

Dr. Z's Math151 Handout #3.8 [Implicit Differentiation]

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Problem Type 3.8.1 : Find dy/dx by implicit differentiation, where you know that $Expression_1(x, y) = Expression_2(x, y)$.

Example Problem 3.8.1: Find dy/dx by implicit differentiation if $x^2 - 2xy + y^3 = 5x$

Steps

1. Make the right side zero by moving everything to the left side. Getting something of the form $F(x, y) = 0$.

2. Use all the rules of differentiation treating y as an abstract function of x , and using the chain rule. So, for example, while $(x^3)' = 3x^2$, $(y^3)' = 3y^2y'$. So $(y^n)' = ny^{n-1}y'$, $(\sin y)' = (\cos y)y'$, $(e^y)' = e^yy'$ etc. In general $(f(y))' = f'(y)y'$, so you can pretend that y is a variable, but whenever you differentiate an expression in y , stick a y' at the end. You usually have to also use the product rule, and sometimes the quotient rule.

3. Use algebra to solve for y' in terms of x and y , by keeping all the terms involving y' on the left, and moving all the other terms to the right. Then factor y' out at the left, and finally solve for y' .

Example

1. $x^2 - 2xy + y^3 - 5x = 0$

2.

$$\begin{aligned}(x^2 - 2xy + y^3 - 5x)' &= (x^2)' - (2xy)' + (y^3)' - (5x)' \\ &= 2x - ((2x)'y + 2xy') + 3y^2y' - 5 =\end{aligned}$$

$$2x - (2y + 2xy') + 3y^2y' - 5 = 2x - 2y - 2xy' + 3y^2y' - 5 = 0$$

3. $-2xy' + 3y^2y' = 2y - 2x + 5$ hence $(3y^2 - 2x)y' = 2y - 2x + 5$ and we get
Ans.: $y' = \frac{2y - 2x + 5}{3y^2 - 2x}$.

Problem Type 3.8.2 : Use implicit differentiation to find the equation of the tangent line to the curve $F(x, y)$ at the point (a, b) .

Example Problem 3.8.2: Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $(0, \frac{1}{2})$.

Steps

1. First (optionally) you may want to make sure that the given point indeed lies on the curve by plugging it in. Then perform implicit differentiation.

2. Now do not solve for y' right away (like you did in the 3.6.1), but you plug-in for (x, y) the given point (a, b) . Then you solve for y' getting a *number*, which is the slope at the given point.

3. As usual the equation of the tangent is $(y - b) = m(x - a)$

Example

1. $0^2 + (1/2)^2 = (2 \cdot 0^2 + 2 \cdot (1/2)^2 - 0)^2$, so the point is OK. Now

$$\begin{aligned} (x^2 + y^2)' - ((2x^2 + 2y^2 - x)^2)' &= \\ 2x + 2yy' - 2(2x^2 + 2y^2 - x)(2x^2 + 2y^2 - x)' &= \\ 2x + 2yy' - 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1) &= 0 \quad . \end{aligned}$$

2. Plugging in $x = 0, y = \frac{1}{2}$ gives

$$2 \cdot 0 + 2(1/2)y' - 2(2 \cdot 0^2 + 2 \cdot (1/2)^2 - 0)(4 \cdot 0 + 4(1/2)y' - 1) = 0 \quad .$$

Hence $y' - (2y' - 1) = 0$ and $-y' + 1 = 0$, so, solving for y' we get $y' = 1$, i.e. the slope of the tangent, m , is 1.

3. $y - \frac{1}{2} = 1(x - 0)$. **Ans.:** $y = x + \frac{1}{2}$.

A problem from a previous Final Exam (Spring 2008, #10 (12 points))

Find the equation of the tangent line at the point $(1, 2)$ to the curve defined by the equation

$$y^2 - 2xy - x^2 - 3x = -4 \quad .$$

Solution: First let's make sure that our point indeed lies on the curve, or else refuse to do it. Plugging-in $x = 1, y = 2$ we need this to be true:

$$(2)^2 - 2(1)(2) - 1^2 - 3(1) = -4 \quad ,$$

Doing the calculations on the left we get $4 - 4 - 1 - 3 = -4$, which is correct, so the problem makes sense.

Now differentiate both sides, using the rules of implicit differentiation.

$$(y^2 - 2xy - x^2 - 3x)' = (-4)' \quad ,$$

and we get

$$2yy' - (2xy' + (2x)'y) - 2x - 3 = 0 \quad .$$

Cleaning-up

$$2yy' - 2xy' - 2y - 2x - 3 = 0 \quad .$$

Now plug-in $x = 1$, $y = 2$ and solve for y'

$$2(2)y' - 2(1)y' - 2(2) - 2(1) - 3 = 0 \quad .$$

Doing the algebra:

$$4y' - 2y' - 4 - 2 - 3 = 0 \quad ,$$

$$2y' - 9 = 0 \quad ,$$

that yields $y' = 9/2$. So the **slope** of the tangent line, at the designated point $(1, 2)$ is $m = 9/2$. Finally use the famous point-slope equation $(y - y_0) = m(x - x_0)$:

$$y - 2 = (9/2)(x - 1) \quad ,$$

which simplifies to $y = 9x/2 - 5/2$.

Ans. The equation of the tangent line is $y = \frac{9x}{2} - \frac{5}{2}$ (or $2y - 9x + 5 = 0$).