

Dr. Z's Math151 Handout #6.1 [Areas between Curves]

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Problem Type 6.1.1 : Find the area of the region enclosed between $y = Expr_1(x)$ and $y = Expr_2(x)$, where they meet at two points.

Example Problem 6.1.1:

Find the area of the region enclosed between $y = 12 - x^2$ and $y = x^2 - 6$.

Steps

1. Find the points of intersection by solving, for x the algebraic equation

$$Expr_1(x) = Expr_2(x) \quad .$$

If you get just one root, then the region is infinite, and there is no (finite) answer. If you get two roots, let's call them $x_1 < x_2$, then the limits of the integral will be x_1 and x_2 . If you get three roots, see Problem 6.1.2 below.

2. For each of the intervals of integration (in this example there is just one), determine which curve is on top, by plugging-in a random value (or sketching).

3. The area is the integral

$$\int_{x_1}^{x_2} [TOP(x) - BOT(x)] \quad .$$

Example

1. $12 - x^2 = x^2 - 6$ means $2x^2 = 18$, i.e. $2(x^2 - 9) = 0$ so $x^2 - 9 = 0$, which means $(x - 3)(x + 3) = 0$, and we have two roots $x = -3$ and $x = 3$.

2. Plugging-in $x = 0$ into $y = 12 - x^2$ gives $y = 12$, while plugging-in $x = 0$ into $y = x^2 - 6$ gives $y = -6$. Since 12 is bigger than -6 , we have that $TOP = 12 - x^2$ and $BOT = x^2 - 6$.

3. The area is

$$\begin{aligned} \int_{-3}^3 [(12 - x^2) - (x^2 - 6)] &= \int_{-3}^3 [18 - 2x^2] = \\ 18x - \frac{2x^3}{3} \Big|_{x=-3}^3 &= 18(3 - (-3)) - \frac{2(3^3 - (-3)^3)}{3} = 108 - 36 = 72 \quad . \end{aligned}$$

Ans.: 72.

Problem Type 6.1.2 : Find the area of the region enclosed between $y = Expr_1(x)$ and $y = Expr_2(x)$, where they meet at three points.

Example Problem 6.1.2:

Find the area of the region enclosed between $y = x^3 - x$ and $y = 3x$.

Steps

1. Find the points of intersection by solving, for x the algebraic equation

$$Expr_1(x) = Expr_2(x) \quad .$$

If you get one root, then the region is infinite, and there is no (finite) answer. If you get two roots, let's call them $x_1 < x_2$, then the limits of the integral will be x_1 and x_2 (like in 6.1.1). If you get three roots, $x_1 < x_2 < x_3$, then you need to do TWO integrals, one $\int_{x_1}^{x_2}$ and the other $\int_{x_2}^{x_3}$.

2. For each of the intervals of integration (in this example there are two: (x_1, x_2) and (x_2, x_3)), determine which curve is on top, by plugging-in a random value (or sketching).

Example

1. $x^3 - x = 3x$ means $x^3 - 4x = 0$, i.e. $x(x-2)(x+2) = 0$ so we have *three* roots $x = -2$, $x = 0$ and $x = 2$.

2. Plugging-in $x = -1$ into $y = x^3 - x$ gives $y = 0$, while plugging-in $x = -1$ into $y = 3x$ gives $y = -3$. Since 0 is bigger than -3 , we have that on the interval $(-2, 0)$, $TOP = x^3 - x$ and $BOT = 3x$.

Regarding the interval $(0, 2)$, plugging-in $x = 1$ into $y = x^3 - x$ gives $y = 0$, while plugging-in $x = 1$ into $y = 3x$ gives $y = 3$. Since 3 is bigger than 0, we have that on the interval $(0, 2)$, $TOP = 3x$ and $BOT = x^3 - x$.

3. The area is the sum of the integrals

$$\int_{x_1}^{x_2} [TOP(x) - BOT(x)] +$$
$$\int_{x_2}^{x_3} [TOP(x) - BOT(x)] \quad ,$$

note that who is TOP and who is BOT (usually) gets switched between these intervals.

3. The area is

$$\int_{-2}^0 [(x^3 - x) - (3x)] + \int_0^2 [(3x) - (x^3 - x)] =$$
$$\int_{-2}^0 [x^3 - 4x] + \int_0^2 [-(x^3 - 4x)] =$$
$$[x^4/4 - 2x^2]_{-2}^0 + [-(x^4/4 - 2x^2)]_0^2 = 4 + 4 = 8 \quad .$$

Ans.: 8.