Dr. Z's Math151 Handout #3.11 [Related Rates]

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Problem Type 3.11.1 : If $F(x,y)=c$ and $dy/dt=a$, find dx/dt when $y=b$.

Example Problem 3.11.1: If $x^3+y^3=9$ and $dy/dt=6$ find dx/dt when $y=2$.

Steps

1. Find the corresponding value of x by solving $F(x,b)=c$.

2. Differentiate the relation $F(x,y)=c$ with respect to t, like in implicit differentiation, treating x and y as unknown functions of t.

3. In the relationship that you got in step 2 between $x, t, dx/dt, dy/dt$, plug-in the values for x, y, and dy/dt and solve for dx/dt.

Example

1. When $y=2$, $x^3+y^3=9$ becomes $x^3+2^3=9$ so $x=1$.

2.

$$\frac{d}{dt}(x^3+y^3)=0 \quad ,$$

so

$$3x^2\frac{dx}{dt}+3y^2\frac{dy}{dt}=0 \quad .$$

3.

$$3\cdot 1^2\frac{dx}{dt}+3\cdot 2^2\cdot 6=0 \quad ,$$

so

$$\textbf{Ans.}: \quad \frac{dx}{dt}=-24 \quad .$$

Problem Type 3.11.2 : A water trough is of length L and a cross-section has the shape of an isosceles trapezoid that has width a at the bottom, b at the top, and height H. If the trough is being filled with water at a rate of d, how fast is the water level rising when the water had depth x ?

Example Problem 3.11.2:

A water trough is of length 30 meters and a cross-section has the shape of an isosceles trapezoid that has width .3 meters at the bottom, .8 meters at the top, and height .5 meters. If the trough is being filled with water at a rate of .2 m^3/min, how fast is the water level rising when the water had depth .3 meters?

Steps

1. If the water-height is x, by similar triangles the top side of the water trapezoid is

$$a + \frac{(b-a)x}{H} \quad ,$$

since when $x = 0$ it is a and when $x = H$ it is b. So the area of the water cross-section is

$$\frac{x}{2}(2a + \frac{(b-a)x}{H}) \quad ,$$

and multiplying by the length, the volume is

$$V(x) = \frac{Lx}{2}(2a + \frac{(b-a)x}{H}) \quad ,$$

this is your expression for the volume.

2. Find dV/dt by implicit differentiation, getting an expression featuring x and dx/dt.

3. Plug-in the known values of x and dV/dt and solve for dx/dt.

Example

1. If the water-height is x, by similar triangles the top side of the water trapezoid is

$$.3 + \frac{(.8 - .3)x}{.5} = .3 + x \quad ,$$

since when $x = 0$ it is .3 and when $x = .5$ it is .8. So the area of the water cross-section is

$$\frac{x}{2}(.3 + (.3 + x)) = \frac{x}{2}(.6 + x) \quad ,$$

and multiplying by the length, the volume is

$$V(x) = \frac{30x}{2}(.6+x) = 15x(.6+x) = 9x+15x^2$$

this is your expression for the volume.

2.

$$\frac{dV}{dt} = \frac{d}{dt}(9x + 15x^2) = 9\frac{dx}{dt} + 30x\frac{dx}{dt}$$

3.

$$.2 = 9\frac{dx}{dt} + 30 \cdot .3\frac{dx}{dt}$$

so

$$.2 = 18\frac{dx}{dt}$$

and

$$\textbf{Ans.}: \quad \frac{dx}{dt} = \frac{.2}{18} = \frac{1}{90} = .0111\, m/min$$

Problem Type 3.11.3 : If two resistors with resistence R_1 and R_2 are connected in parallel, then the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad .$$

If R_1 and R_2 are increasing at a rate of a and b ohms per sec. respectively, how fast is R changing when $R_1 = A$ and $R_2 = B$.

Example Problem 3.11.3: If two resistors with resistence R_1 and R_2 are connected in parallel, then the total resistance is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad .$$

If R_1 and R_2 are increasing at a rate of .6 and .4 ohms per sec. respectively, how fast is R changing when $R_1 = 160$ and $R_2 = 200$.

Steps

1. Differentiate the above relation with respect to t, getting

$$\frac{d}{dt}\frac{1}{R} = \frac{d}{dt}\frac{1}{R_1} + \frac{d}{dt}\frac{1}{R_2} \quad .$$

So

$$\frac{\frac{-dR}{dt}}{R^2} = \frac{\frac{-dR_1}{dt}}{R_1^2} + \frac{\frac{-dR_2}{dt}}{R_2^2}$$

2. Find the value of R corresponding to $R_1 = A$ and $R_2 = B$ by solving for R the equation $1/R = 1/R_1 + 1/R_2$.

Example

1. Same as in general.

2. Find the value of R corresponding to $R_1 = 160$ and $R_2 = 200$ by solving for R the equation $1/R = 1/160 + 1/200 = 9/800$, so $R = 800/9$.

3. Plug the values for R found in step 2, and the values for R_1, R_2 dR_1/dt, dR_2/dt given by the problem into the relationship found in step 1, and solve for dR/dt,

3.

$$\frac{-\frac{dR}{dt}}{(800/9)^2} = \frac{-.6}{160^2} + \frac{-.4}{200^2}$$

So

$$\frac{dR}{dt} = (800/9)^2 \left(\frac{.6}{160^2} + \frac{.4}{200^2} \right)$$

$$= (.6)(5/9)^2 + (.4)(4/9)^2 =$$

$$(15{+}6.4)/81 = 21.4/81 = 107/405\, ohm/s.$$

Problem from a Previous Final Exam (Spring 2008, #14 (11 points))

A spotlight on the ground shines on a wall 20 meters away. A man 2 meters tall walks from the spotlight to the wall, at a speed of 0.4 m./sec. ; his path is perpendicular to the wall. Let x be the distance from his feet to the spotlight and let h be the height of the shadow on the wall. Also let θ be the angle of elevation at the spotlight from the horizontal to the top of his head.

(a) Draw a sketch of the problem, and find a formula relating h and x.

(b) When the woman (sic!) is 4 meters from the wall, find the height of the shadow and the rate of change of the height of the shadow.

(c) What is the rate of change of θ at the time the woman (sic!) is 4 meters from the wall?

Comment: The man became a woman, so if you were a wiseguy you could say "not enough information, you tell me stuff about a man, and you expect me to answer stuff about a woman!", but let's assume that the "woman" is the the man from the beginning.

Solution

(a)

From **similar triangles**

$$\frac{h}{20} = \frac{2}{x} \quad ,$$

so

$$h = \frac{40}{x} \quad .$$

(b) When $x = 4$, $h = 40/4 = 10$. By implicit differentiation

$$\frac{dh}{dt} = \frac{-40}{x^2} \cdot \frac{dx}{dt} \quad .$$

We are told by the problem that $\frac{dx}{dt} = 0.4$ so

$$\frac{dh}{dt} = \frac{-40}{4^2} \cdot (0.4) = -1 \quad .$$

Ans. to (b): When the woman is 4 meters from the wall, the height of the shadow is 10 meters and the rate of change of the height of the shadow is -1 m/sec.

(c) By trig:

$$\tan\theta = \frac{2}{x}$$

By implicit differnetiation

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{-2}{x^2} \cdot \frac{dx}{dt}$$

Now $\tan\theta = 2/4 = 1/2$, so $\sec^2\theta = 1 + (1/2)^2 = \frac{5}{4}$, and we get

$$\frac{5}{4}\frac{d\theta}{dt} = \frac{-2}{4^2} \cdot (0.4) = \frac{-1}{20} \quad ,$$

solving for $\frac{d\theta}{dt}$, we get:

$$\frac{d\theta}{dt} = \frac{\frac{-1}{20}}{\frac{5}{4}} = \frac{-1}{25} \quad .$$

Ans. to (c): When $x = 4$, the rate of change of θ is $-1/25$ radians per seconds.