Dr. Z's Math151 Handout #3.11 [Linear Approximation]

Problem Type 3.11.1: Use the Linear Approximation to estimate $\Delta f = f(a+h) - f(a)$ for the given function $f(x)$, for the given a and h.

Example Problem 3.11.1: Use the Linear Approximation to estimate $\Delta f = f(4+0.01) - f(4)$ for $f(x) = 1/x$.

Steps	**Example**
1. Compute $f'(x)$, and decide who is a and who is h.	**1.** $f'(x) = -x^{-2} = \frac{-1}{x^2}$, $a = 4$, and $h = 0.01$.
2. Set up the formula $$\Delta f \approx f'(a)h \quad ,$$ and do the plugging-in	**2.** $$\Delta f \approx f'(4) \cdot (0.01) = \frac{-1}{4^2} \cdot (0.01) = \frac{-1}{1600} \quad .$$ **Ans.:** $\Delta f = \frac{-1}{1600}$

Problem Type 3.11.2: Estimate the quantity using the Linear Approximation.

Example Problem 3.11.2: Estimate the quantity

$$\frac{1}{\sqrt{97}} - \frac{1}{10} \quad .$$

Steps	**Example**
1. By "pattern-recognition" decide (i) what is the function (ii) what is the "nice" point a (iii) what is the deviation h. Also find $f'(x)$.	**1.** Here $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$. The "nice" point near 97 is $a = 100$ and $h = -3$. $$f'(x) = (-1/2)x^{-3/2} = \frac{-1}{2\sqrt{x}^3} \quad .$$

2. Set up the formula

$$\Delta f \approx f'(a)h \quad ,$$

and implement it for the specific problem.

2.

$$\Delta f \approx f'(100)(-3) = \frac{-1}{2(\sqrt{100})^3} \cdot (-3) = \frac{3}{2000} \quad .$$

Problem Type 3.11.3: Find the linearization at $x = a$, $y = SomeFunction(x)$, $a = SomeNumber$.

Example Problem 3.11.3: Find the linearization at $x = a$, $y = (5 + x^2)^{-1/2}$, $a = 2$.

Steps

Example

1. Find $f'(x)$.

1.

$$f'(x) = ((5+x^2)^{-1/2})' = (-1/2)(5+x^2)^{-3/2}(5+x^2)' =$$

$$(-1/2)(5+x^2)^{-3/2}(2x) = -x(5+x^2)^{-3/2} = \frac{-x}{(\sqrt{5+x^2})^3}$$

2. Plug everything into the formula

$$L(x) = f'(a)(x-a) + f(a) \quad ,$$

2.

$$L(x) = f'(a)(x-a)+f(a) = \frac{-2}{(\sqrt{5+2^2})^3}(x-2)+(5+2^2)^{-1/2}$$

$$= \frac{-2}{27}(x-2)+(9)^{-1/2} = \frac{-2}{27}(x-2)+\frac{1}{3} \quad .$$

Ans.: $L(x) = \frac{-2}{27}(x-2) + \frac{1}{3}$.

Problem from a Previous Final (Spring 2008, #4 (9 points)).

Let $f(x) = \sqrt{1-x}$

(a) (6 points) Using the linear approximation of $f(x)$ at $a = -3$ compute an approximation to $f(-4)$.

(b) (3 points) Use f'' (concavity) to determine whether your approximation is larger or smaller than the true value of $f(-4)$.

Solution

(a) $f(x) = \sqrt{1-x} = (1-x)^{1/2}$. Since $a = -3$, $f(-3) = 4^{1/2} = 2$. We also have

$$f'(x) = (1/2)(1-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{1-x}} \quad .$$

Plugging-in $x = -3$ gives

$$f'(-3) = \frac{-1}{2\sqrt{1-(-3)}} = \frac{-1}{2\sqrt{4}} = \frac{-1}{4} \quad .$$

The linear approximation is

$$L(x) = f'(-3)(x-(-3)) + f(-3) = \frac{-(x+3)}{4} + 2 \quad .$$

And when $x = -4$, we get

$$L(-4) = \frac{-(-4+3)}{4} + 2 = \frac{9}{4} \quad .$$

Ans. to (a): The approximation to $f(-4)$ using the Linear approximation is $\frac{9}{4} = 2.25$.

(b) $f''(x) = (-1/4)(1-x)^{-3/2}$. So $f''(-4) = (-1/4) \cdot 5^{3/2}$ is **negative**, this means that the approximation is **larger** than the true value of $f(-4)$.