

How to Avoid Obvious Nonsense by Dr. Z.

Required Reading for prospective members of the Second Chance Club, and **Highly Recommended Reading** for everyone else.

These pages should be read carefully. There are some problems below for you to do yourself, but don't hand them in. Read everything carefully!

Numbers, Variables, Expressions, and Functions

Arithmetic is all about **numbers**, like 2, 3, 5. Algebra is also about numbers, but **abstract** numbers, denoted by letters, like a , b . Precalculus is all about **functions**, and Calculus is about **derivatives**, **anti-derivatives** (alias *indefinite intergals*), **definite integrals** and their manifold applications: for finding rates of change, maximum and minimum, areas, approximating roots, etc. etc.

What is a function?: A function is a “rule” for assigning numbers to numbers. The **full notation** for a function should be

$$x \rightarrow f(x) \quad ,$$

(Note: this “ \rightarrow ” has nothing to do with the “ \rightarrow ” used in limits).

Strictly speaking, the phrase “the function x^3 ” is grammatically incorrect, it should be “The function that takes any number x to the number x^3 ”. Or

$$f(x) = x^3 \quad .$$

Very Common Error: Confusing Numbers and functions.

Example: $\ln x$ is a function **but** $\ln 10$ is a **number**. Once you plug-in a number into a function, it is another number!

Wrong Wrong Wrong: One student did:

$$(\ln 10)' = \frac{1}{10} \quad .$$

This is a very bad conceptual error. Of course

$$(\ln x)' = \frac{1}{x} \quad ,$$

but $(\ln 10)'$ means derivative of $\ln 10$ with respect to x , and hence is 0, since the derivative of any *number* is always 0.

Do right now: Find the derivative with respect to x of the following

$$\cos((\ln 10)^3) \quad , \quad e^{\sin(11\pi/17)} \quad .$$

Equation of a tangent at a point

Example: Find the equation of the tangent line to $y = x^3$ at the point where $x = 2$.

Wrong Solution: When $x = 2$, $y = 8$, so that point is $(2, 8)$. The slope is the derivative $\frac{dy}{dx} = 3x^2$ so the equation of the tangent line is

$$(y - 8) = 3x^2(x - 2) \quad .$$

Why is it very wrong?: It is not just wrong, but **very** wrong, since this is the wrong **output**. The equation of the tangent line is an equation of a straight line, and hence of the form $y = mx + b$ for m and b specific numbers. It can't have x^2 in it.

Right Solution: When $x = 2$, $y = 8$, so that point is $(2, 8)$. The slope is the derivative $\frac{dy}{dx} = 3x^2$, in general, so at the **designated** point it is $m = 3 \cdot 2^2 = 12$. So the equation of the tangent line is

$$(y - 8) = 12(x - 2) \quad .$$

or

$$y = 12x - 16 \quad .$$

Now this is a **good-looking** equation of a straight line.

When not to use the quotient rule

Suppose that you had to find the derivative of

$$f(x) = \frac{\sin^3 x}{\ln 11}$$

Since this is division, a *quotient*, you may be tempted to use the quotient rule. Stricly speaking, this is legal, if you don't mess-up:

$$\begin{aligned} f'(x) &= \frac{(\sin^3 x)'(\ln 11) - (\sin^3 x)(\ln 11)'}{(\ln 11)^2} = \frac{(3 \sin^2 x)(\cos x)(\ln 11) - (\sin^3 x) \cdot 0}{(\ln 11)^2} = \\ &= \frac{3(\ln 11) \cos x \sin^2 x}{(\ln 11)^2} = \frac{3 \cos x \sin^2 x}{\ln 11} \quad . \end{aligned}$$

But this is a very **stupid** way. At any rate, $(\ln 11)' = 0$ **not** $1/11$, as I have already told you above.

The best way to do this problem is just to take $\frac{1}{\ln 11}$ **out** of the differentiation:

$$\left(\frac{\sin^3 x}{\ln 11} \right)' = \frac{1}{\ln(11)} (\sin^3 x)' = \frac{1}{\ln(11)} (3 \sin^2 x)(\cos x) = \frac{3}{\ln(11)} \cos x \sin^2 x$$

Obvious nonsense to avoid

square-roots of negative numbers (e.g. $\sqrt{-3}$); Log Natural of negative numbers, or 0 (e.g. $\ln(-3)$, $\ln(0)$); Negative distance.

Limit of a function at a point

$$\lim_{x \rightarrow a} f(x)$$

Input: A function $f(x)$ and a number a , or $a = \infty$ or $a = -\infty$.

Output: A number, or DNE (Does Not Exist). Sometimes the DNE can be further explicated to ∞ or $-\infty$.

An example of an obvious conceptual mistake:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} = \frac{-\sin x}{2x} \quad .$$

Why is that the wrong answer?: Because the output (the right side) is a function (depending on x).

Derivatives

Two Problem Types:

One type: Given a function $f(x)$, find $f'(x)$.

Second type: Given a function $f(x)$, find $f'(2)$ (or $f'(3)$ or whatever).

Don't confuse the two types of problems. For the first type, the output is a function. For the second type, the output is a number.

Example: Using the definition of the derivative, find $f'(3)$ if $f(x) = x^2$.

First way: Let's first compute $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x + 0 = 2x \quad . \end{aligned}$$

This is **not** the final answer. The question asked for $f'(3)$. **Ans.:** $f'(3) = 2 \cdot 3 = 6$.

Much better way: Implement $x = 3$ right away!

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} =$$

$$\lim_{h \rightarrow 0} \frac{6h + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(6 + h)}{h} = \lim_{h \rightarrow 0} (6 + h) = 6 + 0 = 6 \quad .$$

Three types of Optimization Problems

First Type: Find the largest area that a rectangle of perimeter 20 inches can have.

Second Type: Find the dimensions of the rectangle of perimeter 20 that has the largest area.

Answer to first type: 25 square-inches (you do it!)

Answer to Second type: 5×5 centimeters (you do it!).

Don't confuse the two types!

Third Type: Find the dimensions of the rectangle of perimeter 20 that has the largest area, and find that largest area.

Answer to Third type: The dimensions are 5×5 centimeters and the largest area is 25 square-centimeters.

Area Bounded by Two Curves

Area is **never** a negative number. If you will give me an answer to an area problem that is negative, I'll be very disappointed at you.

On the other hand, the answer to a definite integral:

$$\int_a^b f(x) dx$$

may be any **number** (positive, negative, or zero). Of course, if a and b are **numbers** than the answer is also a number. If one, or both, of a and b are *expressions* (i.e. depend on letters) than the answer is also an expression.

The following is Complete Nonsense:

$$\int_0^x (x+1)^3 dx \quad .$$

If the integral is with respect to x , that is indicated by what comes after the “d” in dx , then the *limit of integration* are not allowed to use x . Unfortunately, some books have these kind of integrals, but what they mean, and should have said is:

$$\int_0^x (t+1)^3 dt \quad ,$$

Or

$$\int_0^x (z+1)^3 dz \quad ,$$

or whatever, but you can't overwork the letter x .

Newton's Method

The **input** is an equation of the format

$$f(x) = 0 \quad ,$$

so before you do anything else, unless the problem is already phrased in this way, you must phrase the problem in that form, and get the **correct** $f(x)$.

Example: Use one step of Newton's method, with $x_0 = 100$ to find an approximation of $\sqrt{99}$.

First step of Solution: $x = \sqrt{99}$ means $x^2 = 99$ which means $x^2 - 99 = 0$, so $f(x) = x^2 - 99$.

Wrong Wrong Wrong: $f(x)$ is **not** \sqrt{x} .

Linearization

Now the input is a function (not an equation) $f(x)$, and a "nice" number a , such that you can do $f(a)$ without a calculator. You have to find an approximation for $f(b)$ where b is "near" a .

Example: Find the linearization of $f(x) = \sqrt{x}$ at $x = 100$ and use it to approximate $\sqrt{99}$.

Solution: $f(x) = x^{1/2}$, so $f'(x) = (1/2)x^{-1/2}$, and $f(100) = 10$, $f'(100) = 1/20$. The linearization is:

$$L(x) = 10 + \frac{1}{20}(x - 100) \quad .$$

To get an approximation for $\sqrt{99}$ you plug-in $x = 99$ and get:

$$f(99) \approx L(99) = 10 + \frac{1}{20}(99 - 100) = 10 + \frac{1}{20}(-1) = \frac{199}{20}$$

Common Nonsense: plugging-in $x = \sqrt{99}$ instead of the correct $x = 99$.

How would the Qualifying Test for membership at the Second Chance Club be?

It would be multiple-choice. For each problems, there are four choices. Only one of them is the right type. You are to find the one that is the right type.

Example: Find the shortest distance between the point $(1, 0)$ and the curve $y = \sqrt{x}$:

(A) -5.5 (B) $x^2 + y^2$ (C) $\sqrt{3}/2$ (D) -1.13 .

Obviously the answer is (C), since a distance is a **positive** numbers. (A) and (D) are negative numbers, and can never be distances. (B) is an expression, so it is even a bigger nonsense.