

Dr. Z's Math151 Handout #4.6 [Applied Optimization]

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Problem Type 4.6.1 :

A farmer wants to fence an area of A square units and then divide it into $n + 1$ parts by placing n parallel fences parallel to one of the sides of the rectangle. How would he do as to minimize the cost of the fence?

Example Problem 4.6.1:

A farmer wants to fence an area of 5000 square feet and then divide it into 3 parts by placing 2 parallel fences parallel to one of the sides of the rectangle. How would he do as to minimize the cost of the fence?

Steps

1. Give names to the variables of the problems,

2. Write down the constraint, and translate it into mathematics. $xy = A$. Express y in terms of x .

3. Find an expression for the goal function, in this kind of problem, the length of fencing is $2x + 2y + nx$.

Example

1. In this case, since we have to decide about a rectangle, let's call the side parallel to the interior fences x and the other side y .

2. $xy = 5000$. $y = 5000/x$.

3. Find an expression for the goal function.

Goal: $2x + 2y + 2x = 4x + 2y$.

4. Using step 1, express the goal function in terms of x alone, call it $f(x)$.

$$f(x) = (2 + n)x + \frac{2A}{x} .$$

5. Using calculus, minimize $f(x)$ by taking its derivative and setting it equal to zero.

4.

$$f(x) = 4x + \frac{10000}{x} .$$

5.

$$f'(x) = (4x + 10000x^{-1})' = 4 - 10000x^{-2} = 4 - \frac{10000}{x^2}$$

$$f'(x) = 0 \text{ when } x^2 = 2500 \text{ so } x = 50, \text{ and } y = 5000/50 = 100.$$

Ans.: The dimension of the rectangle should be 50×100 . The total length of fencing is then 400 feet.

Problem Type 4.6.2 : Find the point on the ellipse $ax^2 + by^2 = c$ that are furthest from the point (p, q) .

Example Problem 4.6.2: Find the point on the ellipse $x^2 + 4y^2 = 4$ that is furthest from the point $(0, 1)$.

Steps

1. Write down the formula for the distance square from a general point (x, y) and the designated point. This is your goal function.

2. The constraint is the equation of the ellipse. Take one of the variables (whatever is convenient) as the main variable.

Example

1. Goal: $(x - 0)^2 + (y - 1)^2 = x^2 + y^2 - 2y + 1$.

2. $x^2 + 4y^2 = 4$ means that $x^2 = 4 - 4y^2$. It is more convenient to take y as the main variable, so **Goal:** $f(y) = 4 - 4y^2 + y^2 - 2y + 1 = 5 - 2y - 3y^2$.

3. Find the maximum of $f(y)$, by taking its derivative and setting equal to 0.

3. $df/dy = -2 - 6y = 0$, this means $y = -1/3$, and $x^2 = 4 - 4/9 = 32/9$ and $x = \pm 4\sqrt{2}/3$.

Ans.: The farthest points are $(\pm 4\sqrt{2}/3, -1/3)$

Problem Type 4.6.3 : A cylinder can (with or w/o the top) is made to contain $V \text{ cm}^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

[Variations on this problem have two kinds of metal, one for the bottom, and another for the round side, with different prices per unit area, and you have to minimize the cost]

Example Problem 4.6.3:

A cylinder can without the top is made to contain $V \text{ cm}^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

Steps

1. Decide who are the variables?

2. Write down the constraint, and express one of the variables in terms of the other.

Example

1. The variables are the **radius**, r , and the **height** h .

2. The volume is fixed at V . The formula for the volume of the cylinder is $\pi r^2 h$.

constraint: $\pi r^2 h = V$. Hence $h = V/(\pi r^2)$.

3. Your goal is to minimize the amount of metal. The area of the metal consists of two parts: 1) the bottom 2) the sides.

3. The area of the bottom is πr^2 , the area of the side is $2\pi r h$ (if you cut it and roll it flat, you would have a rectangle with sides $2\pi r$ and h). **goal:** $\pi r^2 + 2\pi r h$, and in terms of r alone, using step 2,

$$f(r) = \pi r^2 + 2Vr^{-1}$$

4. Use calculus to minimize the goal function.

4.

$$f'(r) = (\pi r^2 + 2Vr^{-1})' = 2\pi r - 2Vr^{-2}$$

this is zero when $r = (V/\pi)^{1/3}$. By step 2, $h = V/(\pi(V/\pi)^{2/3}) = (V/\pi)^{1/3}$.

Ans.: Both radius and height should be equal to $(V/\pi)^{1/3}$.

A problem from a previous Final (Spring 2008, #7 (12 points))

In the right triangle $\triangle ABC$, the right angle is at C and the legs are $|AC| = 4$ and $|BC| = 12$. A rectangle is to be placed inside the triangle, with one corner at C and the opposite corner on the hypotenuse. What are the dimensions of such a rectangle which has largest area?

Solution: First, draw a picture. It is most convenient to put the point C at the origin $C = (0, 0)$, the point A on the x -axis, $A = (4, 0)$, and the point B on the y -axis, $B = (0, 12)$.

Now we need to find the equation of the hypotenuse. It is the line joining $(0, 12)$ and $(4, 0)$. Its slope is $m = (0 - 12)/(4 - 0) = -3$, so by the famous equation $(y - y_0) = m(x - x_0)$ we get

$$y - 0 = -3(x - 4)$$

that simplifies to

$$y = 12 - 3x \quad .$$

This is your **constraint**.

The goal function is the **area** that equals xy . Using the above constraint, we have that the area, let's call it $f(x)$ is given by

$$f(x) = x(12 - 3x) = 12x - 3x^2.$$

So far it was all geometry and algebra. Now it is time to do **calculus**. We need to find for what value of x is $f(x)$ maximal. Obviously x is between 0 and 4.

Computing $f'(x)$ we get

$$f'(x) = 12 - 6x.$$

Setting it equal to 0

$$12 - 6x = 0$$

and solving we get that x equals 2. The endpoints are 0 and 4, so the finalists are $\{0, 2, 4\}$ and plugging-it in $f(x)$ we get $f(0) = 0, f(2) = 12, f(4) = 0$. So the **absolute maximum value** happens to be 12, and it takes place at $x = 2$.

Finally we need the corresponding y . Plugging-it into $y = 12 - 3x$, we get $y = 12 - 3 \cdot 2 = 6$.

Ans.: The dimensions of such a rectangle which has largest area is 2×6 .

Comment: Some problems **also** ask you “what is the maximal area?”. In that case you would also say, “and the maximum area is 12”. But in **this** problem they didn’t ask it, so don’t bother. In other problems they **only** ask for the maximum area, in that case just say “the maximal area is 12”.