Dr. Z's Math151 Handout #4.7 [L'Hôspital's Rule]

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Problem Type 4.7.1 : Given certain limits of certain functions, $f(x), g(x), \ldots$ at a designated point $x = a$, determine whether the limits (at that very same point $x = a$) of the quotient $f(x)/g(x)$, product $f(x)g(x)$, difference $f(x) - g(x)$, and exponentiation $f(x)^{g(x)}$ are indeterminate limits.

Example Problem 4.7.1: Given that

$$\lim_{x \to a} f(x) = 0 \quad , \quad \lim_{x \to a} g(x) = 0 \quad ,$$

$$\lim_{x \to a} p(x) = \infty \quad , \quad \lim_{x \to a} q(x) = \infty \quad ,$$

which of the following are indeterminate forms. (a) $\lim_{x \to a}[f(x)/g(x)]$, (b) $\lim_{x \to a}[f(x)p(x)]$, (c) $\lim_{x \to a}[p(x) - q(x)]$. (d) $\lim_{x \to a}[f(x)^{g(x)}]$.

Steps

a. For a quotient the limit is *indeterminate* whenever 'plugging in' yields $0/0$ or ∞/∞.

b. For a product the limit is *indeterminate* whenever 'plugging in' yields $0 \cdot \infty$ or $\infty \cdot 0$, (and of course $0 \cdot -\infty$ or $-\infty \cdot 0$).

c. The limit of a difference is indeterminate whenever it is of the type $\infty - \infty$ (or $(-\infty) - (-\infty)$).

Example

a.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad ,$$

hence this is an indeterminate form.

b.

$$\lim_{x \to a}[f(x)p(x)] =$$

$$\lim_{x \to a} f(x) \cdot \lim_{x \to a} p(x) =$$

$$0 \cdot \infty \quad ,$$

hence the limit is indeterminate.

c.

$$\lim_{x \to a}[p(x) - q(x)] = \infty - \infty \quad ,$$

hence the limit is indeterminate.

d. The limit of an exponentiation is indeterminate whenever it is of the type 0^0, ∞^0, or 1^∞

d.

$$\lim_{x \to a} [f(x)^{g(x)}] = 0^0$$

hence the limit is indeterminate.

Problem Type 4.7.2 : Use L'Hospital's rule, if appropriate to evaluate

$$\lim_{x \to a} \frac{TOP(x)}{BOT(x)} \quad .$$

Example Problem 4.7.2: Use L'Hospital's rule, if appropriate to evaluate

$$\lim_{x \to 1} \frac{x^8 - 1}{x^3 - 1} \quad .$$

Steps

1. First plug-in $x = a$ into $TOP(x)/BOT(x)$ and see whether $TOP(a)/BOT(a)$ yields $0/0$ or ∞/∞. If it does, then L'Hospital's rule is applicable.

2. Invoke L'Hospital's rule

$$\lim_{x \to a} \frac{TOP(x)}{BOT(x)} = \lim_{x \to a} \frac{TOP'(x)}{BOT'(x)} \quad .$$

If you still get an indeterminate form (in this example you don't), keep doing it, until you get a doable limit.

3. Evaluate the limit, by simplifying and plugging-in.

Example

1. Plugging-in $x = 1$ into $\frac{x^8-1}{x^3-1}$ gives $0/0$, so L'Hospital's rule is applicable.

2.

$$\lim_{x \to 1} \frac{x^8 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x^8 - 1)'}{(x^3 - 1)'} = \lim_{x \to 1} \frac{8x^7}{3x^2} \quad .$$

3.

$$= \lim_{x \to 1} \frac{8x^5}{3} = \frac{8 \cdot 1^5}{3} = \frac{8}{3} \quad .$$

Ans.: $8/3$.

Problem Type 4.7.3 : Same as 4.7.2, but now you have to use L'Hospital's rule more than once

Example Problem 4.7.3: Use L'Hospital's rule, if appropriate to evaluate

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \quad .$$

Steps

1. First plug-in $x = a$ into $TOP(x)/BOT(x)$ and see whether $TOP(a)/BOT(a)$ yields $0/0$ or ∞/∞. If it does, then L'Hospital's rule is applicable.

2. Invoke L'Hospital's rule

$$\lim_{x\to a} \frac{TOP(x)}{BOT(x)} \quad = \lim_{x\to a} \frac{TOP'(x)}{BOT'(x)} \quad .$$

If you still get an indeterminate form (in this example you do!), keep doing it, until you get a doable limit.

3. Evaluate the limit, by simplifying (if necessary) and plugging-in.

Example

1. Plugging-in $x = 0$ into $\frac{1-\cos x}{x^2}$ gives $0/0$, so L'Hospital's rule is applicable.

2.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{\sin x}{2x} \quad .$$

Now plugging-in $x = 0$ still yields $0/0$, so we have to do L'Hospital again.

$$= \lim_{x\to 0} \frac{(\sin x)'}{(2x)'} = \lim_{x\to 0} \frac{\cos x}{2} \quad .$$

3.

$$= \frac{\cos 0}{2} = \frac{1}{2}.$$

Ans.: $1/2$.

Problem Type 4.7.4 : Use L'Hosptial's rule (or any other method) to evaluate

$$\lim_{x\to\infty}[\,Expression_1(x) - Expression_2(x)\,] \quad ,$$

where one of the expressions is a radical (i.e. involves the square-root sign), and plugging-in gives $\infty - \infty$.

Example Problem 4.7.4: Use L'Hospital's rule, or any other method, to evaluate

$$\lim_{x\to\infty}[\sqrt{x^2+3x} - x]$$

Steps

1. Multiply top and bottom by the *conjugate* $Expression_1(x)+Expression_2(x)$, and simplify as much as you can, using $(a-b)(a+b) = a^2 - b^2$.

2. If you can get by without L'Hospital's rule, don't bother using it (it may be complicated). Try to use any other rules.

Example

1.

$$\lim_{x\to\infty}[\sqrt{x^2+3x} - x] =$$

$$\lim_{x\to\infty}\frac{(\sqrt{x^2+3x} - x)(\sqrt{x^2+3x} + x)}{\sqrt{x^2+3x} + x} =$$

$$\lim_{x\to\infty}\frac{(\sqrt{x^2+3x})^2 - x^2}{\sqrt{x^2+3x} + x} =$$

$$\lim_{x\to\infty}\frac{(x^2+3x) - x^2}{\sqrt{x^2+3x} + x} =$$

$$\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x} + x} \quad .$$

2. In this case you can use the 'only the leading term' counts as $x \to \infty$, what I call 'forget about the little ones'.

$$= \lim_{x\to\infty}\frac{3x}{(\sqrt{x^2} + x)} \quad ,$$

where we ignored $3x$ in view of the much more important x^2, and we get

$$= \lim_{x\to\infty}\frac{3x}{x+x} = \lim_{x\to\infty}\frac{3}{2} = \frac{3}{2} \quad .$$

Ans.: 3/2.

Problem Type 4.7.5 : Use L'Hosptial's rule (or any other method) to evaluate

$$\lim_{x\to\infty} Expression_1(x)^{1/Expression_2(x)} \quad ,$$

where plugging in will give ∞^0.

Example Problem 4.7.5: Use L'Hospital's rule, or any other method, to evaluate

$$\lim_{x\to\infty} x^{1/2x} \quad .$$

Steps

1. Taking natural logarithms, evaluate instead

$$\lim_{x\to\infty} \frac{\ln(Expression_1(x))}{\ln(Expression_2(x))} \quad ,$$

using L'Hospital's rule, if necessary.

2. But what you got now is not the answer but the log-natural of the answer. To get the answer to the problem, you have to undo the effect of ln by exponentiating. So the final answer is $exp(Above\_Limit)$.

Example

1.

$$\lim_{x\to\infty} \ln\left(x^{1/2x}\right) = \lim_{x\to\infty} \frac{\ln x}{2x} =$$

$$\lim_{x\to\infty} \frac{(\ln x)'}{(2x)'} = \lim_{x\to\infty} \frac{1/x}{2} =$$

$$\lim_{x\to\infty} \frac{1}{2x} = 0 \quad .$$

2. **Ans.**: $e^0 = 1$.

A Problem from a Previous Final (Spring 2008, #5 (10 points))

Evaluate the given limits

a) (4 points)

$$\lim_{x\to+\infty} \frac{x^3+2x}{e^x - x}$$

b) (6 points)

$$\lim_{x\to 0^+} x^2 \ln\sqrt{x}$$

Solutions

a) This is an indeterminate form ∞/∞. We have to use L'Hôpital's rule **three** times!

$$\lim_{x\to+\infty} \frac{x^3+2x}{e^x-x} = \lim_{x\to+\infty} \frac{3x^2+2}{e^x-1} = \lim_{x\to+\infty} \frac{6x}{e^x} = \lim_{x\to+\infty} \frac{6}{e^x}$$

Now the top is **not** ∞, and plugging it in, we get

$$= \frac{6}{e^\infty} = \frac{6}{\infty} = 0 \quad .$$

Officially we have to say "because $\lim_{x\to\infty} e^x = \infty$."

Ans. to a): 0.

b) **First** we need to **simplify** to make life easier, using $\ln\sqrt{x} = \ln x^{1/2} = (1/2)\ln x$.

$$\lim_{x\to 0^+} x^2 \ln\sqrt{x} = \lim_{x\to 0^+} (1/2)x^2 \ln x$$

Next we need to bring into a form A/B, by rewriting x^2 as $1/x^{-2}$.

$$\lim_{x\to 0^+} \frac{\ln x}{2x^{-2}} \quad .$$

Now we are ready to apply L'Hôpital's rule.

$$\lim_{x\to 0^+} \frac{\ln x}{2x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-4x^{-3}}$$

Using the algebra of exponents, we get

$$\lim_{x\to 0^+} \frac{x^{-1}}{-4x^{-3}} = \lim_{x\to 0^+} \frac{x^{-1--3}}{-4} = \lim_{x\to 0^+} \frac{x^2}{-4} = \frac{0^2}{-4} = 0 \quad .$$

Ans. to b): 0.