Dr. Z's Math151 Handout #3.2

Problem Type 3.2.1: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x)$ is as given

Example Problem 3.2.1: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x) = x^2$.

Steps

1. Recall the definition of the derivative, using either version (the first version is usually easier), and implement it for the particular function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Use algebra to simplify the expression whose limit you are taking, until it makes sense to plug-in $h = 0$.

Example

1.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad .$$

Here $f(x) = x^2$ so

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \quad .$$

2. Opening-up parantheses and simplifying, we get that this equals

$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{(2x+h)h}{h} = \lim_{h \to 0} (2x + h) = 2x \quad .$$

Ans.: $f'(x) = 2x$.

Problem Type 3.2.2: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x)$ is as given

Example Problem 3.2.2: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x) = x^{-1}$.

Steps

1. Recall the definition of the derivative, using either version (the first version is usually easier), and implement it for the particular function.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

2. Use algebra to simplify the expression whose limit you are taking, until it makes sense to plug-in $h = 0$.

Example

1.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad .$$

Here $f(x) = 1/x$ so

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad .$$

2. Taking common-denonimator at the top we get

$$f'(x) = \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} = \lim_{h\to 0} \frac{\frac{-h}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{(x+0)x} = -\frac{1}{x^2} =$$

Ans.: $f'(x) = -\frac{1}{x^2}$

Problem Type 3.2.3: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x)$ is as given

Example Problem 3.2.3: Compute $f'(x)$ from the limit definition (no credit for other methods!), where $f(x) = \sqrt{x+2}$.

Steps

Example

1. Recall the definition of the derivative, using either version (the first version is usually easier), and implement it for the particular function.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

1.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad .$$

Here $f(x) = \sqrt{x+2}$ so

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \quad .$$

2. Use algebra to simplify the expression whose limit you are taking, until it makes sense to plug-in $h = 0$. In this case, where we have square-roots we use the trick **multiply the top and bottom by the conjugate**.

2. Multiplying top and bottom by the **conjugate**, $\sqrt{x+h+2}+\sqrt{x+2}$, we have

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h+2} - \sqrt{x+2})(\sqrt{x+h+2} + \sqrt{x+2})}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+2})^2 - \sqrt{x+2}^2}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+0+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}} \quad .$$

Ans.: $f'(x) = \frac{1}{2\sqrt{x+2}}$.

A Problem from a Previous Final (Spring 2008, #1)

Write the definition of the derivative $f'(x)$ as a limit and use this definition to find the derivative $f'(2)$ of $f(x) = \frac{2}{1-x}$. Show all your work.

Comment: Note that here we are asked to find the derivative at a **specific** point, so it is slighltly easier (it the kind of problem done in sect. 3.1). The answer should be a **number** not a function. It should have no x in it! Do not confuse the general $f'(x)$ with the specific $f'(2)$. Of course you can first find $f'(x)$ and at the end plug-in $x = 2$, but it is better to plug-in $x = 2$ right away.

Solution The definition of the derivative at a general x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad .$$

Plugging-in $x = 2$ we have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \quad .$$

Our function is $f(x) = \frac{2}{1-x}$. We have, in this problem

$$f'(2) = \lim_{h \to 0} \frac{\frac{2}{1-(2+h)} - \frac{2}{1-(2)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{-1-h} - (-2)}{h} = \lim_{h \to 0} \frac{\frac{-2}{1+h} + 2}{h} = \lim_{h \to 0} \frac{\frac{-2+2(1+h)}{1+h}}{h} = \lim_{h \to 0} \frac{\frac{-2+2+2h}{1+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2h}{1+h}}{h} = \lim_{h \to 0} \frac{2}{1+h} = \frac{2}{1+0} = 2 \quad .$$

Ans.: $f'(2) = 2$.

Warning: Very soon you will learn other ways of finding derivatives, using "differentiation rules". You are not allowed to use them in this kind of problems, where it tells you to use the "defition of the derivative as a limit".