

Dr. Z's Math151 Handout #5.3 [The Fundamental Theorem of Calculus]

By Doron Zeilberger

Problem Type 5.3.1 : Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$f(\text{Variable}_1) = \int_{\text{Number}}^{\text{Variable}_1} \text{Expression}(\text{Variable}_2) d\text{Variable}_2 \quad .$$

Example Problem 5.3.1: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$g(u) = \int_3^u \frac{1}{x+x^2} dx \quad .$$

Steps

1. Make sure that the argument of the function on the left side is the same as the upper limit of the integral on the right side. Also make sure that the lower limit in the integral sign is a number. Also make sure that the expression inside the integral is a function of Variable_2 , where Variable_2 is the variable standing after the 'd' in $d\text{Variable}_2$.

2. To find the derivative of $f(\text{Variable}_1)$ with respect to Variable_1 , all you have to do is replace Variable_2 by Variable_1 in the integrand $\text{Expression}(\text{Variable}_2)$. In other words, the answer is $\text{Expression}(\text{Variable}_1)$.

Example

1. The argument of $g(u)$, u , is the same as the letter appearing at the upper limit of the integral. The lower limit is a number (3), so that's OK too. The variable of integration is x since the integral has a dx in it, and the integrand $\frac{1}{x+x^2}$ is indeed an expression of x . (If it weren't, then you would still consider it as an expression in x , viewing all other variables as silent constants.)

2. Replacing x by u in $\frac{1}{x+x^2}$ yields

Ans.: $g'(u) = \frac{1}{u+u^2}$

Problem Type 5.3.2 : Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$h(Var_1) = \int_{Number}^{g(Var_1)} Expr(Var_2) dVar_2 \quad ,$$

where now the upper limit is an expression $g(Var_1)$ rather than just Var_1 .

Example Problem 5.3.2: Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of

$$h(x) = \int_0^{x^2} \sqrt{1+r^3} dr \quad .$$

Steps

1. Here the upper limit of the integral is an expression in the variable of the named function on the left, not just the variable itself. Rewrite the derivative, using the chain rule, as

$$h'(Var_1) = \frac{d}{dVar_1} g(Var_1) \cdot \frac{d}{dg(Var_1)} \left(\int_{Number}^{g(Var_1)} Expr(Var_2) dVar_2 \right) .$$

2. Do the differentiation $g'(Variable_1)$. The second part is done exactly as in problem 5.3.1, where you replace $g(Variable_1)$ by $Variable_2$ in the integrand.

Example

1. Here the upper limit of the integral is an expression x^2 in the variable of the named function $h(x)$, which is x on the left, not just x itself. Rewrite the derivative, using the chain rule, as

$$h'(x) = \frac{d(x^2)}{dx} \cdot \frac{d}{dx^2} \left(\int_0^{x^2} \sqrt{1+r^3} dr \right) .$$

2. **Ans.:**

$$h'(x) = 2x\sqrt{1+(x^2)^3} = 2x\sqrt{1+x^6} \quad .$$

Comment: If the upper limit of integration is a number but the lower limit of integration is a variable or an expression, first rewrite it as

$$\int_{Expr(Var_1)}^{Number} = - \int_{Number}^{Expr(Var_1)} \quad ,$$

and then proceed as in 5.3.1 or 5.3.2 .

If *both* lower and upper integration limits are expressions, then first rewrite it as

$$\int_0^{Expr_2(Var_1)} - \int_0^{Expr_1(Var_1)} \quad ,$$

and do each piece separately, like in 5.3.2. Example:

$$\begin{aligned} \frac{d}{dx} \left(\int_{x^2}^{x^3} \sin u du \right) &= \frac{d}{dx} \left(\int_0^{x^3} \sin u du \right) - \frac{d}{dx} \left(\int_0^{x^2} \sin u du \right) = \\ \frac{dx^3}{dx} \cdot \frac{d}{dx^3} \left(\int_0^{x^3} \sin u du \right) &- \frac{dx^2}{dx} \cdot \frac{d}{dx^2} \left(\int_0^{x^2} \sin u du \right) = \\ 3x^2 \sin x^3 - 2x \sin x^2 . \end{aligned}$$

Problem Type 5.3.3 : Use Part 2 of the Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

$$\int_a^b f(x) dx \quad .$$

Example Problem 5.3.3: As above with

$$\int_1^4 \frac{1}{\sqrt{x}} dx \quad .$$

Steps

1. First make sure that the function is ‘nice’ in the interval (a, b) , i.e. is defined and does not blow up. If it is, find an antiderivative let’s call it $F(x)$, (a.k.a. *indefinite integral*) by using the integration rules $\int x^n dx = x^{n+1}/(n+1) + C$, $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int e^x dx = e^x + C$, $\int \frac{dx}{x} = \ln|x| + C$, etc. (except that you don’t have to worry about the C , since *any* antiderivative will do, so you may just as well take $C = 0$).

2. The *definite integral* is $F(b) - F(a)$,

Example

1. An antiderivative of $\frac{1}{\sqrt{x}} = x^{-1/2}$ is

$$F(x) = \frac{x^{-1/2+1}}{-1/2+1} = 2x^{1/2} = 2\sqrt{x} \quad .$$

2. The answer is

$$F(4) - F(1) = 2\sqrt{4} - 2\sqrt{1} = 2 \quad .$$

Ans.: 2.