Dr. Z's Math151 Handout # 2.5 [Evaluating Limits Algebraically]

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Problem Type 2.5.1: Evaluate the limit if it exists:

$$\lim_{x \to a} f(x)$$

Example Problem 2.5.1 : Evaluate the limit if it exists:

$$\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$$

Steps

1. Try to plug $x = a$ into the function, if it makes sense (i.e. the denominator is not zero), then the limit is that value. For example, $\lim_{x\to 2}(x^2-1)/(x+1) = (2^2-1)/(2+1) = 1$. If the top is non-zero and the bottom 0, then it does not exists. If you get 0/0 then **SIMPLIFY AS MUCH AS POSSIBLE**.

2. Try to plug in $x = a$ again, if you still get 0/0, go back to step **1**. Otherwise you are done.

Example

1. Plugging in $x = -4$ in $\frac{x^2+5x+4}{x^2+3x-4}$ gives $\frac{(-4)^2+5(-4)+4}{(-4)^2+3(-4)-4}$, which is 0/0. So we must **simplify**. Factoring the top and bottom we get

$$\frac{x^2+5x+4}{x^2+3x-4} = \frac{(x+1)(x+4)}{(x-1)(x+4)} = \frac{x+1}{x-1}$$

2. So

$$\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$$

$$= \lim_{x \to -4} \frac{x+1}{x-1}$$

Plugging in $x = -4$,

$$= \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}, \quad .$$

Final Answer: 3/5.

Problem Type 2.5.2: Evaluate the limit if it exists:

$$\lim_{x \to a} f(x) \quad ,$$

where $f(x)$ features radical (square-root) signs.

Example Problem 2.5.2 : Evaluate the limit if it exists:

$$\lim_{x \to 0} \frac{x}{\sqrt{2+x} - \sqrt{2-x}}$$

Steps

1. In a typical problem, you would have, either at the top or bottom, something of the form $\sqrt{SomeThing} - \sqrt{SomeThingElse}$. The trick is to multiply **both top and bottom** by the **conjugate**:

$$\sqrt{SomeThing} + \sqrt{SomeThingElse} \quad .$$

In other problems, you would have

$$\sqrt{SomeThing} + \sqrt{SomeThingElse} quad,$$

in which case you would multiply both top and bottom by

$$\sqrt{SomeThing} - \sqrt{SomeThingElse} \quad .$$

Example

1. The conjuate of $\sqrt{2+x} - \sqrt{2-x}$ is $\sqrt{2+x} + \sqrt{2-x}$. Sticking it both at the top and bottom gives us:

$$\lim_{x \to 0} \frac{x}{\sqrt{2+x} - \sqrt{2-x}}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{2+x} + \sqrt{2-x})}{(\sqrt{2+x} - \sqrt{2-x})(\sqrt{2+x} + \sqrt{2-x})}$$

2. Use that famous rule $(A-B)(A+B) = A^2 - B^2$ to simplify

$$(\sqrt{SomeThing} + \sqrt{SomeThingElse}) \cdot$$

$$(\sqrt{SomeThing} - \sqrt{SomeThingElse}) =$$

$$SomeThing - SomeThingElse \quad .$$

Simplify further as much as you can.

2.

$$= \lim_{x \to 0} \frac{x(\sqrt{2+x} + \sqrt{2-x})}{(\sqrt{2+x} - \sqrt{2-x})(\sqrt{2+x} + \sqrt{2-x})}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{2+x} + \sqrt{2-x})}{(\sqrt{2+x})^2 - (\sqrt{2-x})^2}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{2+x} + \sqrt{2-x})}{(2+x) - (2-x)}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{2+x} + \sqrt{2-x})}{2x}$$

$$= \lim_{x \to 0} \frac{\sqrt{2+x} + \sqrt{2-x}}{2}$$

3. Plug-in $x = a$.

3.

$$= \frac{\sqrt{2+0} + \sqrt{2-0}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2} \quad .$$

Ans.: $\sqrt{2}$.

A Midterm I style-problem (from Review Problems for Midterm One, Spring 2008)

Evaluate the limit if it exists:

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$$

Solution:

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$$

$$= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{(\sqrt{x+1})^2-2^2}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{(\sqrt{x+1}+2)}$$

$$= \frac{1}{(\sqrt{3+1}+2)} = \frac{1}{(\sqrt{4}+2)} = \frac{1}{(2+2)} = \frac{1}{4} \quad .$$

Ans.: The limit exists and equals $\frac{1}{4}$.