

Dr. Z's Math151 Handout #3.1 [Derivatives of Polynomials and Exponential Functions]

By Doron Zeilberger

Problem Type 3.1.1 : Differentiate the function $y = Expression(x)$ where the expression in x is a sum of terms involving powers (sometimes disguised as roots).

Example Problem x.1: Differentiate the function

$$y = \frac{x^2 - 2\sqrt{x}}{x}$$

Steps

1. Convert all square-roots, cubic-roots etc. into powers using $\sqrt{x} = x^{1/2}$ and, in general, that the n^{th} root of x is $x^{1/n}$, and that $1/x^a = x^{-a}$, and use algebra to bring it into a sum of terms of the form cx^n .

2. Differentiate the expression term by term by using the addition rule, and the rule $\frac{d}{dx}x^n = nx^{n-1}$.

Example

1.

$$\begin{aligned} y &= \frac{x^2 - 2\sqrt{x}}{x} = \frac{x^2 - 2x^{1/2}}{x} = \\ (x^2 - 2x^{1/2})x^{-1} &= (x^2)(x^{-1}) - 2(x^{1/2})(x^{-1}) = \\ x^{2-1} - 2x^{1/2-1} &= x - 2x^{-1/2}. \end{aligned}$$

2.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x - 2x^{-1/2}) \\ &= \frac{d}{dx}(x) - \frac{d}{dx}(2x^{-1/2}) = 1 - 2(-1/2)x^{-3/2} = 1 + x^{-3/2}. \end{aligned}$$

Ans.: $1 + x^{-3/2}$, or in 'high-school' notation $1 + \frac{1}{(\sqrt{x})^3}$.

Problem Type 3.1.2 : Where does the normal line to the parabola $y = Quadratic(x)$ at the point $(a, Quadratic(a))$ intersect the parabola a second time?

Example Problem 3.1.2: Where does the normal line to the parabola $y = 2x - x^2$ at the point $(2, 0)$ intersect the parabola a second time?

Steps

1. Find the slope, let's call it m , of the tangent line of the parabola by differentiating $Quadratic(x)$, and plugging in $x = a$. The slope of the normal line is $-1/m$.

2. Using the slope and the point find the equation of the normal line. Using $(y - y_0) = (slope)(x - x_0)$.

3. Find the point of intersection by solving for x the equation obtained by setting the eq. of the parabola (from the problem) and the equation of the normal line (from step 2) equal to each other.

Example

1. Since $y = 2x - x^2$, $\frac{dy}{dx} = 2 - 2x$. Plugging in $x = 2$ gives that the slope of the tangent is $m = 2 - 2 \cdot 2 = -2$. The slope of the normal is $-1/(-2) = 1/2$.

2. $y - 0 = (1/2)(x - 2)$, so $y = x/2 - 1$.

3. $2x - x^2 = x/2 - 1$ implies $2x^2 - 3x + 2 = 0$. Factorizing, gives $(2x + 1)(x - 2) = 0$ so the solutions are $x = 2$ and $x = -1/2$. We already know about the first one, so the other point has $x = -1/2$, and plugging into the equation of the normal the y -coordinate is $(-1/2)/2 - 1 = -5/4$.

Ans.: $(-1/2, -5/4)$.

Problem Type 3.1.3 :

Let

$$f(x) = \begin{cases} \text{Expression}_1(x), & \text{if } x \leq a; \\ \text{Expression}_2(x), & \text{if } x > a. \end{cases}$$

Is $f(x)$ differentiable at a ?

Example Problem 3.1.3:

Let

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1; \\ x^3, & \text{if } x > 1. \end{cases}$$

Is $f(x)$ differentiable at 1?

Steps

1. First check that the function is continuous at the designated value a by plugging into the two expressions and finding whether they are equal. If they are not, then the function is not even continuous at a , let alone differentiable.

2. Compute the derivative by differentiating each case separately.

3. By plugging in $x = a$ into the two cases above find out whether $f'(x)$ is continuous (if they are equal). If they are not then $f'(x)$ is not continuous at a and hence the function itself $f(x)$ is not differentiable.

Example

1. $(1)^2 = 1$ and $(1)^3 = 1$ are equal, so the function $f(x)$ is continuous at 1.

2.

$$f'(x) = \begin{cases} 2x, & \text{if } x \leq 1; \\ 3x^2, & \text{if } x > 1. \end{cases}$$

3. $2 \cdot 1 = 2$ and $3 \cdot 1^2 = 3$ are *not* the same, hence $f'(x)$ is *not* differentiable at 1.