
x_2 &= c_2.
\end{align*}
\]

This system is so simple that it equals its own solution. Obviously there is a unique solution \( x_1 = c_1, \, x_2 = c_2 \) no matter what \( c_1, \, c_2 \) are. (There are no free variables, of course).

(c) (2 pts) \( m = 3, \, n = 3 \) and the equation \( R \mathbf{x} = \mathbf{0} \) has no solution.

\textbf{Sol. to 6c):} impossible. \( \mathbf{x} = \mathbf{0} \) always has a solution, namely \( \mathbf{0}! \)
7. (10 pts.) **Without first computing** $A^{-1}$, find $A^{-1}B$, if

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}
\]

**Sol,** of 7: We perform Gaussian elimination on $A$, keeping track of the elementary row operations

\[
\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \rightarrow r_2 - 2r_1 \rightarrow r_2 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \rightarrow r_1 + 2r_2 \rightarrow r_1 \rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} - r_2 \rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

Now we **mimick** the same elementary row operations, in the **same order** starting with $B$:

\[
\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \end{bmatrix} \rightarrow r_1 + 2r_2 \rightarrow r_1 \rightarrow \begin{bmatrix} -1 & 3 & -4 \\ -1 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}
\]

**Ans. to 7:**

\[
A^{-1}B = \begin{bmatrix} -1 & 3 & -4 \\ 1 & -2 & 3 \end{bmatrix}
\]
8. (10 pts. altogether, 2 each) **True** or **False**? Give a short explanation!

(a) For any $n \times n$ matrices $A$ and $B$, if $AB = I_n$, then $BA = I_n$.

**Sol. to 8(a): True.** (theorem)

(b) If $A$ and $B$ are invertible $2 \times 2$ matrices, then so is $A + B$.

**Sol. to 8(b): False.** For example if $A = I_2$ and $B = -I_2$.

(c) The sum of *any* two $m \times n$ matrices is always defined.

**Sol. to 8(c): True.**

**Note:** Some people answered “False”, since the two matrices “may not have the same $m$ and $n$”. In a different galaxy they may have been right, but the mathematical language in planet Earth (in the Solar System, Milky Way) implies when you say *two* $m \times n$ matrices, that we are talking about the same $m$ and the same $n$.

(d) The product of *any* two $4 \times 9$ matrices is never well-defined.

**Sol. to 8(d): True.** For a matrix product to be well-defined the number of columns of the left-matrix must equal the number of rows of the right-matrix.

(e) The equation $Ax = b$ is consistent if and only of $b$ is a linear combination of the rows of $A$.

**Sol. to 8(e): False.** The correct statement is with “rows” replaced by columns.
9. (10 pts.) Let \( u \) be a solution of \( Ax = b \) and \( v \) be a solution of \( Ax = 0 \), where \( A \) is an \( m \times n \) matrix and \( b \) is a vector in \( \mathbb{R}^m \). Show that \( u + v \) is a solution of \( Ax = b \).

**Sol. of 9:** We are told that

\[
Au = b, \quad Av = 0
\]

Now, by the distributive property and the data of the problem, we have

\[
A(u + v) = Au + Av = b + 0.
\]

Since adding 0 does not change the vector, this equals

\[
b + 0 = b.
\]

We have just proved that

\[
A(u + v) = b,
\]

but this means that \( u + v \) is a solution of \( Ax = b \).

**Note:** This question is very abstract, and many people didn’t get it. But something very similar was in the Review problems and I posted the answers. I am willing to bet that most of the people who didn’t get it didn’t read the posted answers, that I worked so hard to prepare. Too bad!
10. (10 pts. altogether, 5 each)

(a) What does it mean to say that the vectors \( u_1, \ldots, u_k \) in \( \mathbb{R}^n \) are linearly independent?

**Sol. of 10(a)** \( u_1, \ldots, u_k \) are linearly independent if there is no way that the there are \( k \) real numbers \( c_1, \ldots, c_k \) such that

\[
c_1 u_1 + \ldots + c_k u_k = 0
\]

unless all of them are equal to 0, i.e. \( c_1 = 0, c_2 = 0, \ldots, c_k = 0 \).

(b) What is meant by the *span* of a set of vectors \( S = \{u_1, \ldots, u_k\} \)? Give the precise definition in one or more sentences.

**Sol. of 10b)** the span of \( S = \{u_1, \ldots, u_k\} \) is the set of all linear combinations. In other words, it is the set

\[
\{c_1 u_1 + \ldots + c_k u_k : -\infty < c_1 < \infty, \ldots, -\infty < c_k < \infty \}
\]