Dr. Z's Math151 Handout #2.4 [Limits and Continuity]

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Problem Type 2.4.1 : Explain why the function is discontinuous at $x = a$:

$$f(x) = \begin{cases} Expression(x), & \text{if } x \neq a; \\ \textbf{Number}, & \text{if } x = a. \end{cases}$$

Example Problem 2.4.1: Explain why the function is discontinuous at $x = 1$:

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1}, & \text{if } x \neq 1; \\ 3, & \text{if } x = 1. \end{cases}$$

Steps

1. Find

$$\lim_{x \to a} Expression(x) \quad ,$$

let's call it b.

2. If the function $f(x)$ would have been continuous at $x = a$ then **Number** should have been the limit from part **1**, what we called b. Since they are not the same, $f(x)$ is discontinuous at $x = a$.

Example

1.

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2} \quad .$$

2. If the function $f(x)$ would have been continuous at $x = 1$ then 3 should have been equal to $1/2$. Since they are not the same, $f(x)$ is discontinuous at $x = 1$.

Problem Type 2.4.2 : If $f(x) = Expression(x)$, show that there is a number c such that $f(c) = Number$.

Example Problem 2.4.2: If $f(x) = x^3 - x^2 + 3x$, show that there is a number c such that $f(c) = 20$.

Steps	**Example**
1. Assuming that the function is continous (and usually it is), you have, by trial and error (or by plotting), find two numbers a and b such that $Expression(a) < Number$ and $Expression(b) > Number$.	**1.** Since, $f(x) = x^3 - x^2 + 3x$, $f(0) = 0$ and $f(10) = 930$ (there are many other possibilities, of course).
2. Since $Number$ is between $f(a)$ and $f(b)$, we know for sure that there is a number c such that $f(c) = 20$. In fact there must be at least one such c between a and b.	**2.** Since 20 is between 0 and 930, we know for sure that there is a number c such that $f(c) = 20$. In fact there must be at least one such c between 0 and 10.

Problem Type 2.4.3 : Explain why the function is discontinuous at $x = a$:

$$f(x) = \begin{cases} LeftExpression(x), & \text{if } x < a; \\ RightExpression(x), & \text{if } x \geq a. \end{cases}$$

Example Problem 2.4.3: Explain why the function is discontinuous at $x = 0$:

$$f(x) = \begin{cases} e^x, & \text{if } x < 0; \\ x^2, & \text{if } x \geq 0. \end{cases}$$

Steps

1. Find the limit from the left at $x = a$,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} LeftExpression(x) \quad ,$$

and the limit from the right there:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a} RightExpression(x) \quad .$$

2. If $f(x)$ would have been continous, then these two numbers should have been the same. Since they are not, the function $f(x)$ is discontinuous at $x = a$.

Example

1. In this problem, the limit from the left at $x = 0$ is:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} e^x = e^0 = 1 \quad ,$$

and the limit from the right is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x^2 = 0 \quad .$$

2. Since these two numbers (1 and 0) are not the same, the function is **discontinuous** at $x = 0$.

Problem from a past Final [Spring 2008, #3 (8 points)] : For what value of the constant c is the function f continuous for all x? Here

$$f(x) = \begin{cases} cx^2 + 3, & \text{if } x \geq 5; \\ cx - 3, & \text{if } x < 5. \end{cases}$$

Solution: The limit from the left at $x = 5$ is

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} cx - 3 = 5c - 3 \quad .$$

The limit from the right at $x = 5$ is

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} cx^2 + 3 = c(5)^2 + 3 = 25c + 3 \quad .$$

If the function is going to be continuous at $x = 5$, we must have that the limit from the left equals the limit from the right, so we must have:

$$5c - 3 = 25c + 3 \quad .$$

Solving for c we get: $20c = -6$, So $c = -3/10$.

Ans.: The value of c that makes the function f continuous for all x is $c = -\frac{3}{10}$.