Dr. Z's Math151 Handout #5.1 [The Definite Integral]

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Problem Type 5.1.1 : Use the definition of the integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

(where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$). to evaluate the integral

$$\int_a^b f(x)dx$$

Example Problem 5.1.1: Use the definition of the integral given above to evaluate the integral

$$\int_{-1}^{2} (2+3x)dx \quad .$$

Steps

1. Determine $\Delta x = (b-a)/n$, and write down an expression for $x_i = a + i\Delta x$, that should involve both i and n.

2. Spell-out

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad ,$$

in the present context.

Example

1. $\Delta x = (2-(-1))/n = 3/n$. $x_i = -1 + 3i/n$.

2.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(-1+3i/n)(3/n) =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(-1+3i/n)(3/n) =$$

$$\lim_{n\to\infty} (3/n) \sum_{i=1}^{n} [2+3(-1+3i/n)]$$

$$\lim_{n\to\infty} (3/n) \sum_{i=1}^{n} -1 + 9i/n \quad .$$

3. Evaluate the sum using the formulas (if necessary)

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad .$$

3.

$$\sum_{i=1}^{n} -1 + 9i/n = \sum_{i=1}^{n} -1 \; + \sum_{i=1}^{n} 9i/n =$$

$$-n{+}(9/n)\sum_{i=1}^{n} i = -n{+}(9/n)(n(n{+}1)/2) =$$

$$-n + (9(n+1)/2) = (7n+9)/2 \quad .$$

4. Incorporate the evaluated sum from step 3 into the limit of step 2, and evaluate the limit.

4.

$$\lim_{n\to\infty} (3/n) \sum_{i=1}^{n} -1 + 9i/n =$$

$$\lim_{n\to\infty} (3/n)(7n+9)/2 =$$

$$\lim_{n\to\infty} \frac{3(7n+9)}{2n} =$$

$$\lim_{n\to\infty} \frac{3(7n)}{2n} = 21/2 \quad .$$

Ans.: 21/2.

Problem Type 5.1.2 : Express the integral as a limit of Riemann sums, Do not evaluate the limits. $\int_a^b f(x)dx$.

Example Problem 5.1.2: Express the integral as a limit of Riemann sums, Do not evaluate the limits. $\int_2^6 \frac{x}{3+x^5} dx$.

Steps	**Example**
1. Determine $\Delta x = (b-a)/n$, and write down an expression for $x_i = a + i\Delta x$, that should involve both i and n.	**1.** $\Delta x = (6-2)/n = 4/n$. $x_i = 2+4i/n$.
2. Spell-out $$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad ,$$ in the present context.	**2.** $$\lim_{n\to\infty} \sum_{i=1}^{n} f(2+4i/n)(4/n) =$$ $$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \frac{2+4i/n}{3+(2+4i/n)^5}$$ That's it! Do not try to evaluate.

A problem from a previous Final Exam (Spring 2008, #3a (10 points))

Evaluate the following limit-Riemann sums by any method you like.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2i}{n} - \left(\frac{i}{n}\right)^2 \right] \frac{1}{n} \quad .$$

Solution. This is a trick-question! If you do it directly, it will take you a very long time. The trick is to recognize it as the *area under a curve*, that next time we will learn equals to a **definite integral**.

The given limit has the format:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad ,$$

with $\Delta x = \frac{1}{n}$, $x_i = i/n$, and $f(x) = 2x - x^2$, and the endpoints are $a = 0$ and $b = 1$, so it equals the area under the curve $y = 2x - x^2$, above the x-axis, between $x = 0$ and $x = 1$, and so equals the definite integral

$$\int_0^1 (2x - x^2)\, dx.$$

Very soon we will know how to do it. Take the antiderivative

$$\int_0^1 (2x - x^2) = x^2 - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} - 0 = \frac{2}{3} \quad .$$

Ans.: $\frac{2}{3}$.