Solutions to MATH 151(07-09), Dr. Z. , **First Midterm**, Mon., Oct. 13, 2008.

1. (12 points) Find $f'(4)$, if

$$f(x) = \frac{\sqrt{x}}{2}$$

using the **definition** of the derivative. [No Credit for other methods].

Solution: The **output** is a **number**.

$$f'(4) = \lim_{h \to 0} \frac{\frac{\sqrt{4+h}}{2} - \frac{\sqrt{4}}{2}}{h} = \lim_{h \to 0} \frac{\frac{\sqrt{4+h}}{2} - 1}{h} =$$

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{2h} = \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{(2h)(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{(\sqrt{4+h})^2 - 2^2}{(2h)(\sqrt{4+h} + 2)} =$$

$$\lim_{h \to 0} \frac{4+h-4}{(2h)(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{h}{(2h)(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{(2)(\sqrt{4+h} + 2)} =$$

$$\lim_{h \to 0} \frac{1}{(2)(\sqrt{4+0} + 2)} = \frac{1}{8} \quad .$$

Ans.: $\frac{1}{8}$.

Comments: Most people, who got it right, first found $f'(x)$ and (correctly) got $\frac{1}{4\sqrt{x}}$, and at the very end plugged-in $x = 4$. That's fine, but it is more efficient to plug-in $x = 4$ as soon as possible. Quite a few people, **just** found $f'(x)$. This time I was lenient, and gave them lots of partial credit, but in the future such correct answer to a **different** question will get 0 points. It is better to give the wrong answer to the right question then the right answer to the wrong question, since in the former case, at least, you understood the question. Understanding the question is the secret to success.

2. (12 points) Find the equation of the tangent line to the curve

$$x^3 + 3xy + y^3 + y = 16 \quad ,$$

at the point $(2, 1)$.

Solution to 2: The **output** is an **equation of a straight line**.

Differeniate with respect to x, using implicit differentiation.

$$3x^2 + (3xy)' + (y^3)' + y' = 16' \quad ,$$

Which gives:

$$3x^2 + (3x)'y + (3x)y' + 3y^2y' + y' = 0 \quad ,$$

Which gives:

$$3x^2 + 3y + (3x)y' + 3y^2y' + y' = 0 \quad ,$$

Now plug-in $x = 2$, $y = 1$, to get:

$$3 \cdot (2)^2 + 3 \cdot 1 + (3 \cdot 2)y' + 3 \cdot (1)^2 y' + y' = 0 \quad ,$$

Simplifying:

$$12 + 3 + 6y' + 3y' + y' = 0 \quad ,$$

Yielding:

$$-10y' = 15 \quad ,$$

and solving for y' yields:

$$y' = \frac{15}{-10} = -\frac{3}{2} \quad .$$

So the **slope** of the tangent line is $m = -\frac{3}{2}$. Finally, to get the equation of the tangent-line, you use the famous formula

$$(y - y_0) = m(x - x_0) \quad ,$$

with $(x_0, y_0) = (2, 1)$, getting

$$(y - 1) = -\frac{3}{2}(x - 2) \quad ,$$

and in standard-form

$$y = -\frac{3x}{2} + 4 \quad .$$

Ans.: The equation of the tangent-line at the indicated point is $y = -\frac{3x}{2} + 4$.

Comments: Very few people did it exactly as I did it. The vast majority first found y' in general: From

$$3x^2 + 3y + (3x)y' + 3y^2 y' + y' = 0 \quad ,$$

They did:

$$(3x)y' + 3y^2 y' + y' = -(3x^2 + 3y) \quad ,$$

and then

$$(3x + 3y^2 + 1)y' = -(3x^2 + 3y) \quad ,$$

and expressed y'

$$y' = -\frac{3x^2 + 3y}{3x + 3y^2 + 1} \quad ,$$

and only now plugged-in $x = 2$, $y = 1$. This is correct, but much **less efficient**, especially in more complicated problems.

3.

(a) (6 points) Show that the equation $x^3 - 2 = 0$ has a solution in the open interval $0 < x < 2$.

Sol. of 3a) : The function $f(x) = x^3 - 2$ is continuous in the closed interval $[0, 2]$, (since it is a polynomial). At the left endpoint $x = 0$, $f(0) = -2$. At the right endpoint, $x = 2$,

$f(2) = 2^3 - 2 = 6$. Since the **value** 0 is **intermediate** between the values -2 and 6, it follows by the **Intermediate Value Theorem** that there is a number c in the open interval $(0, 2)$ such that $f(c) = 0$. This is the desired solution.

Comments: About half of the people got it completely right (but with different phrasings). Another quarter started right, but what they wrote didn't make complete sense, and another quarter wrote complete gibberish.

(b) (6 points) Use the **formal definition** of the limit to prove that

$$\lim_{x \to -2} -2x - 3 = 1 \quad .$$

Solution of 3b): The formal definition of

$$\lim_{x \to a} f(x) = L$$

is: For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad if \quad |x - a| < \delta \quad .$$

We want to find an expression for δ in terms of ϵ.
In this problem $f(x) = x - 2$, $a = -2$ and $L = 1$. Implementing these we have to find an expression for δ in terms of ϵ such that

$$| - 2x - 3 - (1)| < \epsilon \quad if \quad |x - (-2)| < \delta \quad .$$

Simplifying inside the **asbsolute value** ($||$), we get that we need

$$| - 2x - 4| < \epsilon \quad if \quad |x + 2| < \delta \quad .$$

Factoring out what's (still) **inside** the $||$, we get

$$| - 2(x + 2)| < \epsilon \quad if \quad |x + 2| < \delta \quad .$$

Using the **very important** rule

$$|ab| = |a||b| \quad ,$$

we get

$$| - 2||x + 2| < \epsilon \quad if \quad |x + 2| < \delta \quad .$$

Of course $| - 2| = 2$, so this is:

$$2|x + 2| < \epsilon \quad if \quad |x + 2| < \delta \quad .$$

Dividing both sides by 2 we get

$$|x + 2| < \epsilon/2 \quad if \quad |x + 2| < \delta \quad .$$

So if we take $\delta = \epsilon/2$ things work out, since in that case we have a **tautology** (something that is obviously true). This completes the proof (since we came out with a δ in terms of ϵ such that the above is true.

Comments: To my great disappointment, very few people got it right. Most people started the right way, but they they committed a **huge** blunder: "taking out" -2 from the $||$. Most people got $\delta = -\epsilon/2$. This is **utter nonsense**, since **both** ϵ and δ are **distances**, and so must be **positive**. A negative distance does not make sense.

4. (16 points ([4 pts each]) Find the derivative $f'(x)$ if:

(a) $f(x) = \frac{x+2}{x^2+5}$

Sol. of 4a):

$$f'(x) = \frac{(x+2)'(x^2+5) - (x+2)(x^2+5)'}{(x^2+5)^2} = \frac{(1)(x^2+5) - (x+2)(2x)}{(x^2+5)^2} =$$

$$\frac{x^2+5-2x^2-4x}{(x^2+5)^2} = \frac{-x^2-4x+5}{(x^2+5)^2} \quad .$$

Ans.: $\frac{-x^2-4x+5}{(x^2+5)^2}$ or $-\frac{x^2+4x-5}{(x^2+5)^2}$

(b) $f(x) = x^3 \sin x + 2x$

Sol. of 4b):

$$f'(x) = (x^3 \sin x + 2x)' = (x^3 \sin x)' + (2x)' = (x^3)' \sin x + (x^3)(\sin x)' + 2$$

$$= 3x^2 \sin x + x^3 \cos x + 2 \quad .$$

Ans. to 4b: $3x^2 \sin x + x^3 \cos x + 2$.

(c) $f(x) = \frac{2+e^x}{1+2e^x}$

Sol. of 4c):

$$f'(x) = \frac{(2+e^x)'(1+2e^x) - (2+e^x)(1+2e^x)'}{(1+2e^x)^2} = \frac{(e^x)(1+2e^x) - (2+e^x)(2e^x)}{(1+2e^x)^2}$$

$$= \frac{e^x + 2e^{2x} - 4e^x - 2e^{2x}}{(1+2e^x)^2} = \frac{-3e^x}{(1+2e^x)^2}$$

Ans. to 4c: $\frac{-3e^x}{(1+2e^x)^2}$ or $-3\frac{e^x}{(1+2e^x)^2}$.

(d) $f(x) = xe^{x^2}$

Sol. of 4d):

$$f'(x) = x'e^{x^2} + x(e^{x^2})' = e^{x^2} + x(e^{x^2}(x^2)') =$$

$$e^{x^2} + x(e^{x^2} 2x) = e^{x^2} + 2x^2 e^{x^2} = (1+2x^2)e^{x^2} \quad .$$

Ans. to 4d): $(1+2x^2)e^{x^2}$.

5.

(a) (6 points) If $f(x) = e^{x^2} + x^2$, find the second derivative $f''(x)$.

Sol. to 5a): First we find the first derivative:

$$f'(x) = (e^{x^2})' + (x^2)' = 2xe^{x^2} + 2x \quad .$$

Now we take the derivative of $f'(x)$.

$$f''(x) = (2xe^{x^2} + 2x)' = (2xe^{x^2})' + (2x)' = (2x)'e^{x^2} + 2x(e^{x^2})' + 2 =$$

$$2e^{x^2} + 2x(2xe^{x^2}) + 2 = 2e^{x^2} + 4x^2e^{x^2} + 2 = 2(1+2x^2)e^{x^2} + 2$$

Ans. to 5a): $2(1+2x^2)e^{x^2} + 2$.
(b) (6 points) If the law of motion is $s = t^7 + t$, find the displacement (position), velocity, and acceleration at $t = -1$. Is it moving forward or backwards then? Is it speeding up or slowing down then?

Sol. to 5b):

$$v = (t^7 + t)' = 7t^6 + 1 \quad ,$$

$$a = (7t^6 + 1)' = 42t^5 \quad .$$

When the time is $t = -1$, we have

$$s(-1) = (-1)^7 + (-1) = -2 \quad , \quad v(-1) = 7 \cdot (-1)^6 + 1 \quad , \quad a(-1) = -42 \quad .$$

Ans.to 5b): At time $t = -1$ the displacement (a.k.a. position) is -2, the velocity is 8 and the acceleration is -42. It is moving **forward** (since the sign of the velocity is **positive**), and it is **slowing down** (since the signs of the velocity and acceleration are opposite each other [one is positive and the other is negative]).

Common Mistakes: 1.‘People said “backwards” because the sign of the displacement is negative (the sign of the displacement does not say anything about the direction of moving, it only tells you where you are located).

2. People said (correctly!) that the particle is slowing-down, but gave the **wrong reason**. They said that it is slowing down since the acceletarion is negative. Ths sign of the acceleration, by itself, doesn’t tell you anything. You have to compare it to the sign of the velocity. The same signs: speeding up, opposite signs: slowing down.

3. People messed up the differentiantions. Come-on, you should be able to do these simple differentiations.

6. (12 points [3 pts each]) Find the limits

$$(a) \qquad \lim_{x \to -1} \frac{x^2+3x+2}{x+1}$$

Sol. of 6a):

$$\lim_{x \to -1} \frac{x^2+3x+2}{x+1} = \lim_{x \to -1} \frac{(x+1)(x+2)}{x+1} = \lim_{x \to -1} x+2 = -1+2 = 1 \quad .$$

Ans. to 6a): 1.

$$(b) \lim_{x \to 0} \frac{x}{\sqrt{9+x}-3}$$

Sol. of 6b):

$$\lim_{x \to 0} \frac{x}{\sqrt{9+x}-3} = \lim_{x \to 0} \frac{x(\sqrt{9+x}+3)}{(\sqrt{9+x}-3)(\sqrt{9+x}+3)} = \lim_{x \to 0} \frac{x(\sqrt{9+x}+3)}{(\sqrt{9+x})^2-3^2}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{9+x}+3)}{(9+x)-9} = \lim_{x \to 0} \frac{x(\sqrt{9+x}+3)}{x} = \lim_{x \to 0} \sqrt{9+x}+3 = \sqrt{9+0}+3) = 3+3 = 6 \quad .$$

Ans. of 6b): 6.

$$\lim_{x \to 0} \frac{3-3\cos x+\sin x}{x}$$

Sol. to 6c):

$$\lim_{x \to 0} \frac{3-3\cos x+\sin x}{x} = \lim_{x \to 0} \frac{3-3\cos x}{x} + \lim_{x \to 0} \frac{\sin x}{x} =$$

$$\lim_{x \to 0} \frac{3(1-\cos x)}{x} + \lim_{x \to 0} \frac{\sin x}{x} = 3 \lim_{x \to 0} \frac{(1-\cos x)}{x} + \lim_{x \to 0} \frac{\sin x}{x} = 3 \cdot 0 + 1 = 1 \quad .$$

Ans. to 6c): 1.

$$(d) \lim_{x \to 0} \frac{16 \sin^5 x}{\sin^2 2x \sin^3 x}$$

Sol. to 6d): **First** use **algebra** to simplify

$$\lim_{x \to 0} \frac{16 \sin^5 x}{\sin^2 2x \sin^3 x} = \lim_{x \to 0} \frac{16 \sin^2 x}{\sin^2 2x} \quad .$$

Now, by Dr. Z's "get-rid of sines rule" we have

$$= \lim_{x \to 0} \frac{16(\sin x)^2}{(\sin 2x)^2} = \lim_{x \to 0} \frac{16(x)^2}{(2x)^2} = \lim_{x \to 0} \frac{16x^2}{4x^2} = \lim_{x \to 0} 4 = 4 \quad .$$

7. (12 points) Find the values of the constants a and b that will make the function

$$f(x) = \begin{cases} x^2, & \text{if } x < 1; \\ ax + b, & \text{if } x \geq 1 \ . \end{cases}$$

differentiable everywhere.

Sol. of 7: This is a **piecewise** function. Both pieces are obviously differentiable, being polynomials. The only issue is at the **transition place**, $x = 1$. We need it to be both continous and differentiable at $x = 1$. For it to be continuous, we need $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, so we have the **first condition**:

$$1 = a + b \quad .$$

Now we have

$$f'(x) = \begin{cases} 2x, & \text{if } x < 1; \\ a, & \text{if } x \geq 1 \ . \end{cases}$$

We also need $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$, so we have the **second condition**

$$2 = a \quad .$$

Going back to the first condition we have $b = 1 - a = -1$. So $a = 2, b = -1$.

Ans. of 7b): $a = 2, b = -1$.

Comment: Only about %20 of the people got it completely right. Most people got as far as getting the first condition, but instead of addmiting that they don't have enough information, they just game **one** solution (usually $a = 1$, $b = 0$). Of course, only using $1 = a+b$ does not determine clear-cut values for a and b, there are infinitely many solutions. You need the second condition as well.

8. (12 points) Find a point on the curve

$$y = -\cos x \quad ,$$

with $0 < x < \pi$, where the tangent line is parallel to the line $y = \frac{x}{2} + 4$. Then find the equation of that tangent line.

Sol. to 8: The slope of the given line is $\frac{1}{2}$, so we need to find a point where the derivative equals $1/2$. Differentiating $y = -\cos x$ gives

$$\frac{dy}{dx} = \sin x \quad ,$$

We need to solve

$$\sin x = \frac{1}{2} \quad .$$

This has two solutions in $0 < x < \pi$, $x = \pi/6$ and $x = 5\pi/6$. Let's pick the first one. The y coordinate when $x = \pi/6$ is $y = -\cos\pi/6 = -\sqrt{3}/2$, so the point is

$$(\frac{\pi}{6}, -\frac{\sqrt{3}}{2}) \quad .$$

This answers the **first part**. To answer the second part, we use the point-slope equation

$$(y - y_0) = m(x - x_0) \quad ,$$

getting

$$(y - -\frac{\sqrt{3}}{2}) = \frac{1}{2}(x - \frac{\pi}{6}) \quad ,$$

Simplifying:

$$y + \frac{\sqrt{3}}{2} = \frac{1}{2}x - \frac{\pi}{12} \quad ,$$

and finally

$$y = \frac{1}{2}x - \frac{\pi}{12} - \frac{\sqrt{3}}{2} \quad .$$

Ans. to second part: The equation of the tangent line at that point is: $y = \frac{1}{2}x - \frac{\pi}{12} - \frac{\sqrt{3}}{2}$