Dr. Z's Math151 Handout #4.9 [Newton's Method]

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Problem Type 4.9.1 : Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places). $f(x) = 0$, $x_1 = Number$.

Example Problem 4.9.1: As above with $x^3 - x^2 - 1 = 0$, $x_1 = 1$.

Steps

1. If not already so, convert the equation to be solved to one in which the right side is 0. The left side is $f(x)$.

2. Find $f'(x)$ and set-up Newton's law

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

1. In the equation $x^3 - x^2 - 1 = 0$, the right side is already 0, so $f(x) = x^3 - x^2 - 1$.

2. Since $f(x) = x^3 - x^2 - 1$, $f'(x) = 3x^2 - 2x$, hence Newton's rule for this particular equation is

$$x_{n+1} = x_n - \frac{{x_n}^3 - (x_n)^2 - 1}{3(x_n)^2 - 2(x_n)}$$

3. Plug in $n = 1$ above to get x_2. Then plug-in $n = 2$ to get x_3.

3.

$$x_2 = x_1 - \frac{{x_1}^3 - (x_1)^2 - 1}{3(x_1)^2 - 2(x_1)} =$$

$$1 - \frac{1^3 - (1)^2 - 1}{3(1)^2 - 2(1)} =$$

$$1 - \frac{1 - 1 - 1}{3 - 2} = 2.$$

Now plug-in $n = 2$ to get

$$x_3 = x_2 - \frac{{x_2}^3 - (x_2)^2 - 1}{3(x_2)^2 - 2(x_2)}$$

$$2 - \frac{2^3 - 2^2 - 1}{3 \cdot (2)^2 - 2 \cdot 2} =$$

$$2 - \frac{3}{3 \cdot 4 - 2 \cdot 2} =$$

$$2 - \frac{3}{12 - 4} =$$

$$2 - \frac{3}{8} = \frac{13}{8} = 1.575$$

Problem Type 4.9.2 : Use Newton's method to to approximate the $\sqrt[m]{a}$ to 8 decimal places,

Example Problem 4.9.2: Use Newton's method to to approximate the $\sqrt[6]{100}$ to 8 decimal places,

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Steps

1. We have to find an approximation to $x^m - a = 0$, so Use Newton's method as above with $f(x) = x^m - a$, and keep doing it until the answers agree to 8 decimal places.

2.

$$x_{n+1} = x_n - \frac{(x_n)^m - a}{m(x_n)^{m-1}}$$

Example

1. We have to find an approximation to $x^6 - 100 = 0$, so Use Newton's method as above with $f(x) = x^6 - 100$, and keep doing it until the answers agree to 8 decimal places.

2.

$$x_{n+1} = x_n - \frac{(x_n)^6 - 100}{6(x_n)^5}$$

3. By trial and error find the best integer to start with, this would be your x_1, then keep applying step 2, until x_{n+1} and x_n agree to the desired accuracy (eight decimal places in this case).

3. $2^6 - 100$ is closer to 0 then $1^6 - 100$ or $3^6 - 100$, so $x_1 = 2$.

$$x_2 = x_1 - \frac{(x_1)^6 - 100}{6(x_1)^5} =$$

$$2 - \frac{(2)^6 - 100}{6(2)^5} =$$

$$2 - \frac{-36}{192} = 2.1875 \quad .$$

$$x_3 = x_2 - \frac{(x_2)^6 - 100}{6(x_2)^5} =$$

$$x_3 = (2.1875) - \frac{(2.1875)^6 - 100}{6(2.1875)^5} = 2.15565929$$

$$x_4 = (2.15565929) - \frac{(2.15565929)^6 - 100}{6(2.15565929)^5} = 2.15443642$$

$$x_5 = (2.15443642) - \frac{(2.15443642)^6 - 100}{6(2.15443642)^5} = 2.15443649$$

Now they agree to eight decimal places, so the approximation to $\sqrt[6]{100}$ is 2.1544364.