

Dr. Z's Math151 Handout #5.2 [The Definite Integral]

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**Problem Type 5.2.1** : Use the definition of the integral

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

(where  $\Delta x = (b - a)/n$  and  $x_i = a + i\Delta x$ ). to evaluate the integral

$$\int_a^b f(x)dx$$

**Example Problem 5.2.1:** Use the definition of the integral given above to evaluate the integral

$$\int_{-1}^2 (2 + 3x)dx \quad .$$

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**Steps**

1. Determine  $\Delta x = (b - a)/n$ , and write down an expression for  $x_i = a + i\Delta x$ , that should involve both  $i$  and  $n$ .

2. Spell-out

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x \quad ,$$

in the present context.

**Example**

1.  $\Delta x = (2 - (-1))/n = 3/n$ .  $x_i = -1 + 3i/n$ .

2.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(-1 + 3i/n)(3/n) =$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(-1 + 3i/n)(3/n) =$$

$$\lim_{n \rightarrow \infty} (3/n) \sum_{i=1}^n [2 + 3(-1 + 3i/n)]$$

$$\lim_{n \rightarrow \infty} (3/n) \sum_{i=1}^n -1 + 9i/n \quad .$$

3. Evaluate the sum using the formulas (if necessary)

$$\sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2 .$$

4. Incorporate the evaluated sum from step 3 into the limit of step 2, and evaluate the limit.

3.

$$\sum_{i=1}^n -1 + 9i/n = \sum_{i=1}^n -1 + \sum_{i=1}^n 9i/n =$$

$$-n + (9/n) \sum_{i=1}^n i = -n + (9/n)(n(n+1)/2) =$$

$$-n + (9(n+1)/2) = (7n+9)/2 .$$

4.

$$\lim_{n \rightarrow \infty} (3/n) \sum_{i=1}^n -1 + 9i/n =$$

$$\lim_{n \rightarrow \infty} (3/n)(7n+9)/2 =$$

$$\lim_{n \rightarrow \infty} \frac{3(7n+9)}{2n} =$$

$$\lim_{n \rightarrow \infty} \frac{3(7n)}{2n} = 21/2 .$$

**Ans.:** 21/2.

**Problem Type 5.2.2 :** Express the integral as a limit of Riemann sums, Do not evaluate the limits.  $\int_a^b f(x)dx$ .

**Example Problem 5.2.2:** Express the integral as a limit of Riemann sums, Do not evaluate the limits.  $\int_2^6 \frac{x}{3+x^5} dx$ .

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**Steps**

1. Determine  $\Delta x = (b - a)/n$ , and write down an expression for  $x_i = a + i\Delta x$ , that should involve both  $i$  and  $n$ .

2. Spell-out

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad ,$$

in the present context.

**Example**

1.  $\Delta x = (6 - 2)/n = 4/n$ .  $x_i = 2 + 4i/n$ .

2.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(2 + 4i/n)(4/n) =$$

$$\lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \frac{2 + 4i/n}{3 + (2 + 4i/n)^5}$$

That's it! Do not try to evaluate.