**Dr. Z's Math151 Handout #5.5 [The Substitution Rule]**

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**Problem Type 5.5.1** : Evaluate the indefinite integral

$$\int COMPLICATED(Var)\, dVar \quad .$$

**Example Problem 5.5.1**: Evaluate

$$\int e^x \sqrt{1+e^x}\, dx \quad .$$

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| **Steps** | **Example** |
|---|---|
| **1.** Try to find a good $u$. Usually what's inside parantheses or square-root (or any root) sign is a good bet. Note that there may be more than one obvious choice, and usually only one of them works. In fact, very often, nothing works. | **1.** Since $1+e^x$ is inside the square-root sign, let's try: $u = 1+e^x$. |
| **2.** Find the derivative of $u$, $du/dx$, and then by 'cross-multiplying' express $dx$ in terms of $du$. | **2.** $$\frac{du}{dx} = e^x \quad hence \quad dx = \frac{du}{e^x} \quad .$$ |
| **3.** Do the translation from the $x$ language to the $u$ language. You should eventually get ONLY $u$. You may need to use some algebra to express the remmaining $x$-stuff in terms of $u$, but often everything is in terms of $u$. | **3.** $$\int e^x \sqrt{1+e^x}\, dx = \int e^x \sqrt{u} \frac{du}{e^x} = \int \sqrt{u}\, du \quad .$$ |

**4.** Do this far simpler $u$ integral. Finally replace $u$ by the expression in $x$ it stands for from step 1. Finally add $+C$.

**4.**

$$\int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} = (2/3)(\sqrt{u})^3 = (2/3)(\sqrt{1+e^x})^3 + C$$

**Problem Type 5.5.2** : Evaluate the definite integral

$$\int_a^b COMPLICATED(Var)\, dVar \quad .$$

**Example Problem 5.5.2**: Evaluate

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\, dx \quad .$$

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| **Steps** | **Example** |
|---|---|
| **1.** As before, try to find a good $u$. | **1.** Since $x^2$ is inside the cos, let's try: $u = x^2$. |
| **2.** Find the derivative of $u$, $du/dx$, and then by 'cross-multiplying' express $dx$ in terms of $du$. | **2.** $$\frac{du}{dx} = 2x \quad hence \quad dx = \frac{du}{2x} \quad .$$ |
| **3.** Do the translation from the $x$ language to the $u$ language. Now you also need to translate the *integration limits*, by plugging $a$ and $b$ into $u$. You should eventually get a *definite* integral involving ONLY $u$. Evaluate it! | **3.** When $x = 0$, $u = 0$. When $x = \sqrt{\pi}$, $u = \pi$. The integral equals $$\int_0^{\pi} x\cos(u)\frac{du}{2x} = \frac{1}{2}\int_0^{\pi}\cos(u)\, du =$$ $$\sin u|_0^{\pi} = \sin\pi - \sin 0 = 0 - 0 = 0 \quad .$$ **Ans.**: 0. |