ON RECURSIVE COMPUTATION

by

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CONTENTS

I.

SUMMARY AND INTRODUCTION

Computation by the use of difference equations in the backward direction was introduced by J.C.P. Miller (1952). In this reference, he applied the method, now sometimes called Miller's algorithm, to the calculation of Bessel functions, and it proceeds essentially as follows.*

Consider the difference equation

$$y(n) - \frac{2(n+1)}{x} y(n+1) - y(n+2) = 0 \ , \ x > 0 \ , \ n \geq 0 \ , \qquad (1.1)$$

which is satisfied by the modified Bessel functions $I_n(x)$ and $(-)^n K_n(x)$.** Let m be an integer ≥ 0 . Put

$$\Lambda_{m+1}(m) = 0 \ , \ \Lambda_m(m) = 1 \ , \qquad (1.2)$$

and calculate $\Lambda_n(m)$ for $0 \leq n \leq m-1$ from (1.1), i.e.,

$$\Lambda_n(m) = \frac{2(n+1)}{x} \Lambda_{n+1}(m) + \Lambda_{n+2}(m) \ , \ 0 \leq n \leq m-1 \ . \qquad (1.3)$$

Now the series

$$1 = \sum_{k=0}^{\infty} (-)^k \epsilon_k I_{2k}(x) \ , \ \epsilon_k = \begin{cases} 1, k=0 \ , \\ 2, k>0 \ , \end{cases} \qquad (1.4)$$

* Miller has stated that he first used the method as an aid in the computation of Airy integrals, see Miller (1946).

** All special functions in this work are defined as in Erdélyi et al (1953).

is known (Erdélyi et al (1953), v. II, p. 7).

Let

$$\Omega(m) = \sum_{k=0}^{[m/2]} (-)^k \epsilon_k \Lambda_{2k}(m) \ , \tag{1.5}$$

where $[m/2]$ means the largest integer not greater than $m/2$. Then, by using the known asymptotic properties of I_n, K_n for large n, one can show that

$$\lim_{m\to\infty} \Lambda_n(m)\big/\Omega(m) = I_n(x) \ , \ n \geq 0 \ , \ x > 0 \ . \tag{1.6}$$

In fact, the asymptotic estimates

$$I_n(x) = \frac{(x/2)^n}{n!}\left[1+O(n^{-1})\right] \ , \ (-)^n K_n(x) = \frac{(-2/x)^n}{2}\,\Gamma(n)\left[1+O(n^{-1})\right] \tag{1.7}$$

follow from the ascending series representations of I_n and K_n, and, since $\Lambda_n(m)$ satisfies (1.1), it can be represented as a linear combination of the (linearly independent) solutions (1.7), see the Appendix. This means that

$$\Lambda_n(m) = \xi_1(m) I_n(x) + \xi_2(m)(-)^n K_n(x) \ . \tag{1.8}$$

From (1.2) and

$$I_m(x)K_{m+1}(x) + I_{m+1}(x)K_m(x) = 1/x \ , \tag{1.9}$$

we conclude that

$$\xi_1(m) = xK_{m+1}(x) \ , \ \xi_2(m) = x(-)^m I_{m+1}(x) \ . \tag{1.10}$$

Thus

$$\left.\begin{aligned} \Lambda_n(m) &= m!(2/x)^m I_n(x)\left[1 + O(m^{-1})\right] \ , \ m \to \infty \ , \\ \Omega(m) &= m!(2/x)^m \left[1 + O(m^{-1})\right] \ , \ m \to \infty \ , \end{aligned}\right\} \tag{1.11}$$

so (1.6) follows.

The above analysis shows clearly why the process converges, and also why it converges to I_n and not to $(-)^n K_n$: I_n is very small compared to K_n as $n \to \infty$. This characteristic of Miller's algorithm, namely, that the solution of the difference equation to which the process converges, if it converges, must, in a certain sense be the smallest solution, remains true when the algorithm is applied to general homogeneous difference equations, see our Theorem 4.3.

A remarkable feature of the Miller algorithm is that no tabular values of I_n are needed in the computations, only a normalization relationship, such as (1.4). Tabular values would be required, of course, if (1.1) were used in the forward direction, and moreover, when (1.1) is used in the forward direction to compute I_n starting with initial values of I_0 and I_1, those small errors inevitably introduced in the course of the computation grow rapidly with n .

Such a phenomenon is called instability.*

The method proposed by Miller created enormous interest, and a number of papers subsequently appeared in which the writers either further treated the application of the method to Bessel functions, or else showed that the method could be used to compute other special functions. Stegun and Abramowitz (1957), Randels and Reeves (1958), Goldstein and Thaler (1959), Corbató and Uretsky (1959), and Makinouchi (1965a,b) all treated the computation of Bessel functions. Rotenberg (1960) showed how the algorithm could be used to compute toroidal harmonics (i.e., Legendre functions) and Miller himself applied the method to parabolic cylinder functions (1964).

Gautschi (1961a) discussed the computation of repeated integrals of the error function

$$y(n) = i^n \operatorname{erfc} x = \left(2/\sqrt{\pi}\, n!\right) \int_x^\infty (t-x)^n e^{-t^2} dt \ , \ n \geq 0 \ , \qquad (1.12)$$

which satisfy

* The computation of $(-)^n K_n$ by using (1.1) in the <u>forward</u> direction with initial values of K_0 and K_1 is stable, i.e., random errors introduced during the computations do not grow with n . In general, a difference equation can be used efficiently in the forward direction only to compute the "largest" solution of the equation. However, the analysis of the forward procedure is rather less of a problem than the analysis of Miller's algorithm, see Gautschi (ca 1962), and will occupy none of our attention here.

$$y(n) - 2xy(n+1) - 2(n+2)y(n+2) = 0 , \qquad (1.13)$$

and in a later paper (1961b) discussed the computation by backward recursion of a number of other functions defined by definite integrals.

The Miller algorithm can be applied to problems other than the computation of the special functions. Recently, it has been employed in such diverse problems as the calculation of successive derivatives of $\left[f(z)/z\right]$, where f is an arbitrary analytic function (Gautschi (1966)) and the computation of coefficients for the Chebyshev polynomial expansions of functions which satisfy differential equations with polynomial coefficients (Clenshaw (1957),(1962)).

Of course, any numerical technique of such general applicability demands a thorough theoretical investigation. Gautschi (1961b), who analyzed its convergence when applied to an arbitrary second order difference equation, seems to have been the first writer to discuss the Miller algorithm from a general point of view. He continues this analysis in two unpublished works (ca 1962, 1963) using as his main tools the theory of continued fractions and the classical asymptotic theory of linear difference equations (e.g., the theorems of Poincaré, Perron, and Kreuser), and he applies his findings to the computation of Bessel functions, Legendre functions, the incomplete Beta function and the numerical computation of Fourier coefficients.

By now a great deal is known about the application of Miller's algorithm to second order difference equations. Conditions on the solutions of the equation which will guarantee the convergence of the algorithm have been given by

Gautschi (1961b) and by Olver (1964). Methods of increasing the efficiency of the algorithm by using the adjoint equation to generate auxiliary sequences have been given by Shintani (1965). (Our Theorem 2.7 is a generalization of one of this author's results.)

On the other hand, little is known concerning application of the algorithm to difference equations of arbitrary order. Gautschi (ca 1962) has touched briefly on the use of difference equations of order $\sigma > 2$, but, unfortunately, the classical asymptotic theory on which his analysis is based does not give very realistic conditions for convergence of the algorithm. More specifically, Gautschi found it necessary to assume the existence of a fundamental set for the equation (see the Appendix) whose members exhibited radically dissimilar behaviour as $n \to \infty$, namely if $\{y_h(n)\}$ were the set in question, and

$$y_h(n+1)/y_h(n) \sim t_h n^{v_h}, \quad t_h \neq 0, \quad v_\sigma > v_{\sigma-1} > \dots > v_1, \tag{1.14}$$

$$n \to \infty, \quad 1 \le h \le \sigma,$$

then it could be shown that Miller's algorithm converged (to $y_1(n)$), provided, of course, that a suitable normalization relationship was known. Needless to say, this condition is excessively stringent, at least for a very wide class of difference equations (cf our Theorem 4.2).

The purpose of this work is to examine the application of Miller's algorithm, as well as several related algorithms, to homogeneous linear difference equations of arbitrary order with coefficients of a fairly general type.

(The related algorithms are modifications of Miller's algorithm which can be used when the equation has no smallest solution.)

First, in Chapter II, we formulate the algorithms as they are applied to difference equations with arbitrary coefficients, and investigate their convergence properties. Conditions for convergence take the form of rather unwieldy restrictions on the growth of solutions of the equation and determinants involving them. To obtain more practical conditions, it is necessary to restrict somehow the form of the coefficients in the equation. This is done in the chapters following by requiring that the coefficients possess certain asymptotic representations as $n \to \infty$. We are then able to use as our principal investigational tool the analytic theory of singular difference equations which was developed by Birkhoff and Trjitzinsky. Virtually all difference equations encountered in practical applications are of the specified form, including all equations with coefficients rational in n. In particular, the computational procedures discussed by the preceding authors are included in our analysis.

Chapter III is devoted exclusively to an asymptotic analysis of the solutions of this difference equation, starting with the above-mentioned theory of Birkhoff and Trjitzinsky. We then prove two new theorems concerning the representation of those solutions whose growth can be described with only algebraic and logarithmic terms.

In Chapter IV, we apply the asymptotic results of the previous chapter to the problem of determining more tractable conditions for the convergence of the algorithms given in Chapter II. One result is that, for the type of difference equation considered, at least one of the algorithms will converge to a solution of that equation, provided that one can find a fundamental set for the adjoint equation in which no more than two solutions have the same rate of growth (in absolute value) as $n \to \infty$. A consequence of this theorem is that, for a second order difference equation of the specified type, Miller's algorithm, or a suitable modification of it, will always converge.

Chapter IV contains a number of examples, among which are the computation of the integral

$$y(n) = \int_0^\infty \exp\left\{-t^\sigma - P(t)\right\} t^n dt \ , \ n \geq 0 \ , \tag{1.15}$$

σ an integer ≥ 2, $P(t)$ a polynomial of degree $(\sigma-1)$, and the computation of a class of hypergeometric functions.

In the Appendix of this work are contained definitions, notation, and those properties of difference equations which are frequently used in the preceding chapters. (References to material in the Appendix are preceded by an A, as A.2, A-VI, etc.)

II.

In this chapter, we will discuss computation by backward recursion based on the linear homogeneous difference equation of order $\sigma \geq 2$

$$\sum_{\nu=0}^{\sigma} C_\nu(n)y(n+\nu) = 0 \ , \ C_0 = 1 \ , \ C_\sigma(n) \neq 0 \ , \tag{2.1}$$

where n is an integer ≥ 0 .

The first algorithm proceeds as follows. Let m be an integer ≥ 0 . Put

$$\Lambda_{m+\sigma-1}(m) = \Lambda_{m+\sigma-2}(m) = \ldots = \Lambda_{m+1}(m) = 0 \ , \ \Lambda_m(m) = 1 \ , \tag{2.2}$$

and calculate $\Lambda_n(m)$ for $0 \leq n \leq m-1$ by recursion from

$$\sum_{\nu=0}^{\sigma} C_\nu(n)\Lambda_{n+\nu}(m) = 0 \ . \tag{2.3}$$

Suppose we are given the convergent series (called a normalization relationship)

$$1 = \sum_{k=0}^{\infty} L_k y_1(k) \tag{2.4}$$

where $y_1(n)$ is a solution of (2.1).

Define

$$\Omega(m) = \sum_{k=0}^{m} L_k \Lambda_k(m) \ , \tag{2.5}$$

and

$$\Gamma_n(m) = \Lambda_n(m) / \Omega(m) \ . \tag{2.6}$$

<u>Definition 2.1</u>

If

$$\lim_{m \to \infty} \Gamma_n(m) = y_1(n) \ , \ n \geq 0 \ , \tag{2.7}$$

then we say the computation of $y_1(n)$ by backward recursion based on (2.1) and (2.4) converges.

Let us analyze the above algorithm. First, by A-IV we note that (2.1) possesses a fundamental set, $\left\{ y_h(n) \right\}$, and since $\Lambda_n(m)$ satisfies the equation, we can write

$$\Lambda_n(m) = \sum_{h=1}^{\sigma} \xi_h(m) y_h(n) \ , \tag{2.8}$$

where ξ_h is independent of n. By setting $n = m \ , m + 1 \ , \ldots \ , m + \sigma - 1$ in (2.8) it is found that

$$\xi_h(m) = T_h(m) \big/ D(m) \ , \tag{2.9}$$

where T_h , D are defined in A-V, A-VI. D is not zero because of properties A-I, A-V, and the fact that $C_\sigma(n) \neq 0$.

Thus

$$\Gamma_n(m) = \frac{\sum_{h=1}^{\sigma} T_h(m) y_h(n)}{\sum_{h=1}^{\sigma} T_h(m) \sum_{k=0}^{m} L_k y_h(k)} \tag{2.10}$$

and this formulation leads to

Theorem 2.1

Let $T_1(m) \neq 0$ for m sufficiently large.

Define

$$R_h \equiv R_h(m) = \frac{T_h(m)}{T_1(m)} \ , \ S_h \equiv S_h(m) = \sum_{k=0}^{m} L_k y_h(k) \ , \ 1 \leq h \leq \sigma \ . \tag{2.11}$$

Now suppose

$$\lim_{m \to \infty} R_h = \lim_{m \to \infty} R_h S_h = 0 \ , \ 2 \leq h \leq \sigma \ . \tag{2.12}$$

Then the computation of $y_1(n)$ by backward recursion based on (2.1) and (2.4) converges.

If $y_1(0)$ is known from some source, the algorithm and the conditions for its validity simplify considerably. This means we can take $L_k = 0$ for $k > 0$, and $L_o = 1/y_1(0)$. We have

Corollary 2.1

Let $y_1(0)$ be known and non-zero and let

$$\lim_{m\to\infty} R_h = 0 \ , \ 2 \le h \le \sigma \ . \tag{2.13}$$

Then

$$\lim_{m\to\infty} \frac{\Lambda_n(m)}{\Lambda_o(m)} = \frac{y_1(n)}{y_1(0)} \ , \ n \ge 0 \ . \tag{2.14}$$

In the application of Theorem 2.1 one will find it more convenient, for large m, to calculate Ω using the following result, rather than (2.5).

Theorem 2.2

Ω satisfies

$$\sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)\Omega(m+\nu) = L_{m+\sigma} \ , \ m \ge 0 \ , \tag{2.15}$$

with the initial values (as obtained from (2.2) - (2.5))

$$\left.\begin{aligned} \Omega(0) = L_o \ , \ \Omega(1) = -C_1(0)L_o + L_1 \ , \\ \Omega(2) = L_o\left[C_1(0)C_1(1) - C_2(0)\right] - L_1C_1(1) + L_2 \ , \\ \cdots \end{aligned}\right\} \tag{2.16}$$

Proof:

By (2.8) and A-VI we have, for k , $m \geq 0$,

$$\sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)\Lambda_k(m+\nu) = 0 \ , \tag{2.17}$$

so

$$\sum_{k=0}^{m+\sigma} \sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)L_k\Lambda_k(m+\nu) = 0 \ . \tag{2.18}$$

Now $\Lambda_k(m+\nu) = 0$ for $m+\nu+1 \leq k \leq m+\nu+\sigma-1$, so if empty sums are interpreted as zero, we can write (2.18) as

$$\sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)\left\{\sum_{k=0}^{m+\nu} L_k\Lambda_k(m+\nu) + \sum_{k=m+\nu+\sigma}^{m+\sigma} L_k\Lambda_k(m+\nu)\right\}$$

$$= L_{m+\sigma}\Lambda_{m+\sigma}(m)C_\sigma(m) + \sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)\Omega(m+\nu) \tag{2.19}$$

$$= - L_{m+\sigma} + \sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(m+\nu)\Omega(m+\nu) = 0 \ .$$

For $\sigma = 2$, this result is given by Shintani (1965).

Now suppose that u of the R_h's behave similarly as $m \to \infty$, but that the ratio of any one of these to each of the $\sigma-u$ other R_h's approaches zero as $m \to \infty$. A generalization of a method due to Luke (unpublished), which was in turn suggested by Clenshaw (1964) for a three-term recursion relation, can often be used to obtain any one of the first u solutions corresponding to the u R_h's . Clenshaw (1962) originally used this method, for $\sigma=2$, to compute coefficients for the Chebyshev polynomial expansions of certain mathematical functions.

Without loss of generality, we may assume $y_1(n)$ is the solution of (2.1) that we wish to compute. The algorithm is then described by the following theorem.

Theorem 2.3

Let the constants $L_{k,j}$ be given for $0 \le k < \infty$, $1 \le j \le u$, and define

$$S_{h,j} \equiv S_{h,j}(m) = \sum_{k=0}^{m} L_{k,j} y_h(k) \ , \ 2 \le h \le \sigma \ , \ 1 \le j \le u \ , \ 2 \le u \le \sigma \ . \tag{2.20}$$

Let R_h be bounded and bounded away from zero as $m \to \infty$ for $2 \le h \le u$, while $R_h \to 0$ as $m \to \infty$ for $u+1 \le h \le \sigma$. Let $\Gamma_n(m)$ be as in (2.6) with $L_k \equiv L_{k,1}$ and let $\Omega(m)$, $T_1(m)$ be non-zero for m sufficiently large. Let also

$$\left.\begin{aligned} &\lim_{m\to\infty} S_{h,j} R_h = 0 \ , \ u+1 \le h \le \sigma \ , \ 1 \le j \le u \ , \\ &\lim_{m\to\infty} S_{h,j} = A_{h,j} < \infty \ , \ 2 \le h \le u \ , \ A_{1,j} = 1 \ , \ 1 \le j \le u \ , \ \left| A_{h,j} \right|_1^u \ne 0 \ . \end{aligned}\right\} \tag{2.21}$$

Then for m_1 sufficiently large, we can determine m_h for $2 \le h \le u$, $m_1 < m_2 < m_3 < \ldots < m_u$, so that the system of equations

$$\left.\begin{aligned} &\sum_{v=1}^{u} \pi_v \sum_{k=0}^{m_v} L_{k,j}\Gamma_k(m_v) = 1 , \quad j = 2,3,\ldots,u; \\ &\sum_{v=1}^{u} \pi_v = 1 , \end{aligned}\right\} \tag{2.23}$$

has a unique solution $\{\pi_h\}$ (depending, of course, on the m_h).

Let $\left|R_h(m_j)\right|_1^u$ be bounded away from zero as $m_1 \to \infty$. Then

$$\lim_{m_1 \to \infty} \sum_{v=1}^{u} \pi_v \Gamma_n(m_v) = y_1(n) , \quad n \ge 0 . \tag{2.24}$$

Proof:

Note that since $T_1(m_v)$, $\Omega(m_v)$ are not zero for m_1 sufficiently large, the system (2.23) is well-defined.

Equation (2.23) is

$$\left.\begin{aligned} &\sum_{v=1}^{u} \pi_v^* \left[\sum_{h=1}^{\sigma} R_h(m_v) S_{h,j}(m_v) \right] = 1 , \quad 1 \le j \le u , \\ &\pi_v^* = \pi_v T_1(m_v) / \Omega(m_v) \quad . \end{aligned}\right\} \tag{2.25}$$

The determinant of (2.25) is

$$\left|R_h(m_j)\right|_1^u \left|A_{h,j}\right|_1^u \Big(1 + o(1)\Big) \ , \ m_1 \to \infty \ . \qquad (2.26)$$

The m_h can be chosen so that (2.26) is not zero, else the R_h are linearly dependent, by A-I, A-II. Since $\left|A_{h,j}\right|_1^u \neq 0$, we conclude that (2.23) uniquely determines π_v for m_1 sufficiently large. Furthermore, when (2.25) is solved for π_v^* , one finds that this quantity (and hence π_v itself) is bounded as $m_1 \to \infty$.

We write

$$\sum_{v=1}^{u} \pi_v \Gamma_n(m_v) = \sum_{s=1}^{\sigma} c_s y_s(n) \ , \ c_s \equiv c_s(m_1, m_2, \ldots, m_u) \ . \qquad (2.27)$$

Then

$$c_s = \sum_{v=1}^{u} \frac{\pi_v R_s(m_v)}{\sum_{h=1}^{\sigma} R_h(m_v) S_{h,1}(m_v)} \ . \qquad (2.28)$$

Clearly

$$\lim_{m_1 \to \infty} c_s = 0 \ , \ u + 1 \leq s \leq \sigma \ . \qquad (2.29)$$

Also, from (2.25) and the boundedness of the π_h we have

$$\sum_{h=1}^{u} A_{h,j} c_h = 1 + o(1) \ , \ m_1 \to \infty \ , \ 1 \leq j \leq u \ , \qquad (2.30)$$

and since $A_{1,j} = 1$, we conclude from this system of equations that

$$\lim_{m_1 \to \infty} c_s = \begin{cases} 1, & s = 1 , \\ 0, & 2 \le s \le u , \end{cases} \tag{2.31}$$

which, when used with (2.29) in (2.27), gives the theorem.

The application of the above theorem requires that we know u normalization relations for the desired solution, $y_1(n)$. If, instead, we know u values of y_1 , then the following result can be used.

Theorem 2.4

Let

$$\lim_{m \to \infty} R_h = 0 \ , \ u + 1 \le h \le \sigma \ , \tag{2.32}$$

while R_h is bounded and bounded away from zero as $m \to \infty$ for $1 \le h \le u$.

Then we can determine k_h , $1 \le h \le u$, $0 \le k_1 < k_2 < k_3 < \dots < k_u$, so that

$$\left| y_h(k_j) \right|_1^u \ne 0 \tag{2.33}$$

and for m_1 sufficiently large, m_h can be determined, $m_1 < m_2 < \dots < m_u$, so that the system of equations

$$\sum_{v=1}^{u} \pi_v \Lambda_{k_j}(m_v) \Big/ \Lambda_{k_1}(m_v) = y_1(k_j) \ , \ 1 \le j \le u \ , \tag{2.34}$$

has a unique solution, π_h , $1 \le h \le u$. Furthermore, suppose $\left| R_h(m_j) \right|_1^u$ is bounded away from zero as $m_1 \to \infty$. Then

$$\lim_{m_1 \to \infty} \sum_{v=1}^{u} \pi_v \Lambda_n(m_v) \Big/ \Lambda_{k_1}(m_v) = y_1(n) \ , \ n \ge 0 \ . \tag{2.35}$$

Proof:

As for Theorem 2.3.

Unless additional assumptions are made about the nature of the coefficients $C_\nu(n)$ in the difference equation, it may not be possible to find a fundamental set so that $T_1(m)$ is non-zero for m large. However, in most applied problems, in particular in all those examples of computation by backward recursion which we discussed in the introduction of this work, the difference equation in question possessed coefficients which were rational functions of n . If this is the case, as we shall see, a fundamental set can always be chosen so that $T_1(m)$ is non-zero for m sufficiently large, and the Miller algorithm at least has a chance of converging. Even if it is only required that $C_\nu(n)$ possess an asymptotic series in powers of $n^{-1/\omega}$, ω an integer ≥ 1 , the same is true.

It is thus natural that we turn our attention to this kind of equation.

In this and the following chapters of this work, the standard form for the difference equation will be

$$\sum_{\nu=0}^{\sigma} C_\nu(n)y(n+\nu) = 0 \ , \ C_0 = 1 \ , \ C_\sigma \neq 0 \ ; \ \sigma \geq 2 \ , \ n \geq 0 \ , \tag{3.1}$$

$$C_\nu \equiv C_\nu(n) \sim n^{K_\nu/\omega} \left\{ c_{0,\nu} + c_{1,\nu} n^{-1/\omega} + c_{2,\nu} n^{-2/\omega} + \ldots \right\} , \ n \to \infty, \tag{3.2}$$

where K_ν is an integer, ω is an integer ≥ 1 , and $c_{0,\nu} \neq 0$ unless $C_\nu \equiv 0$. We also assume that the coefficients are written with the smallest possible value of ω . Note that the equation adjoint to (3.1) also has coefficients of the same general form as (3.2). By (A.12), equation (3.1) has a unique solution $y(n)$ satisfying

$$y(n_0+j) = \alpha_j \ , \ 1 \leq j \leq \sigma \ , \ n_0 \geq 0 \ . \tag{3.3}$$

Thus, for $n \geq n_0 \geq 0$, (3.1) has a fundamental set.

We shall see that there exists a certain fundamental set for (3.1) whose members share an unusual property: each has an asymptotic expansion, valid as $n \to \infty$, which consists of an exponential leading term multiplied by a descending series of the kind (3.2) (where, however, ω may be replaced by an integral multiple of ω , see (3.4)-(3.7) below). Essentially, these series are the same as the so-called subnormal series encountered in the study of ordinary linear differential equations with polynomial coefficients near singular points, see Ince (1956, Ch. 17). This is another example of the close analogy between differential equations and difference equations.

For our purposes, the existence of solutions of the difference equation (3.1) which have such asymptotic representations is quite important, since the very form of the asymptotic series enables us to determine much more practical conditions for the convergence of the class of algorithms discussed in Chapter II.

However, we will have to examine the properties, algebraic and analytic, of these solutions in great detail before we can attack the convergence problem directly, and the present chapter is devoted entirely to this study.

We begin with several definitions.

Consider the series

$$e^{Q(\rho;n)} s(\rho;n) \quad , \tag{3.4}$$

$$Q(n) \equiv Q(\rho;n) = \mu_o n \ln n + \sum_{j=1}^{\rho} \mu_j n^{\frac{\rho+1-j}{\rho}} \quad , \tag{3.5}$$

$$s(n) \equiv s(\rho;n) = n^{\theta} \sum_{j=0}^{t} (\ln n)^j n^{r_{t-j}/\rho} q_j(\rho;n) \quad , \tag{3.6}$$

$$q_j(\rho;n) \equiv q_j(n) = \sum_{s=0}^{\infty} b_{s,j} n^{-s/\rho} \quad , \; . \tag{3.7}$$

where ρ , r_j , $\mu_o\rho$ are integers, $\rho \geq 1$, μ_j , θ , $b_{s,j}$ are complex parameters, $b_{o,j} \neq 0$, unless $b_{s,j} = 0$ for all $0 \leq s < \infty$, $r_o = 0$, $-\pi \leq \text{Im}\, \mu_1 < \pi$.

Definition 3.1

The series (3.4), called a formal series (F.S.), will be called a formal series solution (F.S.S.) of (3.1) if, when it is substituted in (3.1), the equation is divided by $e^{Q(n)}$ and the obvious algebraic manipulations (see below) are performed, then the coefficient of each quantity

$$n^{\theta+r/\rho+s/\omega}(\ln n)^j \; , \; 0 \le j \le t \; , \; r,s = 0,\pm 1,\pm 2, \; \ldots \tag{3.8}$$

is equal to zero.

A concept of formal equality between two F.S. can be defined by requiring that, when the series are written with the same value of ρ (as is always possible), then the parameters t , θ , $b_{s,j}$, r_j , μ_j for both series are the same. Formal equality of formal solutions also arises in the theory of ordinary differential equations (see Coddington and Levinson (1955), p. 114 ff).

The construction of F.S.S. may be carried out by using the identities

$$e^{\mu(n+\nu)^\delta} = e^{\mu n^\delta}\left[1 + \sum_{k,j=1}^{\infty} a_{k,j} n^{k(\delta-1)+1-j}\right] \; , \tag{3.9}$$

$$(n+\nu)^{\mu_o(n+\nu)} = n^{\mu_o n+\mu_o \nu} e^{\mu_o \nu}\left[1 + \frac{\mu_o \nu^2}{2n} + \ldots\right] \; , \tag{3.10}$$

$$(n+\nu)^\alpha = n^\alpha\left[1 + \frac{\alpha\nu}{n} + \ldots\right] \; , \tag{3.11}$$

$$\left[\ln(n+\nu)\right]^r = \left[\ln n + \frac{\nu}{n} - \frac{\nu^2}{2n^2} + \ldots\right]^r \quad , \tag{3.12}$$

although, in practice, it is difficult to obtain by hand other than the first few terms this way, see Birkhoff (1930).*

Very often, we shall let "$Q(n)$," "$s(n)$," be generic symbols for the series (3.5), (3.6). The series so denoted do not necessarily have the same values of the parameters θ , t , $b_{s,j}$, μ_j , when the series occur in different equations. If, however, it is necessary to differentiate between two such series, we shall do so with subscripts, e.g., $Q_1(n)$, $s_1(n)$, $Q_2(n)$, $s_2(n)$, etc. With this convention understood, we see that F.S. possess the following properties:

$$e^{Q(n+\nu)} s_1(n+\nu) = e^{Q(n)} s_2(n) \ , \ \nu = 1,2,\ldots, \tag{3.13}$$

$$e^{Q_1(\rho;n)} s_1(\rho;n) \cdot e^{Q_2(\rho';n)} s_2(\rho';n) = e^{Q_3(\rho^*;n)} s_3(\rho^*;n) \quad , \tag{3.14}$$

where ρ^* is the least common multiple of ρ and ρ' .

The sum of two F.S. is not in general a F.S., but if $Q(n)$, θ are the same for both series, we have

* J.C.P. Miller has brought to our attention the fact that the determination of these series can be done very efficiently by computers.

$$e^{Q(n)}n^{\theta}q_1(\rho;n) + e^{Q(n)}n^{\theta}q_2(\rho';n) = e^{Q(n)}n^{\theta}q_3(\rho^*;n) \quad . \tag{3.15}$$

Definition 3.2

$$f(n) \sim e^{Q(n)}s(n) \ , \ n \to \infty \quad , \tag{3.16}$$

means that, for every $k \geq 1$, we can determine functions $A_{k,j}(n)$, $0 \leq j \leq t$, such that

$$e^{-Q(n)}n^{-\theta}f(n) = \sum_{j=0}^{t} (\ln n)^j n^{r_{t-j}/\rho} \sum_{s=0}^{k-1} b_{s,j} n^{-s/\rho}$$

$$+ n^{-k/\rho} \sum_{j=0}^{t} (\ln n)^j n^{r_{t-j}/\rho} A_{k,j}(n) \quad , \tag{3.17}$$

and $|A_{k,j}|$ is bounded as $n \to \infty$, for all k,j.

See Birkhoff and Trjitzinsky (1932, p. 62). If $t = 0$, this definition coincides with (A.7)-(A.8). Also (3.17) is unique, since it is readily verified that zero has no non-trivial representation of the form (3.16).

Definition 3.3

Let

$$W_k = \left| e^{Q_h(n+j)} s_h(n+j) \right|_1^k \quad . \tag{3.18}$$

By (3.12)-(3.14), W_k is a F.S., and

$$W_k = \exp\left\{\sum_{j=1}^{k} Q_j(n)\right\}\overline{s}(n) = e^{\overline{Q}(n)}\overline{s}(n) \quad . \tag{3.19}$$

We say the k F.S. $\left\{e^{Q_h(n)} s_h(n)\right\}$ are formally linearly independent if $\overline{s}(n) \not\equiv 0$. Otherwise, they are formally linearly dependent.

Definition 3.4

There exist exactly r F.S.S. of a certain type (e.g., with $Q(n) = 0$) for the equation (3.1) means r F.S. of that type can be constructed which are formally linearly independent, and any r+1 such F.S.S. are formally linearly dependent.

Now we may formulate two very important questions about the difference equation (3.1). Does the equation always possess exactly σ F.S.S. of the general type (3.4)? If so, what relationship do these F.S.S. bear to the fundamental sets for the equation?

These questions were answered partially by a number of mathematicians, see Adams (1928) and the references given there. However, only with the advent of two papers, the first by Birkhoff (1930) and the second by Birkhoff and Trjitzinsky (1932) was the theory completed. The results of these two writers yield

Theorem 3.1 (Birkhoff-Trjitzinsky)

There exist exactly σ F.S.S. of equation (3.1) of type (3.4), where $\rho = \nu\omega$, for some integer $\nu \geq 1$, and each F.S.S. represents asymptotically some solution of the equation in the sense of (3.16). The σ solutions so represented constitute a fundamental set for the equation.

Definition 3.5

The particular fundamental sets mentioned in Theorem 3.1 will be called Birkhoff sets. Each member of a Birkhoff set is called a Birkhoff solution.

Now let $\{y_h(n)\}$, $1 \leq h \leq \sigma$, be a Birkhoff set for (3.1), i.e., let

$$y_h(n) \sim e^{Q_h(\rho;n)} s_h(\rho;n) , \quad n \longrightarrow \infty , \quad 1 \leq h \leq \sigma , \tag{3.20}$$

which is permissible, since we can write all the F.S.S. of Theorem 3.1 with a common value of ρ.

Then none of the determinants

$$W_{m_u} = \left| y_{m_h}(n+j) \right|_1^u , \quad 1 \leq m_1 < m_2 < \ldots < m_u \leq \sigma , \tag{3.21}$$

can be zero for n sufficiently large, and in particular, $y_h(n) \neq 0$ for n sufficiently large, $1 \leq h \leq \sigma$. In fact, since the F.S.S. are formally linearly independent,

$$W_{m_u} = ce^{Q(n)}n^{\alpha}(\ln n)^{r}\left[1 + o(1)\right] , \; n \to \infty , \; c \neq 0 , \qquad (3.22)$$

so ultimately W_{m_u} is monotonic in n, as is $|y_h(n)|$.

We now examine more closely the structure of the F.S.S. (3.4).

<u>Theorem 3.2</u>

Let (3.1)-(3.2) hold, and let

$$\varphi_k(n) = e^{Q(\rho;n)}n^{\theta} \sum_{\ell=0}^{k} (\ln n)^{\ell} q_{t+\ell-k}(\rho;n) n^{r_{k-\ell}/\rho}(t+1-k)_{\ell}/\ell! \; , \qquad (3.23)$$

$$0 \le k \le t \; ,$$

r_j an integer.

Then if $\varphi_t(n)$ is a F.S.S. of (3.1), so are $\varphi_h(n)$, $0 \le h \le t-1$.

<u>Proof</u>:

Let

$$\varphi_k(n) = e^{Q(n)}\overline{\varphi}_k(n) \; , \; y(n) = e^{Q(n)}\overline{y}(n) \; . \qquad (3.24)$$

Then

$$e^{Q(n+\nu)-Q(n)} = n^{\mu_o \nu} e^{(\mu_o+\mu_1)\nu}\left[1 + cn^{-1/\rho} + \dots\right] \; ; \qquad (3.25)$$

μ_o is of the form m/ρ , m an integer, so the difference equation for $\bar{y}(n)$ will be of the same kind as (3.1), except with ω replaced by ρ . Let $\bar{C}_\nu(n)$ be the coefficient of the transformed equation and

$$\bar{\varphi}_k(n) = n^\theta \sum_{\ell=0}^{k} (\ln n)^\ell q_{t+\ell-k}(\rho;n) n^{r_{k-\ell/\rho}} (t+1-k)_\ell/\ell! \quad , \tag{3.26}$$

$\bar{\varphi}_t(n)$ being, by hypothesis, a F.S.S. of that equation. Substituting (3.26) with $k = t$ into (3.1) and setting to zero the coefficient of $(\ln n)^u$ gives

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)(n+\nu)^\theta \sum_{\ell=0}^{t-u} q_{\ell+u}(\rho;n+\nu)(u+1)_\ell (n+\nu)^{\rho^{-1} r_{t-\ell-u}} \left[\ln(1+\nu/n)\right]^\ell/\ell! = 0, \tag{3.27}$$

$$0 \le u \le t \quad .$$

Let $u = t-j$:

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)(n+\nu)^\theta \sum_{\ell=0}^{j} q_{\ell+t-j}(n+\nu)(n+\nu)^{r_{j-\ell/\rho}} (t+1-j)_\ell \left[\ln(1+\nu/n)\right]^\ell/\ell! = 0 , \tag{3.28}$$

$$0 \le j \le t \quad .$$

Now writing

$$\left[\ln(1+\nu/n)\right]^\ell = \left[\ln(n+\nu) - \ln n\right]^\ell \quad , \tag{3.29}$$

and expanding by the binomial theorem, we get

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)(n+\nu)^\theta \sum_{u=0}^{j} (\ln n)^u(-)^u/u! \sum_{\ell=0}^{j-u} q_{t+\ell+u-j}(n+\nu)(n+\nu)^{\rho^{-1}r_{j-\ell-u}}$$

$$\times\ (t+1-j)_{\ell+u}\left[\ln(n+\nu)\right]^\ell/\ell! = 0\ ,\quad 0 \le j \le t\ . \tag{3.30}$$

From (3.26) we have

$$\bar{\varphi}_{j-u}(n) = n^\theta \sum_{\ell=0}^{j-u} (\ln n)^\ell q_{t+\ell+u-j}(n) n^{\rho^{-1}r_{j-u-\ell}} (t+1+u-j)_\ell/\ell!\ , \tag{3.31}$$

and so

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n) \sum_{u=0}^{j} (-)^u(\ln n)^u(t+1-j)_u\bar{\varphi}_{j-u}(n+\nu)/u! = 0\ ,\quad 0 \le j \le t\ . \tag{3.32}$$

Now suppose $\bar{\varphi}_o(n), \bar{\varphi}_1(n), \ldots, \bar{\varphi}_k(n)$ are F.S.S. of the transformed equation, and let $j = k+1$ in (3.32). We have

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)\bar{\varphi}_{k+1}(n+\nu) = 0\quad . \tag{3.33}$$

But for $j = 0$, (3.28) gives

$$\sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)(n+\nu)^\theta q_t(n+\nu) = \sum_{\nu=0}^{\sigma} \bar{C}_\nu(n)\bar{\varphi}_o(n+\nu) = 0\ , \tag{3.34}$$

so the induction is complete, and the $\varphi_h(n)$, $0 \le h \le t$, satisfy (3.1).

Birkhoff (1930) has noted that, once a maximum value of t is found so that $e^{Q_o(n)} s_o(n)$ is a F.S.S. of (3.1), then there are t other F.S.S., $e^{Q_o(n)} s_h(n)$, $1 \le h \le t$, and in $s_h(n)$, $\ln n$ occurs to the t-h power. However, the explicit form of the series, i.e., (3.23), seems to be new.

We will need information about the number and form of those F.S.S. of (3.1) whose exponential leading terms are constant. The necessary results are contained in

Theorem 3.3

Let (3.1)-(3.2) hold, and define

$$\left.\begin{aligned} P_k \equiv P_k(n) &= \sum_{\nu=1}^{\sigma} C_\nu \nu^k , \quad k = 1,2,\ldots, \\ P_o \equiv P_o(n) &= \sum_{\nu=0}^{\sigma} C_\nu \quad . \end{aligned}\right\} \qquad (3.35)$$

Then we can write

$$P_k \sim n^{\alpha_k/\omega}\left[a_{o,k} + a_{1,k}\, n^{-1/\omega} + a_{2,k}\, n^{-2/\omega} + \cdots\right], \; n \to \infty \; , \qquad (3.36)$$

α_k an integer, $a_{o,k} \ne 0$ unless $P_k \equiv 0$, in which case we interpret $\alpha_k = -\infty$. Let

$$\tau = \max_{0 \le k \le \sigma} (\alpha_k/\omega - k) \ , \tag{3.37}$$

and let

$$k_0 < k_1 < k_2 < \ldots < k_\psi \tag{3.38}$$

be those values of k for which

$$\alpha_k/\omega - k = \tau \ , \ 0 \le k < \infty \ . \tag{3.39}$$

Then $k_\psi \le \sigma$, and there exist exactly k_ψ F.S.S. of (3.1) of the form $s(\rho;n)$, and $\rho = \infty$. Each represents asymptotically a solution of (3.1) as $n \longrightarrow \infty$, the k_ψ solutions so represented being linearly independent.

Furthermore, $\theta = \theta_h$ is one of the k_ψ values satisfying

$$G_0(\theta) = \sum_{j=0}^{\psi} (-)^{k_j} (-\theta)_{k_j} a_{0,k_j}/k_j! = 0 \ , \tag{3.40}$$

and if no two of the θ_h's differ by integral multiples of $1/\omega$, then the F.S.S. take the form

$$s(n) = n^\theta \left[b_0 + b_1 n^{-1/\omega} + \ldots \right] , \ b_0 \neq 0 \ . \tag{3.41}$$

Logarithmic solutions can occur only if some of the roots of $G_0(\theta)$ differ by integral multiples of $1/\omega$.

Preview of Proof:

First, we show that $\rho = \omega$ in any purely algebraic-logarithmic F.S.S. of the difference equation (3.1).

Second, by considering (3.35) as a system of equations in the "unknowns" $C_\nu(n)$, $0 \le \nu \le \sigma$, we show there exists a finite τ satisfying (3.37) and that $k_\psi \le \sigma$.

Third, the existence of F.S.S. is shown by actual construction, using Frobenius' method, first for the case where none of the roots of $G_0(\theta)$ differ by integral multiples of $1/\omega$, and next for the case where none of the roots of $G_0(\theta)$ are equal, but where there exists a subset of roots differing from each other by integral multiplies of $1/\omega$. It is then shown that in these two cases, any purely algebraic-logarithmic F.S.S. of (3.1) can be expressed as a linear combination of the solutions already constructed, i.e., for these two cases, there are exactly k_ψ such F.S.S.

Lastly, we indicate briefly how similar results are obtained when some of the roots of $G_0(\theta)$ are equal, and the solutions for this case are displayed.

Proof:

Let

$$\hat{\varphi}_k(n) = n^\theta \sum_{\ell=0}^{k} (\ln n)^\ell n^{r_{k-\ell}/\rho} q_\ell(\rho;n) \ , \qquad (3.42)$$

be a F.S.S. of (3.1). By the Euclidean algorithm, we have

$$r_j = \nu m_j + v_j \ , \ 0 \le j \le k \ , \ 0 \le v_j \le \nu - 1 \ , \tag{3.43}$$

where r_j , m_j are integers, and ν is as in Theorem 3.1.

We can also effect the decomposition

$$n^{v_{k-\ell}/\rho} q_\ell(\rho;n) = \sum_{j=0}^{\nu-1} n^{j/\rho - p_{j,\ell}/\omega} q_{j,\ell}(\omega;n) \tag{3.44}$$

or

$$\hat{\varphi}_k(n) = \sum_{j=0}^{\nu-1} s_{j,k}(\omega;n) \tag{3.45}$$

$$s_{j,k}(\omega;n) = n^{\theta^*} \sum_{\ell=0}^{k} (\ln n)^\ell \, n^{r^*_{k-\ell,j}/\omega} q_{j,\ell}(\omega;n) \quad , \tag{3.46}$$

where

$$\left.\begin{aligned} r^*_{k-\ell,j} &= m_{k-\ell} - p_{j,\ell} + p_{j,k} \quad (r^*_{o,j} = 0) \ , \\ \theta^* &= \theta - p_{j,k} + j/\rho \end{aligned}\right\} \tag{3.47}$$

and $s_{j,k}$ is a series of the type (3.6) with $\rho = \omega$ and θ an integral multiple of $1/\omega$. Note that some of these series can be zero.

The difference equation is unchanged by replacing n by $ne^{2h\pi i\omega}$, h an integer, so each of the functions $\hat{\varphi}_k\left[ne^{2h\pi i\omega}\right]$ is also a F.S.S. of (3.1). An application of Theorem 3.2, or a result of Birkhoff (1930, section 2), shows, furthermore, that each of the functions

$$\hat{\varphi}_{h,k}(n) = \sum_{j=0}^{\nu-1} e^{2h\pi ij/\nu} s_{j,k}(\omega;n) \qquad (3.48)$$

is a F.S.S. of (3.1). We can determine unique constants $A_{h,j}$, $0 \le h \le v-1$, where v is the number of non-zero $s_{j,k}$, so that

$$\sum_{h=0}^{v-1} A_{h,j}\hat{\varphi}_{h,k}(n) = s_{j,k}(\omega;n) \ , \ 0 \le j \le \nu-1 \qquad (3.49)$$

since the determinant of the system of equations (3.48) is a Vandermonde determinant which is non-zero.

But (3.49), when substituted in (3.45), shows that every purely algebraic logarithmic F.S.S. of (3.1) can be expressed as a finite linear combintation of similar solutions, each of which can be written with $\rho = \omega$, and the first step of the proof is completed.

Not all the P_k's, $0 \le k \le \sigma$, can be zero unless $C_0 = C_1 = \ldots = C_\sigma = 0$, since the determinant

$$\left| h^j \right|_1^\sigma = 1!2!3!\ldots\sigma! \qquad (3.50)$$

does not vanish. Hence there exists at least one finite α_k , $0 \le k \le \sigma$, and therefore a finite τ satisfying (3.37). Also, from (3.35), we have for $k = 1,2,\ldots,$

$$\begin{vmatrix} P_1 & & & \\ P_2 & & \left[h^j\right]_1^\sigma & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_\sigma & & & \\ P_{\sigma+k} & 1 \quad 2^{\sigma+k} & \ldots & \sigma^{\sigma+k} \end{vmatrix} = 0 \tag{3.51}$$

$$P_{\sigma+k} = \ell_{1,k}P_1 + \ell_{2,k}P_2 + \ldots + \ell_{\sigma,k}P_\sigma , \tag{3.52}$$

and

$$\alpha_{\sigma+k}/\omega \le \max_{1 \le j \le \sigma} \alpha_j/\omega , \; k = 1,2,\ldots, \tag{3.53}$$

or

$$\left[\alpha_{\sigma+k}/\omega - (\sigma+k)\right] < \max_{1 \le j \le \sigma} \left[\alpha_j/\omega - j\right] , \; k = 1,2,\ldots, \tag{3.54}$$

and $k_\psi \le \sigma$.

Now assume

$$y(n) \sim n^\theta \sum_{s=0}^{\infty} b_s n^{-s/\omega} , \; b_o \ne 0 , \tag{3.55}$$

$$y(n+\nu) \sim n^{\theta} \sum_{s=0}^{\infty} b_s n^{-s/\omega} \sum_{k=0}^{\infty} (s/\omega-\theta)_k (-)^k \nu^k / k! n^k \quad . \tag{3.56}$$

Substituting (3.56) in (3.1) gives

$$n^{\theta} \sum_{s=0}^{\infty} b_s n^{-s/\omega} \sum_{k=0}^{\infty} (s/\omega-\theta)_k (-)^k P_k / k! n^k = 0 \quad . \tag{3.57}$$

We can write

$$\sum_{k=0}^{\infty} (s/\omega-\theta)_k (-)^k P_k / k! n^k = n^{\tau} \sum_{m=0}^{\infty} G_m(\theta-s/\omega) n^{-m/\omega} \quad , \tag{3.58}$$

and so

$$\sum_{\nu=0}^{\sigma} C_\nu y(n+\nu) \sim n^{\theta+\tau} \sum_{m=0}^{\infty} n^{-m/\omega} \sum_{s=0}^{m} b_s G_{m-s}(\theta-s/\omega) = 0 \quad . \tag{3.59}$$

We must have

$$\sum_{s=0}^{m} b_s G_{m-s}(\theta-s/\omega) = 0 \ , \ m = 0,1,2,\ldots \quad . \tag{3.60}$$

If none of the roots, θ_h , of $G_o(\theta)$ differ by integral multiples of $1/\omega$, then the construction of the k_ψ solutions can proceed directly from (3.60). If some of the roots of $G_o(\theta)$ do differ by integral multiples of $1/\omega$, then the construction of solutions is done by the method of Frobenius, as follows.

Consider the difference equation

$$\sum_{\nu=0}^{\sigma} C_\nu \psi(n+\nu) = cn^{\theta+\tau} G_o(\theta) \tag{3.61}$$

where

$$\psi(n) \sim n^\theta \sum_{s=0}^{\infty} \beta_s(\theta) n^{-s/\omega} \ ; \ c \neq 0 \ . \tag{3.62}$$

Then the β_h's satisfy the recurrence relation

$$\sum_{s=0}^{m} \beta_s(\theta) G_{m-s}(\theta - s/\omega) = 0 \ , \ m = 1,2,\ldots, \ \beta_o(\theta) = c \ . \tag{3.63}$$

By putting $\theta = \theta_h$, $1 \le h \le k_\psi$, in (3.62)-(3.63) we obtain the previous k_ψ solutions, provided none of the θ_h's differ by integral multiples of $1/\omega$. We can solve the system (3.63) to obtain

$$\beta_s(\theta) = E_s(\theta) \Big/ \prod_{j=1}^{s} G_o(\theta - j/\omega) \ , \ s \ge 1 \ , \tag{3.64}$$

where E_s is a polynomial in θ . Now suppose that

$$\theta_2 = \theta_1 + L_1/\omega \ , \ \theta_3 = \theta_1 + L_2/\omega, \ \ldots \ , \theta_{\chi+1} = \theta_1 + L_\chi/\omega \ , \ \chi+1 \le k_\psi \ , \tag{3.65}$$

where L_h is a positive integer and

$$0 < L_1 < L_2 < \dots < L_\chi \; , \; L_o = 0 \quad , \tag{3.66}$$

but that $\theta_h - \theta_1$ is not an integral multiple of $1/\omega$ for $\chi+2 \le h \le k_\psi$.

Define

$$\overline{\beta}_s(\theta) = \beta_s(\theta) \prod_{j=1}^{L_\chi} G_o(\theta - j/\omega) \; , \; s \ge 0 \quad , \tag{3.67}$$

or

$$\overline{\beta}_s(\theta) = cE_s(\theta) \prod_{j=s+1}^{L_\chi} G_o(\theta - j/\omega) \; , \; 0 \le s \le L_\chi - 1 \quad . \tag{3.68}$$

Then $\overline{\beta}_s(\theta)$ is well defined when $\theta = \theta_h$, $1 \le h \le \chi+1$. We can write

$$\sum_{\nu=0}^{\sigma} C_\nu \overline{\psi}(n+\nu) = cn^{\theta+\tau} \prod_{j=0}^{L_\chi} G_o(\theta - j/\omega) \quad , \tag{3.69}$$

$$= cn^{\theta+\tau} I_\chi(\theta) \prod_{j=1}^{\chi+1} (\theta - \theta_j)^j \tag{3.70}$$

where

$$\overline{\psi}(n) = n^\theta \sum_{s=0}^{\infty} \overline{\beta}_s(\theta) n^{-s/\omega} \quad , \tag{3.71}$$

$I_\chi(\theta)$ a polynomial in θ .

We now differentiate (3.69) h times with respect to θ, set $\theta = \theta_{h+1}$, and use the fact that

$$\frac{\partial^r}{\partial\theta^r} \overline{\beta}_s(\theta)\Big|_{\theta=\theta_{h+1}} = 0 \ , \ 0 \le s \le L_h - L_r - 1 \ , \tag{3.72}$$

to obtain the $\chi+1$ F.S.S.

$$\varphi_{h+1}(n) = n^{\theta_1} \sum_{\ell=0}^{h} (-h)_\ell (-)^\ell (\ln n)^{h-\ell} g_{\ell,h}(n) n^{L_\ell/\omega}/\ell! \ , \tag{3.73}$$

$$g_{\ell,h}(n) = \sum_{s=0}^{\infty} b_{s,\ell,h} n^{-s/\omega} \ , \ b_{s,\ell,h} = \frac{\partial^\ell}{\partial\theta^\ell} \overline{\beta}_{s+L_h-L_\ell}(\theta)\Big|_{\theta=\theta_{h+1}} \ ,$$

$$0 \le h \le \chi \ . \tag{3.74}$$

Since

$$b_{o,h,h} = \frac{\partial^h}{\partial\theta^h} \overline{\beta}_o(\theta)\Big|_{\theta=\theta_{h+1}} \neq 0 \ , \tag{3.75}$$

the above solutions are clearly formally linearly independent, by A-VII.

Let (3.42), with $\rho = \omega$, be a F.S.S. of (3.1), and let $G_o(\theta)$ contain no multiple roots. We wish to show $\hat{\varphi}_k(n)$ can be expressed as a linear combination of the solutions already constructed. Denote by $d_{s,\ell}$ the coefficient of $n^{-s/\omega}$ in $q_\ell(n)$. We can write

$$\hat{\varphi}_k(n) = \sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial\theta^\ell}(n^\theta)\, q_\ell(\omega;n) n^{r_{k-\ell}/\omega} \quad , \tag{3.76}$$

and substituting this in (3.1), we find that θ and $d_{s,\ell}$ must be such that

$$\sum_{\ell=0}^{v} \frac{\Gamma(k+1-\ell)}{\Gamma(v+1-\ell)} n^{r_\ell/\omega} \sum_{m=0}^{\infty} n^{-m/\omega} \sum_{s=0}^{m} d_{s,k-\ell} \frac{\partial^{v-\ell}}{\partial\theta^{v-\ell}} G_{m-s}\left(\theta + \frac{r_\ell - s}{\omega}\right) = 0 \ ,$$

$$0 \le v \le k \ . \tag{3.77}$$

For $v = 0$, the above equation demands that $G_o(\theta) = 0$. Thus assume $\theta = \theta_1$, since the cases $\theta = \theta_h$, $2 \le h \le \chi+1$, can be treated similarly, as can the cases where θ belongs to any other group of roots of $G_o(\theta)$ which differ by integral multiples of $1/\omega$. If $\theta = \theta_\mu$, where θ_μ differs from no other root of $G_o(\theta)$ by an integral multiple of $1/\omega$, then the analysis is quite simple: we must have $k = 0$ in (3.76), and the result is one of the purely algebraic F.S.S. (3.41).

One can show by induction, using formula (3.77), that the r_h's cannot be arbitrary, but must satisfy the relations

$$0 = r_o = L_{j_o} < r_1 = L_{j_1} < r_2 = L_{j_2} < \dots < r_k = L_{j_k} \le L_\chi \quad , \tag{3.78}$$

and $k \le \chi$, $j_1 < j_2 < \dots$. (To avoid double subscripts on the r_h's , it is assumed that any r_h which corresponds to a $q_{k-h}(n) \equiv 0$ is deleted from the chain (3.78).)

Now let μ be the highest power of $\ln n$ which occurs in any of the functions $\varphi_h(n)$, $1 \le h \le \chi+1$. Note that the last term in $\varphi_{h+1}(n)$ is

$$n^{\theta_1+L_h/\omega} g_{h,h}(n) \ , \ g_{h,h}(\infty) \ne 0 \ . \tag{3.79}$$

Hence we may select a $\varphi_h(n)$ which contains the term $n^{\theta_1+r_k/\omega} g_{y,y}(n)$, and subtracting a constant multiple of this function from $\hat{\varphi}_k(n)$ leaves a series of the same kind whose lead term is either zero or $n^{\theta_1+r_k^*/\omega} q_o^*(n)$, $r_k^* < r_k$. But, by (3.78), r_k^* must be one of the L_j . Hence we can find another $\varphi_h(n)$ with lead term $n^{\theta_1+r_k^*}$ and subtract a constant multiple of this series from the above series so that the initial power of n in the lead term is again diminished, to $n^{\theta_1+r_k^{**}}$, $r_k^{**} < r_k^*$. This may be continued until $\hat{\varphi}_k(n)$ is reduced to

$$\hat{\varphi}_k(n) = f_k(n) + g_k(n) \ , \tag{3.80}$$

where $g_k(n)$ is a linear combination of the $\varphi_h(n)$ and

$$f_k(n) = n^{\theta_1} \sum_{\ell=1}^{\overline{k}} (\ln n)^{\ell} \overline{q}_{\ell}(\omega;n) n^{\overline{r}_{\overline{k}-\ell}/\omega} \ , \ \overline{k} = \max\left[k,\mu\right] \ , \tag{3.81}$$

also satisfies the difference equation. It is our intention to prove that any F.S.S. of the form (3.81) must be identically zero.

Denote by $\overline{d}_{s,\ell}$ the coefficient of $n^{-s/\omega}$ in $\overline{q}_{\ell}(n)$. Since $\overline{d}_{s,o} = 0$, the result of writing out (3.77) for (3.81) and $k = v = \overline{k}$ is

$$\sum_{\ell=0}^{\bar{k}-1} n^{\bar{r}_\ell/\omega} \sum_{m=0}^{\infty} n^{-m/\omega} \sum_{s=0}^{m} \bar{d}_{s,\bar{k}-\ell} \frac{\partial^{\bar{k}-\ell}}{\partial\theta^{\bar{k}-\ell}} G_{m-s}\left(\theta + \frac{\bar{r}_\ell - s}{\omega}\right)\Bigg|_{\theta=\theta_1} = 0 \quad , \qquad (3.82)$$

and setting to zero the coefficient of the highest power of n $(n^{\bar{r}_{\bar{k}-1}/\omega})$ gives

$$\bar{d}_{0,1} \frac{\partial}{\partial\theta} G_0(\theta+\bar{r}_{\bar{k}-1}/\omega)\Big|_{\theta=\theta_1} = 0 \qquad (3.83)$$

which is only possible, since $G_0(\theta)$ has no double roots, if $\bar{d}_{0,1} = 0$. Hence, by (3.7), $\bar{q}_1(n) \equiv 0$. The sum (3.81) becomes a sum from $\ell = 2$ to $\bar{k}$. Now write out (3.77) for $v = k-1 = \bar{k}-1$ to show that $\bar{q}_2(n) \equiv 0$. Eventually, we arrive at $\bar{q}_1(n) = \bar{q}_2(n) = \ldots = \bar{q}_{\bar{k}}(n) \equiv 0$, or $f_k(n) \equiv 0$, so $\hat{\varphi}_k(n)$ is a linear combination of the $\varphi_h(n)$, $1 \le h \le \chi+1$, and hence $k \le \mu$.

Thus, if none of the roots of $G_0(\theta)$ are equal, there exist exactly k_ψ algebraic logarithmic solutions of (3.1).

The same analysis can be conducted when some of the θ_h's are allowed to be equal, and the construction of the solutions in this case again is analogous to the procedure used for differential equations, see Ince (1956, Ch. XVI) or Forsyth (1902, Ch. II). Here we simply display the solutions, since their forms will be of importance later.

Let θ_h be as in (3.65)-(3.66), and, furthermore, let it be a root of $G_0(\theta)$ of multiplicity $\delta_h \ge 1$. Define

$$\delta_h^* = \delta_1 + \delta_2 + \cdots + \delta_h \, , \; \delta_o^* = 0 \quad . \tag{3.84}$$

The δ_h F.S.S. corresponding to θ_h are

$$\varphi_{h,j-\delta_{h-1}^*+1}(n) = n^{\theta_h} \sum_{v=0}^{j} \frac{(-j)_v(-)^v}{v!} (\ln n)^{j-v} \sum_{s=0}^{\infty} n^{-s/\omega} \frac{\partial^v}{\partial\theta^v} \overline{\beta}_s(\theta)\Big|_{\theta=\theta_h} ,$$

$$\delta_{h-1}^* \le j \le \delta_h^*-1 \quad . \tag{3.85}$$

Note that

$$\frac{\partial^j}{\partial\theta^j} \; \overline{\beta}_o(\theta)\Big|_{\theta=\theta_h} \neq 0 \; , \; j \ge \delta_{h-1}^* \quad , \tag{3.86}$$

since the multiplicity of $(\theta-\theta_h)$ in $\overline{\beta}_o(\theta)$ is exactly δ_{h-1}^*.

The linear independence of the above solutions can be shown by the same arguments used by Ince, p. 402, and the proof that there are always exactly k_ψ such F.S.S. requires the same type of reasoning as when the roots of $G_o(\theta)$ are simple, but the details are considerably more messy.

Theorem 3.1 is now applied to show that the F.S.S. constructed above represent asymptotically k_ψ linearly independent solutions of (3.1), and the proof of Theorem 3.3 is complete.

<u>Definition 3.6</u>

A Birkhoff set $\left\{y_h(n)\right\}$ for equation (3.1) is a <u>canonical</u> <u>set</u> if

$$y_h(n) = c_h M_h(n)\left[1 + o(1)\right] , \; n \to \infty , \; c_h \neq 0 , \tag{3.87}$$

where

$$M_h(n) = e^{Q_h(n)} n^{\theta_h} (\ln n)^{p_h} , \tag{3.88}$$

p_h a positive integer, and $M_h = M_j$ for $n = 1,2,3,\ldots$, if and only if $h = j$.

By the construction (3.85) and Theorem 3.1, every equation (3.1) has a canonical set, and so does the equation adjoint to (3.1). No two members of a canonical set display the same asymptotic behaviour as $n \to \infty$, so not every Birkhoff set is canonical, e.g., $\left\{1, n^2, n^2+n\right\}$ is a Birkhoff set for

$$y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = 0 \tag{3.89}$$

but not a canonical set, as is $\left\{1, n, n^2\right\}$ or $\left\{1, n+1, n^2+n\right\}$.

Also, let S denote a subset of a canonical set, all the members of which correspond to the same $Q(n)$ and to a group of θ_h's which differ by integral multiples of $1/\omega$. Then S contains a smallest member $z_1(n)$ and a largest member $z_2(n)$, i.e.,

$$\lim_{n \to \infty} \frac{z_1(n)}{y_1(n)} = \lim_{n \to \infty} \frac{y_2(n)}{z_2(n)} = 0 , \tag{3.90}$$

where $y_1(n)$ is any member of S other than z_1 and $y_2(n)$ is any member of S other than z_2 .

For the construction (3.85), z_1 and z_2 correspond to F.S.S. $\varphi_{1,1}(n)$ and $\varphi_{\chi+1,\delta_{\chi+1}}(n)$, respectively, and

$$z_1(n) = c_1 n^{\theta_1}\left[1 + o(1)\right] \quad , \tag{3.91}$$

$$z_2(n) = c_2 n^{\theta_{\chi+1}} (\ln n)^{\delta_{\chi+1}-1}\left[1 + o(1)\right] \quad . \tag{3.92}$$

(In the general case, a term $e^{Q(n)}$ will appear on the right-hand side of (3.91)-(3.92).)

<u>Theorem 3.4</u>

Let $\left\{y_h(n)\right\}$ be a Birkhoff set for (3.1),

$$y_h(n) \sim e^{Q_h(\rho;n)} s_h(\rho;n) \ , \ 1 \le h \le \sigma \quad . \tag{3.93}$$

Then:

i)
$$D(n) \sim e^{Q(\omega;n)} s(\omega;n) \quad , \tag{3.94}$$

where in $Q(n)$

$$\left.\begin{aligned} &\mu_0 = -K_\sigma/\omega \ , \ e^{\mu_1} = (-)^\sigma e^{K_\sigma/\omega}/c_{0,\sigma} \ , \ \mu_2 = -\omega c_{1,\sigma}\Big/\left[(\omega-1)c_{0,\sigma}\right] , \\ &\mu_3 = \frac{\omega}{(\omega-2)}\left[(c_{1,\sigma}^2/2c_{0,\sigma}^2) - (c_{2,\sigma}/c_{0,\sigma})\right] , \ldots , \end{aligned}\right\} \tag{3.95}$$

where $c_{j,\sigma}$ is as in (3.2), and $s(\omega;n)$ is free from logarithms;

ii) $$Q(\omega;n) = \sum_{h=1}^{\sigma} Q_h(\rho;n) \quad ; \tag{3.96}$$

iii) $$y_h^*(n) = T_h(n)/D(n) \sim e^{-Q_h(n)} s_h^*(\rho;n) \ , \ 1 \le h \le \sigma \ , \tag{3.97}$$

are a set of Birkhoff solutions for the equation adjoint to (3.1).

Proof:

The proofs of i) and iii) are purely computational, i) following from the difference equation for $D(n)$, (A.15). No logarithms appear in $s(\omega;n)$ since, by Theorem 3.2, the F.S.S. of a first order difference equation can never contain logarithms.

To prove iii), note that from A-VI,

$$\Big[T_h(n+j-1)/D(n+j-1)\Big]_1^{\sigma} \times \Big[y_j(n+h-1)\Big]_1^{\sigma} = \Big[e_{jh}\Big]_1^{\sigma} \ , \tag{3.98}$$

where

$$e_{jh} = \begin{cases} 1 \ , \ j = h \ , \\ 0 \ , \ h > j \ , \end{cases} \tag{3.99}$$

and so

$$\left| y_h^*(n+j-1) \right|_1^\sigma = D(n)^{-1} . \tag{3.100}$$

Since the $y_h^*(n)$ can be represented by the F.S. (3.97) which are formally linearly independent, by (3.100), it follows that they are Birkhoff solutions.

We close this chapter with a theorem on exponential sums.

Theorem 3.5

Let

$$S \equiv S(n) = \sum_{k=0}^{n} f(k) \ , \ n \geq 0 \ , \tag{3.101}$$

where

$$f(n) = e^{Q(n)} v(n) \left[1 + o(1) \right] \ , \ n \to \infty \ , \tag{3.102}$$

$$v(n) = n^{\theta} (\ln n)^{r} \ , \tag{3.103}$$

r is a non-negative integer, and $Q(n)$ is given by (3.5).

Let μ^* be the first non-zero element in the sequence $\{\text{Re } \mu_h\}$, $h = 0,1,2,\ldots,\rho$ and $\mu^* = 0$ if all the $\text{Re } \mu_h$ are zero.

Then:

i) if $\mu^* < 0$, we can write

$$A = \sum_{k=0}^{\infty} f(k) \quad , \tag{3.104}$$

and

$$S(n) = A + O\left\{e^{Q(n+1)} n v(n)\right\} \quad , \; n \to \infty \; ; \tag{3.105}$$

ii) if $\mu^* > 0$,

$$S(n) = O\left\{e^{Q(n)} n v(n)\right\} \quad , \; n \to \infty \quad ; \tag{3.106}$$

iii) if $\mu^* = 0$,

$$S(n) = \begin{cases} O\left\{n v(n)\right\} \; , \; \text{Re}\;\theta > -1 \; ; \\ O\left\{(\ln n)^{r+1}\right\} \; , \; \text{Re}\;\theta = -1 \; ; \\ O(1) \; , \; \text{Re}\;\theta < -1 \; . \end{cases} \tag{3.107}$$

Proof:

We can write

$$f(n) = c e^{Q(n)} v(n)\left[1 + K(n)\right] \; , \; n \geq 1 \; , \tag{3.108}$$

where $K(n) \to 0$ as $n \to \infty$.

For $\mu^* < 0$, the series converges absolutely. Define

$$T(n) = \sum_{k=n+1}^{\infty} f(k) \quad . \tag{3.109}$$

We have

$$\left| e^{-Q(n+1)} T(n) \right| \le |c| \sum_{k=n+1}^{\infty} g(k) k^{-\nu} |1 + K(k)| \tag{3.110}$$

$$\le c' \sum_{k=n+1}^{\infty} g(k) k^{-\nu} \quad , \tag{3.111}$$

where

$$g(k) = |v(k)| \; k^{\nu} e^{\text{Re } Q(k) - \text{Re } Q(n+1)} \tag{3.112}$$

and we take ν real and > 1 .

Now

$$\frac{dg(x)}{dx} = g(x) \left\{ \text{Re } Q'(x) + \frac{1}{x}\left(\frac{r}{\ln x} + \theta + \nu\right) \right\} \quad . \tag{3.113}$$

But $g(x) \neq 0$ for $x > 1$, and any zero of the quantity in brackets above cannot depend on n . Since $g(x) \to 0$ as $x \to \infty$, it follows that g is monotone decreasing, $x > x_0$, or, for n sufficiently large

$$g(k) \le (n+1)^{\nu} |v(n+1)| \le Mn^{\nu} |v(n)| \; , \; k \ge n+1 \; , \tag{3.114}$$

and so

$$\left| e^{-Q(n+1)} T(n) \right| \le \bar{M} \, |v(n)| \, n^{\nu} \sum_{k=n+1}^{\infty} k^{-\nu} \tag{3.115}$$

$$\le \bar{M} \, |v(n)| \, n^{\nu} \int_{n}^{\infty} x^{-\nu} dx = \frac{\bar{M} n v(n)}{(\nu-1)} \quad , \tag{3.116}$$

which gives i).

The result ii) follows by a simple majorization, while for iii) it suffices to consider

$$\sum_{k=1}^{n} |v(k)| \le (\ln n)^{r} \sum_{k=1}^{n} k^{\mathrm{Re}\,\theta} \quad . \tag{3.117}$$

When $\mathrm{Re}\,\theta \ge 0$, the series on the right is easily bounded. Otherwise we use

$$\sum_{k=1}^{n} k^{\mathrm{Re}\,\theta} < 1 + \int_{1}^{n} x^{\mathrm{Re}\,\theta} dx \ , \ \mathrm{Re}\,\theta < 0 \quad . \tag{3.118}$$

IV.

By using the asymptotic theory of the last chapter, we can now apply the general theory of computation by backward recursion, as developed in Chapter II, to difference equations of the kind (3.1).

Throughout this chapter, we assume the difference equation in question to be (3.1).

Theorem 4.1

There exists a canonical set $\left\{\varphi_h^*(n)\right\}$ for the equation adjoint to (3.1) and an integer $u \geq 1$ such that

$$\lim_{m \to \infty} \left| \varphi_h^*(m) \big/ \varphi_1^*(m) \right| = \begin{cases} c_h \neq 0 \;, \; 1 \leq h \leq u \;, \\ 0 \;, \; u+1 \leq h \leq \sigma \;. \end{cases} \tag{4.1}$$

Furthermore, if $u = 1$ then for some n^*, $0 \leq n^* \leq \sigma-1$,

$$\lim_{m \to \infty} \Lambda_n(m) \big/ \Lambda_{n^*}(m) = y(n) \;, \; n \geq n^* \;, \tag{4.2}$$

exists, is not identically zero and satisfies (3.1), while, if $u = 2$, then m_1, m_2 can be chosen so that (2.35) holds for some solution, $y_1(n)$, of (3.1). Furthermore, $y_1(n)$, the function to which the algorithm converges, is independent of the particular set $\varphi_h^*(n)$.

Proof:

Equation (4.1) is obvious, since, clearly, the absolute value of the ratio of any two canonical solutions must either approach zero, a constant, or infinity. Let $\left\{y_h(n)\right\}$ be that (unique) fundamental set for the original equation (3.1) which is obtained by letting $T_h(n)/D(n) = \varphi_h^*(n)$ in (A.18), then solving for $y_h(n)$ for $n \geq \sigma-1$, and finally using (3.1) to compute $y_h(n)$ for all $n \geq 0$, $1 \leq h \leq \sigma$. We have

$$\Lambda_n(m) = \sum_{h=1}^{\sigma} \varphi_h^*(m) y_h(n) \tag{4.3}$$

and the statement for $u = 1$ follows immediately. Note that $\Lambda_{n^*}(m)$ cannot be zero for more than $\sigma-1$ consecutive values of n^*, for it satisfies (3.1) and is not identically zero. Since $\Lambda_n(m)$ is fully determined by the conditions (2.2), it is independent of $\varphi_h^*(n)$, and so is $y_1(n)$.

Now

$$\Delta(m_1) = \left|R_h(m_j)\right|_1^2 = \begin{vmatrix} 1 & \alpha e^{iQ(m_1)} m_1^{i\theta}\left(1 + o(1)\right) \\ 1 & \alpha e^{iQ(m_2)} m_2^{i\theta}\left(1 + o(1)\right) \end{vmatrix} , \tag{4.4}$$

where Q, θ are real, $|\alpha| = c_1$, and μ_o in Q is zero. Also we may assume $-\pi \leq \mu_1 < \pi$, or else replace iQ by $iQ \pm 2r\pi i$, r a suitably chosen integer.

One finds

$$|\Delta(m_1)| = 2c_1 \left| \sin \left\{ \frac{Q(m_2)-Q(m_1)}{2} + \frac{\theta}{2} (\ln m_2 - \ln m_1) \right\} \right| + o(1) \quad . \quad (4.5)$$

First, assume $Q(m) \not\equiv$ const., $\mu_1 = 0$, so

$$Q(m) = \sum_{r=k}^{\rho} \mu_r m^{1+\frac{1-r}{\rho}} , \; 2 \le k \le \rho , \; \mu_k \ne 0 \quad . \quad (4.6)$$

Let

$$\left. \begin{aligned} m_2 &= \left[m_1 + K m_1^{\frac{k-1}{\rho}} \right] = m_1 + K m_1^{\frac{k-1}{\rho}} + O(1) , \\ K &= \left| \pi \Big/ \left(\mu_k \left(1 + \frac{1-k}{\rho} \right) \right) \right| , \end{aligned} \right\} \quad (4.7)$$

and $[x]$ means the largest integer not greater than x . We have

$$\left. \begin{aligned} \frac{Q(m_2)-Q(m_1)}{2} &= \tfrac{1}{2} \sum_{r=k}^{\rho} \mu_r \left\{ m_2^{1+\frac{1-r}{\rho}} - m_1^{1+\frac{1-r}{\rho}} \right\} \\ &= \tfrac{1}{2} K \sum_{r=k}^{\rho} \mu_r \left(1 + \frac{1-r}{\rho} \right) m_1^{\frac{k-r}{\rho}} \left(1 + O(m_1^{-1/\rho}) \right) \\ &= \frac{\mu_k}{2} K \left(1 + \frac{1-k}{\rho} \right) + O(m_1^{-1/\rho}) = \pm \frac{\pi}{2} + O(m_1^{-1/\rho}) , \end{aligned} \right\} \quad (4.8)$$

so

$$\lim_{m_1 \to \infty} |\Delta(m_1)| = 2c_1 \ , \tag{4.9}$$

which assures us that (2.34) has a unique solution π_1, π_2, and also that (2.35) holds.

Next, we note that if Q is constant, we cannot have $\theta = 0$, else φ_2^* is proportional to φ_1^* . Thus, let

$$m_2 = \left[m_1 e^{\pi/|\theta|} \right] = m_1 e^{\pi/|\theta|} + O(1) \ . \tag{4.10}$$

Then (4.9) again holds.

Lastly, if $\mu_1 \neq 0$ in Q , let $m_2 = m_1 + r$, r a positive integer.

$$|\Delta(m_1)| = c_1 \left| e^{ir\mu_1} - 1 + o(1) \right| \ . \tag{4.11}$$

Since r can be chosen so that $e^{ir\mu_1} \neq 1$, (4.9) again follows, and, by Theorem 2.4, the proof is complete.

<u>Theorem 4.2</u>

Let there exist a canonical set $\left\{ y_h(n) \right\}$ for (3.1) such that one of its members, say $y_1(n)$, has the following property:

$$\lim_{n \to \infty} y_1(n) n^{\kappa} / y_h(n) = 0 , \quad 2 \le h \le \sigma , \tag{4.12}$$

for all κ . Then

$$\lim_{m \to \infty} \Lambda_n(m)/\Lambda_{n^*}(m) = y_1(n)/y_1(n^*) \tag{4.13}$$

for all $n \ge n^*$ and some n^* , $0 \le n^* \le \sigma-1$.

If, in addition to (4.12) we are given the series

$$1 = \sum_{k=0}^{\infty} L_k y_1(k) \quad , \tag{4.14}$$

where

$$\lim_{k \to \infty} k^{\kappa} L_k y_1(k) = 0 \tag{4.15}$$

for all κ , then the computation of $y_1(n)$ by backward recursion based on (3.1) and (4.14) converges.

Proof:

Note that y_1 , if it exists, is unique, apart from a constant multiple, by A-VII. Since

$$y_h(n) = c_h e^{Q_h(n)} n^{\theta_h} (\ln n)^{p_h} \bigl(1 + o(1)\bigr) \tag{4.16}$$

we have, by (3.97)

$$T_h(m)/T_1(m) = \bigl(y_1(m)/y_h(m)\bigr)\, O(m^{\alpha_h}) \ , \ 2 \le h \le \sigma \ , \tag{4.17}$$

for some real α_h , and the first part of (2.12) follows.

Now, from (4.15), we deduce

$$|L_k| \le M e^{-\mathrm{Re}\ Q_1(k)} k^{\beta} \ , \ k > 0 \ , \ \beta = -\alpha_h - 2 - \mathrm{Re}\ \theta_1 \ , \tag{4.18}$$

so

$$|S_h(m)| \le M_h \sum_{k=1}^{m} e^{\mathrm{Re}\ \left(Q_h(k) - Q_1(k)\right)} k^{\mathrm{Re}\ \theta_h + \beta} (\ln k)^{p_h} \ , \tag{4.19}$$

and we may apply ii) of Theorem 3.5, since (4.12) implies that $\mu_{o,1} < \mu_{o,h}$. The result is that we can determine constants K_h so that, for $m > \bar{m}$,

$$|S_h(m)| \le K_h y_h(m) m^{-\alpha_h - 1} \big/ y_1(m) \ . \tag{4.20}$$

Thus, by (4.17)

$$\lim_{m \to \infty} R_h S_h = 0 \ , \ 2 \le h \le \sigma \tag{4.21}$$

and the proof is complete.

The condition (4.12) is rather stringent, and for $\sigma = 2$ or 3 it can be weakened considerably. We have, in fact,

Theorem 4.3

Let $\sigma = 2$ or 3 , and let $\left\{y_1(n), y_2(n)\right\}$ be a fundamental set for (3.1) if $\sigma = 2$ and let $\left\{y_1(n), y_2(n), y_3(n)\right\}$ be a canonical set for (3.1) if $\sigma = 3$.

Let

$$\lim_{n\to\infty} n^{L(\sigma)}(\ln n)^{\sigma-2} y_1(n)\Big/y_h(n) = 0 \ , \ 2 \le h \le \sigma \ , \tag{4.22}$$

where $L(\sigma)$ is a positive number depending only on σ , $L(2) = 0$, $L(3) = 1$.

Then for some n^* , $0 \le n^* \le \sigma-1$,

$$\lim_{m\to\infty} \Lambda_n(m)/\Lambda_{n^*}(m) = y_1(n)/y_1(n^*) \ , \ n \ge n^* \ . \tag{4.23}$$

Let, in addition to the above, (4.14) hold, with

$$L_k = O\left(e^{Q(k)} k^{\alpha}\right) \tag{4.24}$$

for some $Q(k)$, α such that

$$e^{Q(k)} k^{L(\sigma)+\alpha+1} (\ln k)^{\sigma-1} y_1(k) = o(1) \ . \tag{4.25}$$

Then the calculation of $y_1(n)$ by backward recursion based on (3.1) and (4.14) converges.

Proof:

We have, from Theorem 3.5,

$$S_h(m) = \begin{cases} A_h + o(1) \, , \sum L_k y_h(k) \text{ convergent,} \\ O\left(e^{Q(m)} m^{\alpha+1} \ln m \; y_h(m)\right) \, , \text{ otherwise .} \end{cases} \tag{4.26}$$

It remains only to show that

$$R_h(m) = O\left(m^{L(\sigma)} \ln m^{\sigma-2} y_1(m) \big/ y_h(m)\right) \, , \; 2 \le h \le \sigma \, . \tag{4.27}$$

Then (2.12), and hence the Theorem, will follow from (4.22) and (4.25). For $\sigma = 2$, (4.27) is obvious, since

$$T_h(m) = (-)^{h-1} y_{3-h}(m+1), \; h = 1,2 \; . \tag{4.28}$$

Let, then, $\sigma = 3$. It suffices to consider only R_2 . Now $\{y_h(n)\}$ is canonical, so, by the construction of Theorem 3.3, we can write

$$y_h(n) = e^{P_h(n)} \left[1 + \frac{d_{1,h}}{\ln n} + \frac{d_{2,h}}{(\ln n)^2} + \cdots \right] \, , \tag{4.29}$$

$$P_h(n) = Q_h(n) + \theta_h \ln n + p_h \ln(\ln n) + \ln c_h \; . \tag{4.30}$$

Since

$$\left[\ln(n+1)\right]^{-j} = \left[\ln n\right]^{-j}\left[1 - \frac{j}{n \ln n} + \cdots\right] \qquad (4.31)$$

we have

$$y_h(n+1)/y_h(n) = e^{\Delta_n P_h(n)}\left[1 + O\left(\frac{1}{n(\ln n)^2}\right)\right] \quad , \qquad (4.32)$$

where Δ_n is the forward difference operator, Milne-Thomson (1960). Thus

$$R_2(n-1) = - y_1(n)A(n)\Big/\left[y_2(n)B(n)\right] \quad , \qquad (4.33)$$

$$A(n) = 1 - e^{\Delta_n \left(P_1(n)-P_3(n)\right)}\left[1 + O\left(\frac{1}{n(\ln n)^2}\right)\right] \quad , \qquad (4.34)$$

$$\left.\begin{aligned} B(n) &= 1 - e^{\Delta_n \left(P_2(n)-P_3(n)\right)}\left[1 + O\left(\frac{1}{n(\ln n)^2}\right)\right] \\ &= 1 - e^{\Delta_n \left(Q_2(n)-Q_3(n)\right)}\left[1 + \frac{(\theta_2-\theta_3)}{n} + \frac{(p_2-p_3)}{n \ln n} + O\left(\frac{1}{n(\ln n)^2}\right)\right] . \end{aligned}\right\} \qquad (4.35)$$

If $\mu_{o1} \neq \mu_{o2}$, then (4.22) shows that $\mu_{o1} < \mu_{o2}$ and so $R_2(m) \to 0$. Hence assume $\mu_{o1} = \mu_{o2}$. If $\mu_{o1} \neq \mu_{o3}$, then $\mu_{o1} < \mu_{o3}$ and the leading term of both $A(n), B(n)$ is unity, so again, by (4.22), $R_2(m) \to 0$. Hence assume $\mu_{o1} = \mu_{o3}$. Then $A(n)$ is bounded, and $B(n)$ is asymptotically smallest in the case where $Q_2 = Q_3$, $\theta_2 = \theta_3$. Then it approaches zero as

$1/(n \ln n)$, since we cannot have, furthermore, $p_2 = p_3$. Thus (4.27) follows, and R_h , $R_h S_h$, $h = 2,3$, approach zero. This proves the theorem.

We conjecture that Theorem 4.3 is true for all σ , i.e., that there exists an $L(\sigma)$ for (4.22) which will insure (4.23) and (2.7).

For $\sigma = 2$, (4.23) follows from a result Gautschi (1961) proved for a second order difference equation with arbitrary coefficients, and also for this equation, Olver (1964) has determined other conditions on y_1, y_2 which will guarantee the convergence of Miller's algorithm, based on (4.14), to $y_1(n)$.

We now present several examples of applications of the previous material.

Let

$$y(n) = \int_0^\infty e^{-t^\sigma - P(t)} t^n dt \ , \ n = 0,1,2,\dots \ , \qquad (4.36)$$

$$P(t) = \sum_{r=1}^{\sigma-1} a_r t^r \ , \ \sigma \geq 2 \ . \qquad (4.37)$$

A single integration by parts shows that $y(n)$ satisfies the difference equation

$$y(n) - (n+1)^{-1} \sum_{\nu=1}^{\sigma} \nu a_\nu y(n+\nu) = 0 \ , \ a_\sigma = 1 \ . \qquad (4.38)$$

From de Bruijn (1961 , p. 119) we have

$$\left. \begin{aligned} & y(n) \sim \frac{\Gamma\left(\frac{n+1}{\sigma}\right)}{\sigma} \exp\left\{\mu_2 n^{1-\frac{1}{\sigma}} + \mu_3 n^{1-\frac{2}{\sigma}} + \cdots + \mu_\sigma n^{\frac{1}{\sigma}}\right\} q(\sigma,n), \ n \to \infty \ , \\ & \mu_2 = - a_{\sigma-1} \sigma^{\frac{1}{\sigma}-1} \ , \\ & \mu_3 = \left[\frac{a_{\sigma-1}^2}{2}\left(1 - \frac{1}{\sigma}\right)^2 - a_{\sigma-2}\right] \sigma^{\frac{2}{\sigma}-1} \ . \end{aligned} \right\} \qquad (4.39)$$

When $\sigma = 2$, $y(n)$ is essentially the parabolic cylinder function $U(n+\frac{1}{2}, a_1/\sqrt{2})$, the theory and computation of which are discussed by Miller (1964).

Now, by Birkhoff (1930, section 2) , we know there exists a fundamental set for (4.38), $\{y_h(n)\}$, where y_h has as an asymptotic expansion the F.S. on the right of (4.39) with n replaced by $ne^{2\pi i(h-1)}$, $1 \le h \le \sigma$. This gives σ F.S.S., so there are no more. Furthermore, by Definition 3.6, $\{y_h(n)\}$ is canonical. We can identify $y(n)$ with $y_1(n)$.

We have

$$|y_1(n)/y_{h+1}(n)| = \exp\left\{2|a_{\sigma-1}|(n/\sigma)^{1-1/\sigma} f_h(\gamma)\left[1 + O(n^{-1/\sigma})\right]\right\} \ , \qquad (4.40)$$

$$f_h(\gamma) = \sin(\gamma - h\pi/\sigma)\,\sin(h\pi/\sigma)\;,\; \gamma = \arg a_{\sigma-1}\;. \tag{4.41}$$

For $|\gamma| < \pi$, f_h has zeros only at the points $h\pi/\sigma$ and $h\pi/\sigma - \pi$, so f_h is of one sign for γ between these points. Since

$$f_h(0) = -\sin^2(h\pi/\sigma) < 0 \;, \tag{4.42}$$

we have

$$f_h(\gamma) < 0\;,\; h\pi/\sigma - \pi < \gamma < h\pi/\sigma \tag{4.43}$$

or

$$f_h(\gamma) < 0\;,\; -\pi/\sigma < \gamma < \pi/\sigma\;,\; 1 \le h \le \sigma-1 \;. \tag{4.44}$$

and so, for these values of $\arg a_{\sigma-1}$, condition (4.12) of Theorem 4.2 is fulfilled.

Let

$$e^{P(t)} = \Gamma(1+1/\sigma) \sum_{k=0}^{\infty} L_k t^k \;. \tag{4.45}$$

Since the left-hand side is an entire function of order $\sigma-1$, we have, by Boas (1954, p. 10) ,

$$|L_k| < C\, k^{-k/\sigma}\;,\; k \ge 1 \;, \tag{4.46}$$

so

$$k^{\kappa}|L_k||y_1(k)| \le M_1 k^{\kappa+1/\sigma-\frac{1}{2}} \exp\left\{-\frac{k}{\sigma}(1+\ln\sigma)\left[1+M_2k^{-1/\sigma}\right]\right\} = o(1) \quad , \qquad (4.47)$$

for all κ .

Also

$$\sum_{k=0}^{\infty} L_k y_1(k) = \sum_{k=0}^{\infty} L_k y(k) = 1 \quad . \qquad (4.48)$$

Thus, by Theorem 4.2, the computation of the integral (4.36) by backward recursion based on (4.38) and (4.48) converges when $|\arg a_{\sigma-1}| < \pi/\sigma$.

Our next example is a class of hypergeometric functions. (For notation, see Erdélyi et al, 1953; v. I.) Let

$$y(n) = \frac{(-)^n\lambda^n(\beta+1)_n}{(\gamma)_{2n}} \frac{\prod_{j=1}^{Q+1}(a_j)_n}{\prod_{j=1}^{Q}(b_j)_n} \; {}_{Q+2}F_{Q+1}\left(\begin{matrix} n+a_1,\ldots,n+a_{Q+2} \\ n+b_1,\ldots,n+b_Q,2n+\gamma+1 \end{matrix}\middle|\lambda\right) \quad , \qquad (4.49)$$

where Q,n are non-negative integers, $\beta+1$, γ , a_i , b_j are complex constants, $(a_{Q+2} = \beta+1)$, none of which are negative integers or zero, and λ is a complex variable, finite, $\neq 0$, $|\arg(1-\lambda)| < \pi$.

Then (Wimp (1966)) $y(n)$ satisfies the difference equation

$$\sum_{\nu=0}^{Q+2} \left[M_\nu + N_\nu/\lambda \right] y(n+\nu) = 0 \quad , \tag{4.50}$$

where $M_0 = 1$, $N_0 = N_{Q+2} = 0$, and

$$M_\nu = \frac{(-)^Q (2n+\gamma)_{Q+3} \prod_{j=1}^{Q+2} (n+\gamma+\nu-a_j)}{\Gamma(Q+3-\nu)(2n+\gamma+\nu)_{\nu+1} \prod_{j=1}^{Q+2} (n+a_j)}$$

$$\times \; {}_{Q+4}F_{Q+3}\left(\begin{matrix} \nu-Q-2, 2n+\gamma+\nu, n+\gamma+\nu+1-a_1, \ldots, n+\gamma+\nu+1-a_{Q+2} \\ 2n+\gamma+2\nu+1, n+\gamma+\nu-a_1, \ldots, n+\gamma+\nu-a_{Q+2} \end{matrix} \middle| 1 \right) \quad , \tag{4.51}$$

for $1 \le \nu \le Q+2$, and

$$N_\nu = \frac{(-)^Q (2n+\gamma)_{Q+3} \prod_{j=1}^{Q} (n+\gamma+\nu+1-b_j)}{\Gamma(Q+2-\nu)(2n+\gamma+\nu+1)_\nu \prod_{j=1}^{Q+2} (n+a_j)}$$

$$\times \; {}_{Q+2}F_{Q+1}\left(\begin{matrix} \nu-Q-1, 2n+\gamma+\nu+1, n+\gamma+\nu+2-b_1, \ldots, n+\gamma+\nu+2-b_Q \\ 2n+\gamma+2\nu+1, n+\gamma+\nu+1-b_1, \ldots, n+\gamma+\nu+1-b_Q \end{matrix} \middle| 1 \right) \quad , \tag{4.52}$$

for $1 \le \nu \le Q+1$.

Note that (4.51)-(4.52) are terminating hypergeometric functions, and for any value of ν , they are rational functions of n . Thus (4.50) is of the form (3.1) with $\omega = 1$.

A result of Wimp and Luke (1962, the Corollary on p. 7, with $m = 0$ and then $\omega = 0$) shows that

$$1 = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} y(k) \quad . \tag{4.53}$$

Since

$$M_\nu = \frac{(-Q-2)_\nu}{\nu!} + O(n^{-1}) \ , \ n \to \infty \quad , \tag{4.54}$$

$$N_\nu = \frac{4(-Q)_{\nu-1}}{\Gamma(\nu)} + O(n^{-1}) \ , \ n \to \infty \tag{4.55}$$

there are two linearly independent solutions of (4.50) with the behaviour

$$y_1(n) \sim (-)^n \left[\zeta_1(\lambda)\right]^n s_1(n) \ , \ n \to \infty \quad , \tag{4.56}$$

$$y_2(n) \sim (-)^n \left[\zeta_2(\lambda)\right]^n s_2(n) \ , \ n \to \infty \quad , \tag{4.57}$$

see Wimp (1966), where

$$\zeta_1(\lambda) = \left[1 - \sqrt{1-\lambda}\right]^2 / \lambda \ , \ \zeta_2(\lambda) = \left[1 + \sqrt{1-\lambda}\right]^2 / \lambda \quad , \tag{4.58}$$

and we define

$$\sqrt{1-\lambda} = \left|1-\lambda\right|^{\frac{1}{2}} e^{\frac{i}{2}\left[\arg(\lambda-1)-\pi\right]} \ , \ 0 < \arg(\lambda-1) < 2\pi \quad . \tag{4.59}$$

It can be verified that ζ_1 maps the λ plane cut along the real axis from 1 to ∞, (i.e., $|\arg(1-\lambda)| < \pi$) onto the interior of the unit circle $|\lambda| < 1$, while ζ_2 maps the cut plane onto $|\lambda| > 1$. Thus for the values of $\arg\lambda$ considered,

$$|\zeta_1(\lambda)| < 1 \ , \ |\zeta_2(\lambda)| > 1 \ , \tag{4.60}$$

and so (4.12) holds not only for $h = 2$, but also when y_h is any purely algebraic-logarithmic solution of (4.50). We now proceed to show that all the remaining members of any canonical set $\{y_h(n)\}$ to which y_1, y_2 belong are purely algebraic-logarithmic.

Consider

$$\Omega_M(m) = \sum_{\nu=m}^{Q+2} (-\nu)_m M_\nu \ , \ 0 \le m \le Q+2 \ , \tag{4.61}$$

$$\Omega_N(m) = \sum_{\nu=m}^{Q+1} (-\nu)_m N_\nu \ , \ 0 \le m \le Q+1 \ . \tag{4.62}$$

Using (4.51) in (4.61), interchanging the order of summation and evaluating a ${}_2F_1$ of unit argument, we have.

$$\Omega_M(m) = \frac{(-)^Q m!(2n+\gamma)_{Q+3}}{\prod_{j=1}^{Q+2} (n+a_j)\Gamma(Q+3-m)} \Delta_x^{(Q+2-m)} \left\{ \frac{\prod_{j=1}^{Q+2} (a_j-\gamma-m-x)}{(2x+\gamma+m)_{m+1}} \right\}_{x=n} \tag{4.63}$$

where $\Delta_x^{(Q+2-m)}$ is the forward difference operator, see Milne-Thomson (1960). But the quantity in brackets above may be decomposed into a polynomial of degree $Q+1-m$ in x, the indicated difference of which is zero, and a sum of terms

$$\sum_{j=0}^{m} d_j/(2x+\gamma+m+j) \quad . \tag{4.64}$$

The d_j may be calculated by multiplying the bracketed quantity in (4.63) by $2x+\gamma+m+j$ and letting $x \to -(\gamma+m+j)/2$. Then substituting (4.64) in (4.63) gives

$$\Omega_M(m) = \frac{(-)^{m+Q}(2n+\gamma)_{Q+3}}{2^{Q+3}\prod_{j=1}^{Q+2}(n+a_j)} \sum_{r=0}^{m} \frac{(-m)_r}{r!} \frac{\Gamma\left(n+\frac{\gamma+m+r}{2}\right)\prod_{j=1}^{Q+2}(\gamma+m-r-2a_j)}{\Gamma\left(n+\frac{\gamma-m+r}{2}+Q+3\right)} \tag{4.65}$$

and an order estimate for this sum is easily obtained by using

$$\Gamma(n+\alpha)/\Gamma(n+\beta) = n^{\alpha-\beta}\left[1 + O(n^{-1})\right] \quad . \tag{4.66}$$

The result is

$$\Omega_M(m) = (-)^{Q+m} n^{m-Q-2} \Delta_\gamma^{(m)} \left\{\prod_{j=1}^{Q+2}(\gamma-2a_j)\right\}\left[1 + O(n^{-1})\right] \quad . \tag{4.67}$$

Similarly, one finds that

$$\Omega_N(m) = \begin{cases} 4(-)^{Q+m} n^{m-Q} \Delta_\gamma^{(m)} \left\{ \prod_{j=1}^{Q} (\gamma+2-2b_j) \right\} \left[1+O(n^{-1})\right], & 0 \le m \le Q , \\ O(1) , & m = Q+1 . \end{cases} \tag{4.68}$$

Next, we have

$$\nu^k = \sum_{m=0}^{k} \frac{(-k)_m (-\nu)_m}{m!} B_{k-m}^{(-m)} , \tag{4.69}$$

the B's being Bernoulli numbers, see Nörlund (1954, p. 150).

We may use the above results to finally arrive at the estimate for P_k in Theorem 3.3,

$$n^{-k} P_k = \begin{cases} 4\lambda^{-1}(-)^Q n^{-Q} \Delta_\gamma^{(k)} \left\{ \prod_{j=1}^{Q} (\gamma+2-2b_j) \right\} \left[1+O(n^{-1})\right], & 0 \le k \le Q , \\ O(n^{-Q-1}) , & Q+1 \le k \le Q+2 . \end{cases} \tag{4.70}$$

Thus, in Theorem 3.3 we have $\rho = 1$, $\tau = -Q$, $k_j = j$, $k_\psi = Q$, and

$$\left. \begin{aligned} G_0(\theta) &= 4\lambda^{-1}(-)^Q \sum_{j=0}^{Q} \frac{(-\theta)_j}{j!} \sum_{r=0}^{j} \frac{(-j)_r}{r!} \prod_{s=1}^{Q} (\gamma+r+2-2b_s) \\ &= 4\lambda^{-1}(-)^Q \prod_{s=1}^{Q} (\gamma+\theta+2-2b_s) , \end{aligned} \right\} \tag{4.71}$$

and so

$$\theta_h = 2b_h - \gamma - 2 \ , \ h = 1,2,\ldots,Q \ . \tag{4.72}$$

According to that theorem, there are exactly Q algebraic-logarithmic solutions of (4.50), and, if none of the θ_h differ by integers, these are of the form

$$y_{h+2}(n) = n^{\theta_h}\left[1 + O(n^{-1})\right] \ , \ 1 \le h \le Q \ . \tag{4.73}$$

Otherwise, logarithmic terms may appear.

We have thus determined a canonical set for (4.50), $\left\{y_h(n)\right\}$. A result of Luke (1968), which is a generalization of Watson's result for a ${}_2F_1$ (see Erdélyi et al (1953); v. I, p. 77, formula (16)), enables us to identify $y(n)$ with a constant multiple of $y_1(n)$. Lastly, by examining M_{Q+2} as given by (4.51), we see that if

$$a_j \ne -\gamma-Q-2, -\gamma-Q-3, \ldots, \ 1 \le j \le Q+2 \ , \tag{4.74}$$

then $M_{Q+2} \ne 0$ for $n \ge 0$. If this condition is satisfied, as well as the conditions immediately following (4.49), then Theorem 4.2 may be invoked: <u>the hypergeometric functions (4.49) may be computed by backward recursion from (4.50) and (4.53).</u>

In particular, we have demonstrated a way of computing Gauss' function wherever it is analytic, i.e., for $|\arg(1-\lambda)| < \pi$, as is seen by letting $Q = 0$, $a_1 = a$, $a_2 = \beta+1 = b, \gamma+1 = c$ in (4.49)-(4.52):

$$y(n) = \frac{(-)^n \lambda^n (a)_n (b)_n}{(c-1)_{2n}} \, {}_2F_1\left(\begin{matrix} n+a, n+b \\ 2n+c \end{matrix} \middle| \lambda \right) \quad , \tag{4.75}$$

$$\left. \begin{aligned} M_o &= 1 \; ; \; M_1 = - \frac{(2n+c-1)\left[2n^2+2n(c+1)+c(a+b+1)-2ab\right]}{(2n+c)(n+a)(n+b)} \quad , \\ M_2 &= \frac{(n+c+1-a)(n+c+1-b)(2n+c-1)(2n+c)}{(n+a)(n+b)(2n+c+2)(2n+c+3)} \; , \; N_1 = \frac{(2n+c-1)(2n+c)}{(n+a)(n+b)} \quad . \end{aligned} \right\} \tag{4.76}$$

To further illustrate the power of the method, we compute the function at a point on the circle of convergence of its Taylor series

$$\lambda = e^{\pi i/3} = (1 + \sqrt{3}i)/2 \quad , \tag{4.77}$$

with $a = 2/3$, $b = 1$, $c = 4/3$. It is known, e.g., Erdélyi et al (1953, v. I, p. 105 (55))

$$y(0) = {}_2F_1\left(\begin{matrix} 2/3, 1 \\ 4/3 \end{matrix} \middle| e^{\pi i/3} \right) = \frac{e^{\pi i/6} 2\pi \Gamma(1/3)}{9\Gamma(2/3)^2} \sim 0.88331\;9376 + 0.50998\;4679i \; . \tag{4.78}$$

Computation of $y(0)$ by backward recurrence using (4.50), (4.53) yields the following table:

$\underline{m}$	$\Gamma_o(m)$	$\epsilon_m = y(0)-\Gamma_o(m)$
5	0.88239 8541 + 0.50945 3036i	$(9.2 + 5.3i) \times 10^{-4}$
9	0.88331 4192 + 0.50998 1687i	$(5.2 + 3.0i) \times 10^{-6}$
13	0.88331 9347 + 0.50998 4663i	$(3.0 + 1.8i) \times 10^{-8}$

and $\Gamma_o(15)$ agrees with $y(0)$ to all the places (4.78).

The series for $y(0)$ converges only conditionally, Knopp (1947, p. 401), but this is irrelevant, because the Miller algorithm will work whether the Taylor series converges or not, as long as $|\arg(1-\lambda)| < \pi$, and, as is easily seen from the formulation (2.10), the convergence is exponential,

$$y(n) = \Gamma_n(m)\left[1 + O\left(m^c \zeta_1(\lambda)^m\right)\right], \quad m \to \infty \quad . \tag{4.79}$$

In this case

$$\zeta_1(\lambda) = i(2 - \sqrt{3}) \ , \ |\zeta_1(\lambda)| = 0.268 \cdots \quad . \tag{4.80}$$

As our final example, we take the confluent hypergeometric function

$$y_1(n) = \frac{\lambda^\delta (b)_n (\delta)_n}{n!} \Psi(n+\delta, \delta+1-b; \lambda) \quad . \tag{4.81}$$

(Our notation and subsequent analysis draw heavily on the material contained in Erdélyi et al (1953, v. I, Ch. 6).)

$y_1(n)$ satisfies the difference equation

$$y(n) - \frac{(n+1)\left[(2n+\delta+b+1)+\lambda\right]}{(n+\delta)(n+b)} y(n+1) + \frac{(n+1)(n+2)}{(n+\delta)(n+b)} y(n+2) = 0 , \qquad (4.82)$$

and

$$y_1(n) = V(\lambda)n^{(\delta+b)/2-5/4}e^{-2n^{\frac{1}{2}}\lambda^{\frac{1}{2}}}\left[1 + O(n^{-\frac{1}{2}})\right] , \qquad (4.83)$$

$$|\arg \lambda| \le \pi , \; V(\lambda) = \sqrt{\pi}\, \lambda^{(\delta+b)/2-\frac{1}{4}}e^{\lambda/2}\Big/ \Gamma(\delta)\Gamma(b) , \qquad (4.84)$$

$$1 = \sum_{k=0}^{\infty} y_1(k) , \; |\arg \lambda| < \pi . \qquad (4.85)$$

Another function satisfying (4.82) is

$$y_2(n) = \frac{(\delta)_n}{n!} \Phi(n+\delta, \delta+1-b;\lambda) , \qquad (4.86)$$

$$= \frac{\lambda^{-\delta}}{2\pi} \Gamma(b)\Gamma(\delta+1-b)V(\lambda)n^{(\delta+b)/2 - 5/4}e^{2n^{\frac{1}{2}}\lambda^{\frac{1}{2}}}\left[1+O(n^{-\frac{1}{2}})\right],$$

$$|\arg \lambda| \le \pi . \qquad (4.87)$$

It is thus seen from Theorem 4.2 that the computation of y_1 by backward recursion based on (4.82), (4.85) converges for $|\arg \lambda| < \pi$. To illustrate this, let $\delta = b = \frac{1}{2}$, so

$$y_1(0) = \lambda^{\frac{1}{2}}\Psi(\tfrac{1}{2},1;\lambda) = \lambda^{\frac{1}{2}}e^{\lambda/2}K_0(\lambda/2)\Big/ \sqrt{\pi} , \qquad (4.88)$$

$$y_1(1) = \frac{\lambda^{\frac{1}{2}}}{4}\,\Psi(3/2,1;\lambda) = \lambda\,\frac{d}{d\lambda}\left[\lambda^{\frac{1}{2}}\Psi(1/2,1;\lambda)\right] \tag{4.89}$$

$$= \frac{\lambda}{\sqrt{\pi}}\,\frac{d}{d\lambda}\left[\lambda^{\frac{1}{2}}e^{\lambda/2}K_o(\lambda/2)\right] = \frac{\lambda^{3/2}}{2\sqrt{\pi}}\,e^{\lambda/2}\left[-K_1(\lambda/2)+K_o(\lambda/2)(1+1/\lambda)\right] \quad . \tag{4.90}$$

For $\lambda = 4$, standard tables give

$$y_1(0) = 0.94960\,8042 \cdots , \; y_1(1) = 0.04171\,2616 \cdots \quad . \tag{4.91}$$

Taking $m = 10$ in (2.2) to (2.6) yields

$$\Gamma_o(10) = 0.94961\,1302 \cdots , \; \Gamma_1(10) = 0.04171\,2759 \cdots , \tag{4.92}$$

with approximate absolute errors of 3.3×10^{-6} and 1.4×10^{-7} , respectively. It is interesting that the difference equation (4.82) serves to compute K_ν , while the usual recursion relation does not.

If $\arg\lambda = \pi$, then we can define

$$\varphi_1^*(m) = m^c e^{2im^{\frac{1}{2}}|\lambda|^{\frac{1}{2}}}\left[1 + O(m^{-\frac{1}{2}})\right] \quad , \tag{4.93}$$

$$\varphi_2^*(m) = m^c e^{-2im^{\frac{1}{2}}|\lambda|^{\frac{1}{2}}}\left[1 + O(m^{-\frac{1}{2}})\right] , \tag{4.94}$$

and Theorems 4.1 and 2.4 apply, with $\rho = k = 2$, $\mu_2 = -4|\lambda|^{\frac{1}{2}}$, $\mu_1 = 0$, and

$$m_2 = \left[m_1 + \pi m_1^{\frac{1}{2}} \Big/ \left(2|\lambda|^{\frac{1}{2}}\right)\right] \quad . \tag{4.95}$$

Thus both the functions (4.81) and (4.86) can be computed in this case, provided that suitable normalization relationships (or initial values) are known, since it is clear the series (4.85) will, in general, no longer suffice.

An even more efficient algorithm for the calculation of the Ψ function can be based on the third order difference equation satisfied by the functions

$$z(n) = \frac{(-)^n(2n+a-1)\Gamma(n+a-1)}{\Gamma(a)\Gamma(\delta)\Gamma(n+b)} G_{23}^{31}\left(\lambda \middle| \begin{matrix} 1-n, n+a \\ a, \delta, b \end{matrix}\right) \tag{4.96}$$

$$= \frac{(\delta)_n}{\Gamma(a)} (2n+a-1)\Gamma(n+a-1)\lambda^{n+a} \frac{d^n}{d\lambda^n}\left\{\lambda^{\delta-a}\Psi(n+\delta, \delta+1-b; \lambda)\right\} , \tag{4.97}$$

(see Wimp (1966, 1967) and Luke and Wimp (1963)). This is because (4.96) behaves as

$$z(n) = c\, n^d e^{-3\lambda^{1/3} n^{2/3}}\left[1 + O(n^{-1/3})\right] , \; n \to \infty , \tag{4.98}$$

for some c , d (independent of n), $|\arg \lambda| < 3\pi/2$. (A canonical set for the equation is readily obtained by replacing n above by $ne^{2h\pi i}$, $h = 1,2$.)

In fact, the hypergeometric functions discussed in the above three references, which are of the form

$$G_{Q+2,P+1}^{P+1,1}\left(\lambda \middle| \begin{matrix} 1-n, n+\gamma+1, b_Q \\ a_P, \beta+1 \end{matrix}\right) , \; n = 0,1,2,\ldots , \tag{4.99}$$

can all be computed by Miller's algorithm, using the recursion relations and normalization series given by Wimp (1966), provided $|\arg \lambda| < \pi$ for $P > Q+1$, and $|\arg(1+\lambda^{-1})| < \pi$ for $P = Q+1$, in which case (4.99) is related to (4.49). For $P > Q+1$, the difference equation for (4.99) has Q algebraic-logarithmic solutions with the same values of θ_h as given by (4.72), plus an additional $P+1-Q$ anormal F.S.S.

$$y_h(n) \sim c_h n^{d_h} \exp\left\{-(P+1-Q)(n^2 \lambda e^{2h\pi i})^{1/(P+1-Q)}\right\}, \quad h = 0,1,\ldots,P-Q, \tag{4.100}$$

while if $P < Q$, then $\rho = \omega = 1$ in all the F.S.S. of the difference equation. Thus the recursion relation for (4.99) has canonical sets whose components exhibit widely varying behaviour, depending on the relation between P and Q, and the equation may be expected to furnish a number of additional interesting applications of the theory developed in this work.

APPENDIX

Here we set forth the notation used in the body of the thesis, and list some frequently invoked results from the theory of linear difference equations.

For determinants and matrices, we use the notation

$$\left|\alpha_{j,h}\right|_1^\tau = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\tau} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\tau} \\ \vdots & \vdots & & \vdots \\ \alpha_{\tau 1} & \alpha_{\tau 2} & \cdots & \alpha_{\tau\tau} \end{vmatrix} , \tag{A.1}$$

$$\left[\alpha_{j,h}\right]_1^\tau = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1\tau} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\tau} \\ \vdots & \vdots & & \vdots \\ \alpha_{\tau 1} & \alpha_{\tau 2} & \cdots & \alpha_{\tau\tau} \end{bmatrix} , \tag{A.2}$$

$\alpha_{j,h}$ always denoting the element in the j^{th} row and h^{th} column of the determinant or matrix.

Let φ_n , ψ_n be (complex-valued) functions of $n = 0,1,2,\ldots$. We write

$$\varphi_n = O(\psi_n) \ , \ n \to \infty \ , \tag{A.3}$$

if $\psi_n \neq 0$ for $n > n_o$ and if a positive A exists for which

$$|\varphi_n/\psi_n| < A \ , \ n > n_o \quad , \tag{A.4}$$

and

$$\varphi_n = o(\psi_n) \ , \ n \to \infty \quad , \tag{A.5}$$

if, given $\epsilon > 0$, there exists an $N(\epsilon) \geq n_o$ such that

$$|\varphi_n/\psi_n| < \epsilon \quad \text{for all} \quad n > N \ . \tag{A.6}$$

Let ρ be an integer ≥ 1.

$$\varphi_n \sim \psi_n\left[c_o + c_1 n^{-1/\rho} + c_2 n^{-2/\rho} + \ldots\right], \ c_o \neq 0 \ , \ n \to \infty \quad , \tag{A.7}$$

means that

$$\left| \varphi_n/\psi_n - \sum_{k=0}^{r} c_k n^{-k/\rho} \right| = O\left(n^{-(r+1)/\rho}\right) , \ n \to \infty \quad , \tag{A.8}$$

for each $r = 0,1,2,\ldots$.

Occasionally φ, ψ will be functions of a complex variable λ belonging to some sector S in the complex plane. Then

$$\varphi(\lambda) = O\left(\psi(\lambda)\right), |\lambda| \to \infty \ , \ \lambda \epsilon S \quad , \tag{A.9}$$

etc., are interpreted similarly. See Erdélyi (1956) or de Bruijn (1961) for details.

Suppose we have a set of (complex-valued) functions $\left\{ y_h(n) \right\}$, $1 \le h \le \sigma$, defined for $n = 0,1,2,\ldots$. The functions are called linearly dependent if and only if a relation

$$c_1 y_1(n) + c_2 y_2(n) + \cdots + c_\sigma y_\sigma(n) = 0 \ , \ n = 0,1,\ldots \ , \qquad \text{(A.10)}$$

holds for some constants (independent of n) c_j which are not all zero. Otherwise, the functions are linearly independent.

Good sources for the following material are Gautschi (ca 1962), Milne-Thomson (1960) and Nörlund (1954).

A-I. The functions $\left\{ \varphi_h(n) \right\}$, $1 \le h \le \sigma$ are linearly dependent if and only if

$$D(n) = \left| \varphi_h(n+j-1) \right|_1^\sigma = 0 \ , \ n = 0,1,\ldots \ . \qquad \text{(A.11)}$$

A-II. The functions in any subset of linearly independent functions are linearly independent.

A-III. For any integer $k \ge 0$, the difference equation of order $\sigma (\ge 1)$

$$\sum_{\nu=0}^{\sigma} C_\nu(n) y(n+\nu) = 0 \ , \ n = 0,1,2,\ldots, \ C_0 = 1 \ , \ C_\sigma(n) \ne 0 \ , \qquad \text{(A.12)}$$

possesses a unique solution satisfying the conditions

$$y(k+\nu) = \alpha_\nu \ , \ \nu = 0,1,2,\ldots,\sigma-1 \ . \tag{A.13}$$

A-IV. Equation (A.12) possesses a linearly independent set of solutions $\{y_h(n)\}$, $1 \le h \le \sigma$, called a fundamental set, and any solution of (A.12), such as (A.13), can be expressed as a linear combination of these functions.

A-V. Let

$$D(n) = \left| y_h(n+j-1) \right|_1^\sigma \ . \tag{A.14}$$

Then

$$D(n+1) = (-)^\sigma D(n)/C_\sigma(n) \ , \ n = 0,1,2,\ldots \ . \tag{A.15}$$

A-VI. The equation

$$\sum_{\nu=0}^{\sigma} C_{\sigma-\nu}(n+\nu) y^*(n+\nu) = 0 \ , \ n = 0,1,2,\ldots \ , \tag{A.16}$$

is called the equation adjoint to (A.12), and the functions $T_h(n)/D(n)$, $1 \le h \le \sigma$ are linearly independent and satisfy (A.16) where

$$T_h(n) = (-)^{h-1} \begin{vmatrix} y_1(n+1), \cdots y_{h-1}(n+1), y_{h+1}(n+1), \cdots, y_\sigma(n+1) \\ \vdots \\ y_1(n+\sigma-1), \cdots y_{h-1}(n+\sigma-1), y_{h+1}(n+\sigma-1), \cdots, y_\sigma(n+\sigma-1) \end{vmatrix} . \tag{A.17}$$

Thus

$$\sum_{h=1}^{\sigma} T_h(n)y_h(n+r)/D(n) = \begin{cases} 1, \ r = 0 \ , \\ 0, \ 1 \le r \le \sigma-1 \ , \\ -1/C_\sigma(n) \ , \ r = \sigma \ . \end{cases} \qquad \text{(A.18)}$$

A-VII. Let the functions $\left\{y_h(n)\right\}$, $1 \le h \le \sigma$, be such that

$$\lim_{n\to\infty} y_h(n)/y_{h+1}(n) = 0 \ , \ 1 \le h \le \sigma-1 \ . \qquad \text{(A.19)}$$

Then the functions are linearly independent.

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