Dr. Z's Math151 Handout #2.1 [Limits, Rates of Change, and Tangent Lines]

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Problem Type 2.1.1 : The point $P(a, f(a))$ lies on the curve $y = f(x)$. (a) If Q is the point $(x, f(x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x: (i) x_1 (ii) a+tiny (iii) a+'very tiny'

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(a, f(a))$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P = (a, f(a))$.

Example Problem 2.1.1: The point $P(2, 8)$ lies on the curve $y = x^3$. (a) If Q is the point (x, x^3), use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x: (i) 3 (ii) 2.001 (iii) 2.0001.

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, 8)$.

(c) Using the slope from part (b), find the equation of the tangent line to the curve at $P = (2, 8)$.

Steps	**Example**
1. (a) The slope of the secant line between $P(a, f(a))$ and $Q(x, f(x))$ is $$(f(x) - f(a))/(x - a) \quad .$$	**1. (a)** (i) $(3^3 - 2^3)/(3 - 2) = (27 - 8)/1 = 19.$; (ii)$((2.001)^3 - 2^3)/(2.001 - 2) = (8.012006 - 8)/.001 = 12.006001$; (iii)$((2.0001)^3 - 2^3)/(2.0001 - 2) = (8.00120006 - 8)/.0001 = 12.00060$;
2. (b) The answers to (ii) and (iii) should be very close to each other and if they are both close to a 'nice' value, that would be a good guess.	**2. (b)** guessed slope=12.

3. (c) $(y - f(a)) = (slope)(x - a)$

3. (c) $(y - 8) = (12)(x - 2)$, hence,

$y = 12x - 24 + 8 = 12x - 16$.

Answer: $y = 12x - 16$.

Problem Type 2.1.2 : (a) In a certain planet the height of a stone thrown vertically upwards with velocity v_0 m/s is given by $h = v_0 t - At^2$. Find the average velocity in the time intervals (i) $[a, b]$ (ii) $[a, a + tiny]$ (iii) $[a, a + very \quad tiny]$.

Example Problem 2.1.2: (a) In a certain planet the height of a stone thrown vertically upwards with velocity 100 m/s is given by $h = 100t - t^2$. Find the average velocity in the time intervals

(i) $[1, 2]$ (ii) $[1, 1.01]$ (iii) $[1, 1.001]$.

(b) Estimate the instantenous velocity after one second.

Steps	**Example**
1. (a) The average velocity of a particle whose height (or distance) is given by $h(t)$, over a time interval $[a, b]$ is $$\frac{h(b) - h(a)}{b - a}.$$	**1. (a)** $h(t) = 100t - t^2$ so (i) $(h(2)-h(1))/(2-1) = ((100(2)-2^2)-(100(1) - 1^2))/1 = (196 - 99)/1 = 97$. (ii) $(h(1.01)-h(1))/(1.01-1) = ((100(1.01)-(1.01)^2) - (100(1) - 1^2))/.01 = 97.99$. (iii) $(h(1.001)-h(1))/(1.001-1) = ((100(1.001)-(1.001)^2) - (100(1) - 1^2))/.001 = 97.999$.
2. (b) The answers to (ii) and (iii) should be very close to each other, and if they are both close to a 'nice' value, that would be a good estimate.	**2. (b)** 98.