Dr. Z's Math151 Handout #4.2 [Extreme Values]

By Doron Zeilberger

Problem Type 4.2.1 : Find the critical numbers of the function $f(x) = Expression(x)$.

Example Problem 4.2.1: Find the critical numbers of the function $f(x) = x^3 + 3x^2 - 24x$.

Steps	**Example**
1. Find the derivative of $f(x)$, and set it equal to 0.	**1.** $f'(x) = (x^3 + 3x^2 - 24x)' =$ $3x^2 + 6x - 24$. We have to solve $3x^2 + 6x - 24 = 0$.
2. Solve this equation, by any method.	**2.** $3x^2 + 6x - 24 = 0$ is the same as $3(x^2 + 2x - 8) = 0$, which is the same as $3(x + 4)(x - 2) = 0$. There are two roots $x = -4$ and $x = 2$.
3. If applicable, find when $f'(x)$ is undefined (usually because it blows up, for example for $f(x) = x^{1/2}$, $f'(x) = x^{-1/2} = 1/x^{1/2}$, and $f'(x)$ is undefined at $x = 0$, so in that case 0 would be a critical point.) In our problem $f'(x)$ is always defined (since it is a polynomial), so this other kind of critical numbers do not show up. The critical numbers are obtained by combining these two kinds.	**3.** $f'(x) = 3x^2 + 6x - 24$ is always defined, so **Ans.:** The set of critial numbers is $\{-4, 2\}$.

Problem Type 4.2.2 : Find the absolute maximum and absolute minimum values of $f(x) = \mathit{Expression}(x)$ on the closed interval $[a, b]$.

Example Problem 4.2.2: Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x + 2$ on the interval $[0, 2]$.

Steps	Example
1. Find the critial numbers of $f(x)$ by solving $f'(x) = 0$.	**1.** $$(x^3 - 3x + 2)' = 3x^2 - 3 = 3(x-1)(x+1) \quad .$$ Solving $3(x-1)(x+1) = 0$ gives the critical numbers $x = -1$ and $x = 1$.
2. The slate of **finalists** consists of those critical numbers from step 1 that **belong** in the specified interval **plus** the **endpoints** a and b.	**2.** $x = -1$ does not belong to the interval $[0, 2]$ but $x = 1$ does. The set of **finalists** in the contest is $\{0, 1, 2\}$ consisting of the two endpoints 0 and 2 and the critical number $x = 1$ that lies in the interval $[0, 2]$.
3. For each of the finalists, plug-in into $f(x)$, and compare the scores. The smallest of these is the **absolute minimum value** and The largest of these is the **absolute maximum value**. **Note:** Sometimes they may ask about the **absolute minimum** and **absolute maximum** of the function in the given interval. These are the correspoding locations. Here the abslute mimimum is at $x = 1$ and the abslute maximum is at $x = 2$.	**3.** $f(x) = x^3 - 3x + 2$. So $$f(0) = 2 \quad ,$$ $$f(1) = 1^3 - 3 \cdot 1 + 2 = 0 \quad ,$$ $$f(2) = 2^3 - 3 \cdot 2 + 2 = 4 \quad .$$ It follows that the **absolute minimum value** is 0 and the **absolute maximum value** is 4 .

Problem from a Previous Final Exam (Spring 2008 #6 (10 points))

Find the absolute maximum and the absolute minimum of $f(x) = x^{2/3} - 2x^{1/3}$ for $-1 \leq x \leq 8$.

Solution $f'(x) = (2/3)x^{-1/3} - 2(1/3)x^{-2/3} = (2/3)(x^{-1/3} - x^{-2/3})$. The critical points are where $f'(x)$ is not defined (i.e. $x = 0$ in this case) and where $f'(x) = 0$. Solving $f'(x) = 0$ gives

$$x^{-1/3} = x^{-2/3}$$

Raising both sides to the power (-3) we get

$$(x^{-1/3})^{-3} = (x^{-2/3})^{-3} \quad ,$$

so

$$x = x^2 \quad ,$$

so

$$x - x^2 = 0$$

so

$$x(x-1) = 0$$

and we get $x = 0$ and $x = 1$.

Including these plus the **endpoints** gives you that the **finalists** are $\{-1, 0, 1, 8\}$. Now we are ready for the **final contest**.

$f(-1) = (-1)^{2/3} - 2(-1)^{1/3} = 1 + 2 = 3$

$f(0) = 0$

$f(1) = 1^{2/3} - 2 \cdot 1^{1/3} = -1$

$f(8) = 8^{2/3} - 2 \cdot 8^{1/3} = 4 - 2 \cdot 2 = 0$

The **absolute maximum value** is 3 at $x = -1$.

The **absolute minimum value** is -1 at $x = 1$.

Ans. The **absolute maximum value** is 3 at $x = -1$ and the **absolute minimum value** is -1 at $x = 1$.